

**Special Topic Paper: “On the theory of dispersion of
X-rays” and the Kramers-Kronig Relation**
by R. de L. Kronig

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Introduction to Kronig:

Ralph de Laer Kronig has had a long standing relationship with the field of quantum mechanics and material sciences. Receiving his Ph.D in 1925 from Columbia University in New York, he was beginning his career as a physicist in the era of extreme expanse in the knowledge of the quantum realm. He was often writing with many other professionals of the time such as Heisenberg, Pauli, Bohr, and Kramers. Kronig was in collaboration with Heisenberg as the latter was coining the basis for quantum mechanics. In developing major findings in quantum and solid state physics such as the Kronig-Penny model, the initial idea of particle spin, and his theory of X-ray dispersion, Ralph Kronig has cemented himself in history as a juggernaut in the physicist world.

Kronig Paper:

In one of his earliest, and most cited, papers [2] Kronig discusses how quantum dispersion theory can be applied to the x-ray region of electromagnetic radiation and x-ray refraction on a bulk material. Kronig first discusses general dispersion theory and how external radiation can affect a bulk material using classical electrodynamics. He assumes that atoms contain electric charges elastically and are isotropically bound to equilibrium positions with damping terms proportional to velocity. Externally incident radiation causes oscillations at what he calls the impressed frequency, which is the frequency of the electromagnetic radiation. These atomic oscillations then emit spherical wavelets with an electric (dipole) moment parallel to and almost in phase with the incident electric field when the impressed frequency, ν , is sufficiently different from any natural frequency of the atomic resonators. These spherical wavelets are the scattered radiation. For $\nu \approx \omega_i$ which is the natural frequency of the i 'th resonator, the amplitude of the oscillations becomes larger and considerably out of phase from the incident field. The scattering thus increases and due to the phase shift the wavelets produce a decrease in the amplitude of the original incident wave which is also known as material absorption. If the incident electric field is sinusoidal, then Kronig states that the induced electric moment is

$$M(t) = E \sum_{i=1}^f \frac{e_i^2 \cos(2\pi\nu t)}{4\pi^2 m_i (\omega_i^2 - \nu^2)} \quad \text{Eq(1)}$$

for an atom containing charges e_1, e_2, \dots, e_f with masses m_1, m_2, \dots, m_f and having resonant frequencies $\omega_1, \omega_2, \dots, \omega_f$ and $\nu \neq \omega_i$. The amount of energy absorbed on average by one of these resonators during a time dt from the incident radiation field whose density, $\rho(\nu)$, does not vary much about the resonant frequency is given as

$$\alpha_i c \rho(\omega_i) dt = \frac{\pi e_i^2}{m_i} \rho(\omega_i) dt \quad \text{Eqn(2)}$$

where α_i is the total atomic absorption coefficient of resonator i . The induced electric moment can then be described by the absorption coefficient:

$$M(t) = E \sum_{i=1}^f \frac{c\alpha_i \cos(2\pi\nu t)}{4\pi^3(\omega_i^2 - \nu^2)} \quad \text{Eqn(3)}$$

Kronig explains that wave theory is the only theory that describes the connection between absorption and dispersion well enough and therefore he applies this classical electrodynamic relation to every single atom in the material in order to make a quantum theory of dispersion.

Kramers had been working on a similar feat just the year before (1924) and had advised Kronig on completing this portion of the paper [3]. Kramers wrote a paper [4] describing the perturbations that come forth when a periodic system (a crystalline solid) experiences an incident sinusoidal electric field. Kramers found the induced scattering moment of the atoms by introducing quantum analogues to the quantities of the periodic system resulting in the quantum average scattering moment per atom of

$$M(t) = E(t) \sum_i \frac{f_i e^2 \cos(2\pi\nu t)}{4\pi^2 m(\omega_i^2 - \nu^2)} \quad \text{Eqn(4)}$$

where m and e are the mass and charge of an electron and $f_i = \frac{mh\omega_i b_i}{\pi e^2}$ where h is Planck's constant and b_i is defined by

$$P_i dt = b_i \rho(\omega_i) dt \quad \text{Eqn(5)}$$

This is the probability of an atom being excited from the ground state to the state i in a time dt by an electromagnetic field with energy density between the frequencies of ω_i and $\omega_i + d\omega$.

Kronig relates this parameter b_i that Kramers uses to the absorption coefficient of a material by stating that $b_i h \omega_i \rho(\omega_i) dt$ is the energy absorbed on average per atom in a time dt and is thus equal to $\alpha_i c \rho(\omega_i) dt$. This means that the parameter f_i simplifies to

$$f_i = \frac{m\alpha_i c}{\pi e^2} \quad \text{Eqn(6)}$$

When this form of f_i is plugged into Kramers' function for the average scattering moment per atom it simplifies to Kronig's derived version of the scattering moment of the atom. This parameter f_i was named the 'number of dispersion electrons' by Drude; however, Pauli and even now it is called the 'oscillator strength'. It describes how strong or probable a transition between two energy levels of a system is. This is a very important parameter when designing laser or light emitting materials as one wants a material with a very strong oscillator strength to improve efficiency of the device.

Ladenburg Paper:

Although Kramers and Kronig have both now shown the relationship between the quantum scattering moment and the classical electrodynamic scattering moment, this was first proclaimed by Rudolph Ladenburg in 1921 [5]. Ladenburg published a paper in 1921 titled "The Quantum-theoretical interpretation of the number of dispersion electrons". In this he sought to determine a quantum mechanical version of the number of dispersion electrons per unit volume, which he called R , and is related to the absorption of isolated spectral lines. That is, it is related to the absorption and emission spectra that is unique to every element in the periodic table. Ladenburg begins similarly as Kronig did by using classical electrodynamic, but to derive the energy radiated by N molecules that have R oscillatory electrons per second.

$$J_{EI} = \frac{\bar{U}R}{\tau}, \quad \tau = \frac{3mc^3}{8\pi^2 e^2 \nu_0^2} \quad Eqn(7)$$

where τ is the decay time of the electron oscillation which is assumed to be damped only by emission and \bar{U} is the mean energy. Ladenburg assumes that the molecules are in thermal equilibrium and that the electrons have three degrees of freedom, thus yielding the mean energy to be $\bar{U} = \frac{3c^3}{8\pi\nu_0^2} u_0$, where u_0 is the radiation density. Thus the classical derivation for the radiated energy per second of this oscillating electron system is

$$J_{EI} = \frac{\pi e^2}{m} R u_0 \quad Eqn(8)$$

Ladenburg then uses the quantum mechanical theories set forth by Bohr and Einstein to determine a quantum mechanical version of the radiated energy per second of this system. By this time it is understood that the emission of radiation from atoms could come from either spontaneous transitions from a state k to a state i , or from interactions from a radiation field present in the system under question. He gives probability constants to these two types of emission: the first having $a_{k \rightarrow i}$ and the second having $b_{k \rightarrow i}$ and he assumes these are both temperature independent. If each transition is related to an energy $h\nu_{ik}$ of emission then the total energy emitted per second is given by

$$J_Q = h\nu_{ik} N_k (a_{k \rightarrow i} + b_{k \rightarrow i} u_{ik}) \quad Eqn(9)$$

where N_k is the number of oscillators in the state k and is related to the number of oscillators in state i via Boltzmann relations (which is an approximation made by Ladenburg).

$$\frac{N_k}{N_i} = \frac{g_k [\exp(-E_k/kT)]}{g_i [\exp(-E_i/kT)]} \quad Eqn(10)$$

Here the terms $g_{i,k}$ are degeneracy terms which Ladenburg calls a statistical weight. From Einstein's theory of spontaneous emission, the transition probabilities $a_{k \rightarrow i}$ and $b_{k \rightarrow i}$ are related via

$$a_{ki} = b_{ki} \frac{8\pi h \nu_{ik}^3}{c^3} \quad \text{and} \quad g_i b_{ik} = g_k b_{ki} \quad Eqn(11)$$

Through these equations, the quantized energy emitted or absorbed can be written as

$$J_Q = N_i \frac{g_k}{g_i} a_{ki} \frac{c^3}{8\pi \nu_{ik}^2} u_{ik} \quad Eqn(12)$$

To determine the relation between this quantum version of the energy emitted by quantum oscillators and the classical electrodynamic version of the energy radiated, simply set them equal to each other:

$$\begin{aligned} J_{EI} &= \frac{\pi e^2}{m} R u_0 = J_Q = N_i \frac{g_k}{g_i} a_{ki} \frac{c^3}{8\pi \nu_{ik}^2} u_{ik} \\ \Rightarrow R &= N_i h \nu_{ik} b_{ik} \frac{m}{\pi e^2} = N_i \frac{g_k}{g_i} a_{ki} \frac{mc^3}{8\pi^2 e^2 \nu_{ik}^2} u_{ik} \end{aligned} \quad Eqn(13)$$

Again R is the number of dispersion electrons per volume which is now shown to be proportional to the spontaneous emission rate and the frequency of the oscillator. Therefore Ladenburg had derived the quantum mechanical equivalent of the number of dispersion electrons per unit volume. Comparing to the oscillator strength that Kramers and Kronig use ($f_i = \frac{m h \omega_i b_i}{\pi e^2}$), it is easy to see that $R = N_i f_i$.

In electron theory the number of dispersion electrons per unit volume, R , is related to not only the spectral lines of absorption, but also describes anomalous dispersion and magnetic

rotation. Therefore if using quantum mechanics can relate R to the transition probabilities of spontaneous emission and absorption, then there must be a quantum equivalent for anomalous dispersion as well. However, at the time of Ladenburg's 1921 paper there was no quantum theory of anomalous dispersion. So to accommodate the lack of detailed theory, Ladenburg describes that if molecules observe a change in their electric dipole moment from an incident wave or radiation will cause modified states to occur. Transitions between these closely spaced levels are related via the spontaneous emission and absorption frequencies of the material. In addition to these effects, the velocity of the radiation will change and this change is more pronounced the closer the radiation frequency is closer to the natural frequency of the material and thus describes an anomalous dispersion effect. As such, the scattering of the incident radiation by the material is constrained by radiation damping. In this case the damping is the inverse of the spontaneous emission rate, a_k .

Ladenburg finally uses experimental means to calculate the number of dispersion electrons, R , from magnetic rotation measurements from alkali metals such as sodium and is related to the equation of R that he had determined.

Refractive Index:

The refractive index of a material is the ratio of how fast a lightwave propagates in the material compared to the speed of light in vacuum. Kronig uses the scattering moment he derived (with the help of Kramers and Ladenburg) to determine the refractive index of a material. He describes a system of which a slab of material is perpendicular to the propagation axis and is thin compared to the wavelength of incident light such that $dz = q\lambda$ where q is a small. He then attempts to calculate the electric field emanating from the atoms in the slab at a point P a distance l away from the slab. Using spherical and cylindrical coordinates, the slab can be divided into cylinders of radius ρ about the z (optical) axis. He then assumes that the phase of the spherical waves must match when they are arising from the inner and outer boundaries of the material: $dr = q\lambda$ where r is distance from P to the edges of the cylinder. In order to calculate the field at the point P , a finite number of atoms must reside within the cylindrical volume of the material ($2\pi\rho dr dz = 2\pi l q^2 \lambda^2$), thus a constraint is placed on the distance l :

$$2\pi l q^2 \lambda^2 N = n$$

where n is the total number of atoms in the cylindrical volume and N is the number of atoms per unit volume. Finally, Kronig states that if all of these radiation fields are integrated then at distances larger than l the scattered wave from the material "irons out" by interference and becomes a plane wave. Thus, if a refractive medium is made of these types of slabs and the polarization of the medium produced by the incident electric field is small compared to the incident field then the refractive index (μ) is obtained from the scattering moment M and the constraint on l as

$$\mu - 1 = \delta = \sum_i \frac{N e^2 f_i}{2\pi m (\omega_i^2 - \nu^2)} \quad Eqn(14).$$

This again assumes that $\mu - 1 \ll 1$ and that all of the atoms are of the same kind.

Kronig performs further analysis on the refractive index of a material by enforcing that the wavelength of light incident on the material in question is in the X-ray range ($\lambda \sim 1 \text{ Angstrom}$). It is important to keep in mind that the separation of atoms in a solid are also on the order of Angstroms and thus the wavelength of incident radiation and the atomic spacing are comparable. This means if one draws out a cube with sides equal to the incident wavelength within the solid, then there will only be a few atoms that can fit within this wavelength cube. Ordinarily in the field of Optics and in dispersion theory it is necessary that there is a large number of atoms within a wavelength cube in order for the formulas of the polarization of a sphere in a uniform electric field to be applied to the system (making the problems simplify vastly). This is of course if the material polarization (i.e the sum of electric dipole moments of the atoms in the material) is comparable to the incident electric field. If the material polarization field is however much smaller than the incident electric field, then a large number of atoms within the wavelength cube is not necessarily needed to hold as an assumption. Because $\mu - 1 \approx 10^{-6}$ and $l \approx 10^{-3} \text{ cm}$ for solids, Equation (14) holds for the analysis of X-ray dispersion.

The electrons that will interact with the X-rays can be excited into different energy states depending on the electron's initial state and its velocity; therefore the excited energy states are described by (v, n, k, j) or (v, r) where the quantum numbers n, k, j are compacted into a single quantum number r . The velocity of the excited electron is proportional to the frequency of the transition to the excited state which means the state is described by (ω, r) . Thus the continuous spectrum of states (ω, r) make an absorption band of the material.

Kronig finally links the formula he derived for the refractive index of a slab of material to the absorption band or absorption coefficient of the same material by stating that within a frequency interval of $d\omega$ there is a resonator with charge $f_r(\omega)d\omega * e$ and mass $f_r(\omega)d\omega * m$ where $f_r(\omega)$ is the frequency dependent oscillator strength (number of dispersion electrons). Using Equation (6) for the oscillator strength, the frequency dependent version is

$$f_r(\omega) = \frac{cm\alpha_r(\omega)d\omega}{\pi e^2} \quad Eqn(15)$$

Thus using this frequency dependent version of the oscillator strength, the refractive index of a solid can be related to the absorption band of the solid as :

$$\mu - 1 = \delta = \sum_r \int \frac{Nc \alpha_r(\omega)}{2\pi^2(\omega^2 - \nu^2)} d\omega \quad Eqn(16)$$

Kramers-Kronig Relation:

Not even one year later, Hendrik Anthony Kramers published a paper [6] detailing similar findings of Kronig and even expanding upon them. Due to the fact that the paper is in French and my knowledge of the French language is quite limited, I can only put together pieces of Kramers 1927 paper on the dispersion of light from atoms. Kramers uses electrodynamics to relate an incident electric field and the material polarization field to a complex value $\zeta = \xi + i\eta$

which Kramers calls the polarization coefficient. This value is similar to the refractive index term δ that Kronig derived. Kramers uses the Sellmeier formula to state that

$$\zeta = \sum_k \frac{e^2}{m} \frac{f_k}{(\omega_k^2 - \omega^2)} \quad Eqn(17)$$

Here f_k is the oscillator strength and ω_k are the resonant frequencies as in [2]. Kramers then introduces the atomic absorption coefficient α and states that it is dependent on both the frequency of incoming radiation and the imaginary part of ζ as

$$\alpha(\omega) = -\frac{4\pi\omega}{c} \eta \quad Eqn(18).$$

Kramers then goes on to state that the frequency dependent absorption must integrate to a constant value over all frequencies (which Kronig also discerned [2]) to $a_k = \int \alpha(\omega) d\omega$.

He then makes the same equivalence as Kronig did for the oscillator strength and the material absorption: $f_k = \frac{mc\alpha_k}{2\pi e^2}$, then only difference from Kronig's evaluation being the subscript on f and α and the factor of 2 in the denominator. Finally the complex value of the polarization coefficient is determined to be

$$\zeta = \frac{c}{2\pi^2} \int_0^\infty \frac{\alpha(\omega')}{\omega'^2 - \omega^2} d\omega' \quad Eqn(19).$$

He uses the Cauchy principle and the fact that the system is causal (does not depend on future time instances) to show that the real part of the polarization coefficient is equal to

$$Re\{\zeta\} = \xi = \frac{c}{2\pi^2} \oint_0^\infty \frac{\alpha(\omega')}{\omega'^2 - \omega^2} d\omega' \quad Eqn(20)$$

which is exactly what Kronig found in Equation (16) but not on a per unit volume basis and he even cites Kronig's 1926 paper on this. This is where Kramers expands upon Kronig's findings on the dispersion of incident light by a refracting medium. Kramers uses the frequency dependent absorption that he derived previously (18) to relate the real and imaginary parts of the complex polarization coefficient

$$\xi(\omega) = -\frac{2}{\pi} \oint_0^\infty \frac{\omega' \eta(\omega')}{\omega'^2 - \omega^2} d\omega' \quad Eqn(21).$$

It is easy to see that the real part of the polarization coefficient is even with respect to ω as $\xi(-\omega) = \xi(\omega)$. From the fact that $\zeta = \xi + i\eta$, it is safe to say that because η is $\pi/2$ out of phase from ξ then η must be an odd function with respect to ω . The real polarization coefficient can be simplified by using the fact that it is even

$$\xi(\omega) = -\frac{2}{\pi} \oint_0^\infty \frac{\omega' \eta(\omega')}{\omega'^2 - \omega^2} d\omega' = -\frac{2}{\pi} \oint_0^\infty \frac{\omega' \eta(\omega')}{(\omega'+\omega)(\omega'-\omega)} d\omega' = -\frac{1}{\pi} \oint_{-\infty}^\infty \frac{\omega' \eta(\omega')}{(\omega'+\omega)(\omega'-\omega)} d\omega' = \frac{1}{\pi} \int_{-\infty}^\infty \frac{\eta(\omega')}{\omega'-\omega} d\omega' \quad Eqn(22)$$

Using the fact that ξ and η are 90 degrees out of phase, then

$$\eta(\omega) = -\frac{1}{\pi} \int_{-\infty}^\infty \frac{\xi(\omega')}{\omega'-\omega} d\omega' \quad Eqn(23)$$

Thus Kramers (and by his previous work, Kronig) have derived a correlation between the real and imaginary part of the complex polarization coefficient given by the two equations (22) and (23). Because of this triumph to material sciences and optics, these relations have since been known as the Kramers-Kronig relation. The Kramers-Kronig relation is invaluable in the field of

optics as one can experimentally determine a material's refractive index via absorption measurements. Using a tunable light source, various wavelengths of light can be made incident upon a lossy material of known length. By measuring the intensity of the output light after propagating through the material, the absorption coefficient at various wavelengths can be determined via Beer's Law assuming that the loss is linear:

$$I(\omega, z) = I_0 \exp[-\alpha(\omega) z] \Rightarrow \alpha(\omega) = -\frac{1}{L} \ln\left(\frac{I(\omega, L)}{I_0}\right) \quad Eqn(24)$$

From these absorption measurements one can then use the Kramers-Kronig relation to determine the real refractive index of the material. Although Kronig derived his form of the refractive index of a solid in terms of its absorbance with the thoughts of X-ray dispersion in mind, the Kramers-Kronig relation is actually valid for any range of wavelengths. The only fault is that the integration in equations (22) and (23) are bound from negative infinity to infinity. This means to get an exactly correct result one would need a lightsource with infinite bandwidth in wavelength or infinitely small resolution in wavelength such as to sweep over all wavelengths and frequencies. In practice this is never the case. Therefore there can be some error in experiment if the range of frequencies used to characterize the attenuation of the media is not fine or broad enough. However, nonlinear media can still take advantage of Kramers-Kronig relation as the absorption of nonlinear optical media is continuous in short bands of frequency.

Current Research:

Material Classification:

As previously stated, both linear and nonlinear optical responses of materials can be classified by using the Kramers-Kronig relationship. In Kramers and Kronig's time, they were describing dispersion via the polarization coefficient. In current research and teachings the Kramers-Kronig relation is most often used for the complex susceptibility of a material, χ . The complex electric susceptibility is defined as the proportionality constant (or tensor) that relates the electric field and the material polarization field: $\vec{P} = \epsilon_0 \chi \vec{E}$. In general χ can be complex and a function of time or frequency. In the case of a causal linear system, the susceptibility can be seen as the response of the material. In general, a material cannot respond instantaneously to the incident radiation and therefore the polarization field is defined to be a convolution of the time dependent susceptibility and the time varying electric field

$$P(t) = \epsilon_0 \int_0^t \chi(t - \tau) E(\tau) d\tau = \epsilon_0 \chi(t) * E(t) \quad Eqn(25)$$

Performing a Fourier Transform of this equation takes it into the frequency domain. It is common knowledge that the Fourier Transform of a convolution is equivalent to a product of two functions in the frequency domain. Therefore $P(\omega) = \epsilon_0 \chi(\omega) E(\omega)$. In this form it is clear to see that the susceptibility is the transfer function of this material system (the output to input ratio).

As noted this is a complex term as shown in [2,6]. The modern version of the Kramers-Kronig relation for the electric dipole material response is thus [7]

$$\chi'(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\chi''(\Omega)}{\omega - \Omega} d\Omega$$

$$\chi''(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\chi'(\Omega)}{\omega - \Omega} d\Omega$$

where $\chi(\omega) = \chi'(\omega) + i\chi''(\omega)$ *Eqn(26a – 26c)*

In the field of optics, the real part of the complex susceptibility is related to the phase or refractive index of the material and the imaginary part is related to the loss or absorption coefficient. In Mathematics and signal processing, the Kramers-Kronig relation is known as the Hilbert Transform.

Acoustic Waves and Compressibility:

In 1980, M. O'Donnell et. al [8] derived an acoustic version of the Kramers-Kronig relation by following the formalism and derivations of Kramers and Kronig but applying it to an acoustic medium that obeys Hooke's Law and analyzed the response of the medium when an acoustic wave is incident (instead of an Electromagnetic wave as done by Kramers and Kronig). They use the ultrasonic equation of motion to derive a linear acoustic response in the form of

$$s(t) = \int_{-\infty}^{\infty} K(t-t')p(t')dt' \quad \text{Eqn(27)}$$

where s is the condensation, K is the compressibility, and p is the pressure. In this acoustic version, K takes the role of the generalized susceptibility relating the response of the material to the stimulus. They then use the same formalism as Kramers [6] to derive the real and imaginary components of the complex compressibility (susceptibility) to form an acoustic Kramers-Kronig relation:

$$K_1(\omega) = \frac{2}{\pi} P \int_0^{\infty} \frac{\omega' K_2(\omega')}{\omega'^2 - \omega^2} d\omega'$$

$$K_2(\omega) = -\frac{2}{\pi} P \int_0^{\infty} \frac{\omega' K_1(\omega')}{\omega'^2 - \omega^2} d\omega'$$

$$K(\omega) = K_1(\omega) + iK_2(\omega) \quad \text{Eqn(28a – 28c)}$$

where P is a factor from the integration. Using Kramers-Kronig relations O'Donnell et al were able to relate the phase and attenuation of acoustic waves through acoustic media together (as just in the optical case, the real part of the compressibility is related to the phase that the acoustic wave experiences and the imaginary part is related to the attenuation that the acoustic wave experiences in the material. These acoustic Kramers-Kronig relations were then used to determine the dispersion and attenuation of different material systems such as $CoSO_4$ solution in water and polyethylene. The validity of their derived relations proved sound as experiments measured the attenuation and dispersion of $CoSO_4$ solution from 1 MHz to 10 MHz, then they used the above relations to transform the measured attenuation into dispersion data. The dispersion data matches well within error when compared to the directly measured dispersion of the $CoSO_4$ solution, thus the acoustic Kramers-Kronig relations hold valid.

Communication Systems:

In even more recent research, Antonio Mecozzi et al have developed a communication system using what they call Kramers-Kronig coherent receiver [9]. They do so by describing that a signal that is considered to have minimum phase means that the phase of the signal can be uniquely extracted from the measured intensity of the signal. If the signal is a minimum phase signal, then the phase and magnitude of the signal are related via the Hilbert Transform (Kramers-Kronig relation). Therefore if they have an Electric field as the signal which is sideband modulated with an optical bandwidth B and a real constant amplitude of E_0 , then the modulated signal is given by

$$E(t) = E_s(t) + E_0 \exp[i\pi Bt] \quad Eqn(29)$$

where $I(t)$ is the current measured by the photodetector and is proportional to the electric field intensity, and $\phi_E(t)$ is the phase of the phase. If E_0 is large enough such that the demodulated signal $E(t) \exp[-i\pi Bt] = E_0 + E_s(t) \exp[-i\pi Bt]$ is a minimum phase signal, then the phase and amplitude of the signal can be reconstructed using:

$$E_s(t) = \{\sqrt{I(t)} \exp[i\phi_E(t)] - E_0\} \exp[i\pi Bt]$$

$$\phi_E(t) = \frac{1}{2\pi} p.v \int_{-\infty}^{\infty} dt' \frac{\log[I(t')]}{t-t'} \quad Eqn(30a - 30b)$$

They use digital signal processing to desample the directly detected modulated signal and then use the Hilbert Transform shown above to reconstruct the signal.

Conclusion:

Ralph de Laer Kronig has contributed many grand enterprises to the field of physics as a whole. He is credited for being the first person to introduce particle spin, introduced the Kronig-Penny model of describing solids with periodic attractive potentials, and propelled the field of X-ray absorption spectroscopy. In his 1926 paper "On the dispersion of X-rays", Kronig discusses general dispersion theory using classical electrodynamics. Then he produces a quantum equivalent of the electric dipole moment in terms of a material's oscillator strength, absorption coefficient, and natural resonance frequency with the help of Kramers and Ladenburg (1921). Kronig also shows how one can derive the refractive index of a material in relation to the materials absorption as a function of frequency. Kramers expanded upon this finding by deriving explicit formulae for the real and imaginary part of the complex susceptibility of a material response to incident radiation. These formula are today known as the Kramers-Kronig relations named so for both Kramers and Kronig's contribution to the derivation of the relations.

These relations have been used countlessly in experiment to determine the refractive index (real part of electric susceptibility) of an unknown material by measuring the intensity of light after having propagated through the absorbing material with various wavelengths of light. The findings of Kramers and Kronig will most likely be used time and time again in experiment to not only classify materials, but also their derivation laid down the groundwork for nuclear magnetic resonance measurements, acoustic wave attenuation and phase measurements, and even coherent detectors.

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