Statistical Dynamics of Classical Systems*

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The statistical dynamics of a classical random variable that satisfies a nonlinear equation of motion is recast in terms of closed self-consistent equations in which only the observable correlations at pairs of points and the exact response to infinitesimal disturbances appear. The self-consistent equations are developed by introducing a second field that does not commute with the random variable. Techniques used in the study of the interacting quantum fields can then be employed, and systematic approximations can be obtained. It is also possible to carry out a "charge normalization" eliminating the nonlinear coupling in favor of a dimensionless parameter which measures the deviation from Gaussian behavior. No assumptions of spatial or time homogeneity or of small deviation from equilibrium enter. It is shown that previously inferred renormalization schemes for homogeneous systems were incomplete or erroneous. The application of the method to classical microscopic systems, where it leads from first principles to a coupled-mode description is briefly indicated.

I. INTRODUCTION

Despite the deluge of papers that have been written over many years on the organization and calculation of the statistical properties of classical systems, there is not to our knowledge a satisfactory theory with the utility, generality, and precision of the quantum-field theories. In particular there is no parallel to the functional equations of Schwinger,[1] or the equivalent diagrammatic techniques of Feynman[2] for expressing the statistical and dynamical properties of a classical system, conservative or dissipative, in terms of closed albeit complicated equations involving the first few of the exact correlation functions for the system. As a consequence there is no renormalized perturbation theory for a classical system.

The lack of such a theory has led to certain bizarre and devious calculational procedures in conservative systems. In plasmas,[3] and even in conservative systems with only short-ranged non-singular forces,[4] approximate classical kinetic equations have been derived using quantum-mechanical techniques and the classical result deduced by setting $\hbar \to 0$ in the final result. There has been even less systematic study of the dynamic properties of dissipative systems for which no quantum Hamiltonian exists.

Although a classical procedure does not exist, the value of a practical scheme would be enormous. In problems involving fluids, for example, it would provide a framework for calculating transport and other kinetic properties near or far from equilibrium, from a microscopic starting point. In other problems where macroscopic equations like the Navier–Stokes equations provide a satisfactory starting point, such a theory would permit an attack on problems such as turbulence. And in simpler problems still, like the statistical treatment of nonlinear damped oscillators, it would provide a systematic self-consistent treatment in terms of physical properties. Indeed, the utility of such an approach is so great, and its existence so likely, that without demonstrating its correctness authors have used it implicitly with notable success. Thus, on the basis of equations intuitively obtained with a mixture of classical and quantum notions, Kadanoff and Swift,[5] Kawasaki,[6] and Résibois[7] and co-workers have produced mode-coupling schemes that have enabled them to predict qualitatively the properties of transport coefficients near the critical point.

The most serious attempts of which we are aware, to deduce such equations for classical systems from "first principles" are those of Kraichnan,[8] Wyld,[9] and Edwards[10] for the problem of turbulence. Unfortunately, their approaches have proven too cumbersome and insufficiently systematic to be carried to completion. Indeed, as we shall show, they have only been treated correctly to fourth order in the anharmonicity.[11]

It is the purpose of this paper to present what we believe to be the elusive generalization which is necessary for deriving a renormalized set of

equations and thus to deduce the renormalized statistical theory of a classical field satisfying a nonlinear dynamical equation. Before we go further let us emphasize the word *believe*. Our procedure is far from rigorous from a mathematical point of view. Certain formal steps are involved which are even less well justified than those physicists normally call mathematics. We believe these formal devices can be justified and provide no serious obstacle. We have checked them to a few orders in perturbation theory and resolved several apparent internal inconsistencies, but we have no categorical proof of their correctness.

The formal quantity which will play a central role in our discussion is an operator[12] which serves to infinitesimally change the classical random variable at a given point in space and time. With the aid of this quantity we will be able to ask questions about the response of the system in a representation-free fashion and thus, to determine the response in a state, the details of which are only determined at the end of an exact (or approximate) self-consistent calculation. From our procedure will emerge differences between classical and quantum systems which suggest reasons for the difficulties that have previously been encountered and why the curious methods that have sometimes been used to circumvent these difficulties, did so.

We have not fully assimilated all of the differences, but two seem worthy of special mention. The first is that the operators that displace the value of the classical field need not change the energy by a discrete amount. In quantum systems, "second quantization" imposes the restriction that the modulus squared of the field (and therefore the energy or more precisely number of quanta) change discretely. Local changes in the expression for the classical energy or particle intensity are not restricted in this fashion. They are described classically by a real field (not the square of a complex field), which can be altered continuously. The second is that the classical phase space for the corresponding problem is infinitely larger because the position and momentum of a single particle can be independently specified. It seems that this feature is connected with the fact that in our discussion of conservative classical systems there appears in a natural fashion a non-unitary transformation, or an effective non-Hermitian Hamiltonian on the Hilbert space which includes both the physical field and the operator which increases or decreases it. As a consequence, the equations of motion for both conservative and dissipative classical systems are naturally described in terms of a free-energy functional.[13] (The reader may not be surprised to learn that we were led to the de-

scription of conservative systems after studying problems involving dissipation, that is, problems in which a non-Hermitian "Hamiltonian" was expected for the physical field itself because the underlying equations were irreversible. The adjoint operation we introduce to define the Hermitian conjugate is not related to complex conjugation as in quantum mechanics.)

We shall introduce the extra field we have been discussing into our equations in a formal fashion. It should be recognized, however, that it has a physical basis. In order to discuss statistical problems it is necessary to calculate both the response of the system to a fluctuation and the likelihood of fluctuations. In a quantum system both effects can be characterized by products of the same pair of quantum operators. Fluctuations involve the anticommutator and response is related to the commutator. Classically the commutator vanishes, but the physical response is given by the Poisson bracket, in which each of the observables is a different function of the dynamical variables than it is in the operator product which gives the fluctuations.[14] Our additional operators are not necessary but they permit us to calculate the response conveniently. While it is not in any way a physical justification for their introduction, it is noteworthy that independent attacks on this problem by Kraichnan, Wyld, Edwards, and ourselves have all implicitly or explicitly predicted a renormalized theory in which more than one (in principle four, and in practice two) propagator occurs; our procedure defines three propagators nonperturbatively. It appears that the failure to recognize this "operator doubling" in a classical theory has been the major stumbling block to the development in terms of closed equations, or equivalently, to all orders of a completely renormalized many-time classical theory (in which $\hbar$ nevers appears). It is perhaps also worth noting that this "doubling" is not necessary for the static-equilibrium properties of classical systems. Symptomatic of the operator doubling is the occurence of both the energy (Hamiltonian) and the Liouville operator in classical statistical dynamics; only the former is necessary for classical thermostatics or quantum dynamics and thermostatics.

Up to this point we have been discussing the questions we shall consider as if the reader were familiar with the notions of renormalization and quantum-field theory. The point of our work, however, has been to develop an equivalent non-quantum theory and our hope is to convince a wider audience that these techniques are the proper ones for a great variety of problems. We shall therefore attempt, in Sec. II, to summarize and explain the notions that are entailed in quantum-field-

theory renormalization and what they accomplish. In Sec. III we derive fundamental classical equations by functional techniques. In Sec. IV we discuss the diagrammatic rules, and how certain simple approximations emerge. We will refrain from describing all the available techniques that this rearrangement makes possible—the study of low-energy theorems, the renormalization group, variational formulations in terms of entropy production—partly because it seems more worthwhile to discuss them in the context of particular problems, and partly because much work along these lines remains undone. We merely note that most relations of this type are derived by imposing general symmetries on the exact correlation functions that occur in the renormalized perturbation theory.

II. DESCRIPTION OF PROBLEM AND RESULTS

The problem in which we are interested is the determination of average properties of a classical random variable whose time dependence is governed by a prescribed nonlinear differential equation. A typical situation might be one in which we know the differential equation but do not know, and wish to average over initial conditions, perhaps with some constraints. The same is, of course, true in quantum systems. In both cases we have an equation of the form

$$\dot{\psi}(1) - \int U_2(12)\psi(2)\,d2$$

$$- \int U_3(123)\psi(2)\psi(3)\,d2\,d3 = U_1(1) \quad . \qquad (2.1)$$

The index 1 refers to the time and to the other space and internal indices on which the random variable depends. The integration implies a summation for internal coordinates and the functions $U_i(1 \cdots i)$ are prescribed.

Let us illustrate the formula with a few examples. The simplest might be a classical damped-nonlinear one-dimensional forced system[15] for which ψ has two components $\psi(\uparrow t) = x(t)$ and $\psi(\downarrow t) = p(t)$ and for which the equations of motion are

$$\dot{x}(t) - p(t)/m = 0 \quad ,$$

$$\dot{p}(t) + \gamma p(t) - \lambda x^2(t) + m\omega_0^2 x(t) = f(t) \quad .$$

In the general notation above we would have

$$U_1(\uparrow t_1) = f(t_1) \ , \quad U_2(\uparrow t_1;\downarrow t_2) = (1/m)\delta(t_1 - t_2) \ ,$$

$$U_2(\downarrow t_1;\downarrow t_2) = -\gamma\delta(t_1 - t_2) \ ,$$

$$U_2(\downarrow t_1;\uparrow t_2) = -m\omega_0^2\delta(t_1 - t_2) \ ,$$

$$U_3(\downarrow t_1;\uparrow t_2;\uparrow t_3) = \lambda\delta(t_1 - t_2)\delta(t_1 - t_3) \ ,$$

and all other elements of $U(\alpha_1 t_1; \alpha_2 t_2; \alpha_3 t_3)$ with

$\alpha = \uparrow$ or $\downarrow$ vanish.

A second example of our basic equation is the Navier–Stokes equation for an incompressible fluid. Here the field ψ depends on space and time and has vectorial components, i.e.,

$$\psi(1) = v_{i_1}(\vec{r}_1 t_1) \ ,$$

$$U_1(1) = f_{i_1}(\vec{r}_1 t_1) \ ,$$

$$U_2(12) = \nu\nabla_{\vec{r}_1}^2\delta(\vec{r}_1 - \vec{r}_2)\delta(t_1 - t_2)\delta_{i_1 i_2} \ , \qquad (2.2)$$

$$U_3(123) = \nabla_{\vec{r}_3}^{i_2}\delta(\vec{r}_1 - \vec{r}_2)\delta_{i_1 i_3}$$

$$\times \delta(\vec{r}_1 - \vec{r}_3)\delta(t_1 - t_2)\delta(t_1 - t_3) \ ,$$

plus similar terms with interchanged vector indices which come from eliminating the pressure.

A third example is the Liouville equation for a fluid of interacting particles. If $\vec{r}^\alpha(t)$ and $\vec{p}^\alpha(t)$ are the random variables that are associated with the position and momentum of the αth particle at time t, we may introduce the random variable[16]

$$\psi(1) \equiv f(\vec{r}_1 \vec{p}_1 t_1) = \sum_\alpha \delta(\vec{r}_1 - \vec{r}^\alpha(t_1))\delta(\vec{p}_1 - \vec{p}^\alpha(t_1)) \ ,$$

$$(2.3a)$$

which describes the single-particle phase-space density. In this problem it is not usual to introduce a source of particles, i.e., a potential $U_1(\vec{r}\,\vec{p}t)$, but we often have a streaming term and a one-body external potential $V^{(1)}$ of the form

$$U_2(\vec{r}_1\vec{p}_1 t_1; \vec{r}_2\vec{p}_2 t_2) = [-(\vec{p}_1/m)\cdot\nabla_{\vec{r}_1} + \nabla_{\vec{r}_1}V^{(1)}(\vec{r}_1)\cdot\nabla_{\vec{p}_1}]$$

$$\times \delta(\vec{r}_1 - \vec{r}_2)\delta(\vec{p}_1 - \vec{p}_2)\delta(t_1 - t_2) \ ,$$

$$(2.3b)$$

and a two-body interaction $V^{(2)}(\vec{r}_1 - \vec{r}_2)$ of the form

$$U_3(\vec{r}_1\vec{p}_1 t_1; \vec{r}_2\vec{p}_2 t_2; \vec{r}_3\vec{p}_3 t_3) = \nabla_{\vec{r}_1}V^{(2)}(\vec{r}_1 - \vec{r}_2)\cdot\nabla_{\vec{p}_1}$$

$$\times \delta(\vec{r}_1 - \vec{r}_3)\delta(\vec{p}_1 - \vec{p}_3)$$

$$\times \delta(t_1 - t_3)\delta(t_1 - t_2) \ . \qquad (2.3c)$$

In all three of the above examples we may look upon each of the infinitely many variables $\psi(1)$ as a scalar and all of the variables commute. In quantum-field theory each random variable ψ is an infinite dimensional matrix or operator and typically there is a second set of operators $\psi^\dagger(1)$ which does not commute with the ψ. Indeed, in contrast with what happens in classical physics, the operators ψ at different times do not commute. The quantities $\psi^\dagger$ satisfy equations of the same form as those satisfied by $\psi(1)$. In addition the quantity $[\psi(1), \psi^\dagger(1')]$ is directly related to the response of the interacting system[17] to an external disturbance, i.e., to the effect at point 1 in the correlated sys-

tem produced by an infinitesimal change in the force $U_1(1')$. If we make the additional notation change that $\psi(1)$ and $\psi^\dagger(1)$ are the two components of a single field with an additional index α taking on the values 1 and 2 and correspondingly understand that the potentials U_1 depend on the indices $\alpha_1, \ldots, \alpha_i$, the equations above give a complete description of the quantum system.

The basic problem that concerns us both classically and quantum mechanically is the calculation of mean values and correlations of the field ψ. Whatever the state, i.e., the probability distribution, we may write

$$\langle \dot{\psi}(1) \rangle - \int U_2(12)\langle \psi(2) \rangle$$

$$- \int U_3(123)\langle \psi(2)\psi(3) \rangle \, d2 \, d3 = U_1(1) \quad ,$$

or with $\psi(1) \equiv \langle \psi(1) \rangle + \delta\psi(1)$ so that $\langle \delta\psi(1) \rangle = 0$,

$$\langle \dot{\psi}(1) \rangle - \int \overline{U}_2(12)\langle \psi(2) \rangle \, d2$$

$$- \int U_3(123)\langle \delta\psi(2) \, \delta\psi(3) \rangle \, d2 \, d3 = U_1(1) \quad .$$

In this equation we have introduced the "mean field" $\overline{U}_2(12)$,

$$\overline{U}_2(12) = U_2(12) + \tfrac{1}{2} \int \, [U_3(123) + U_3(132)]\langle \psi(3) \rangle \, d3 \quad .$$

Of course, this equation is not closed. To evaluate $\langle \psi(1) \rangle$ we must know $\langle \delta\psi(2) \, \delta\psi(3) \rangle$, which satisfies the equation

$$\langle \delta\dot{\psi}(1) \, \delta\psi(1') \rangle - \int \overline{U}_2(12)\langle \delta\psi(2) \, \delta\psi(1') \rangle \, d2$$

$$- \int U_3(123)\langle \delta\psi(2) \, \delta\psi(3) \, \delta\psi(1') \rangle \, d2 \, d3 = 0 \quad .$$

$$(2.4)$$

Virtually no attempt to calculate correlations does not make this first step. That is to say, all methods eliminate completely uncorrelated events by working in terms of cumulants or linked diagrams. But up to this stage we have done very little. Indeed since our last equation contains a new unknown, one could say that we have done nothing.

Of course we can always put off our problem by writing equations for $\langle (\delta\psi)^3 \rangle$ in terms of $\langle (\delta\psi)^4 \rangle$, etc. The approximations come in when we replace products like $\langle \delta\psi(1) \, \delta\psi(2) \, \delta\psi(3) \, \delta\psi(4) \rangle$ by lower cumulants.[18] Thus the first approximation that is often made (a Hartree–Fock[19] or Gaussian[20] approximation) is the replacement

$$\langle \delta\psi(1) \, \delta\psi(2) \, \delta\psi(3) \, \delta\psi(4) \rangle \to \langle \delta\psi(1) \, \delta\psi(2) \rangle \langle \delta\psi(3) \, \delta\psi(4) \rangle$$

$$+ \langle \delta\psi(1) \, \delta\psi(4) \rangle \langle \delta\psi(2) \, \delta\psi(3) \rangle$$

$$+ \langle \delta\psi(1) \, \delta\psi(3) \rangle \langle \delta\psi(2) \, \delta\psi(4) \rangle. \quad (2.5)$$

When we do so we obtain a closed equation for the matrix $\langle (\delta\psi)^2 \rangle$ which, in a sense, is correct to order $[U_3(123)]^2$. There are, however, often serious

difficulties with such a procedure. In particular, in a system whose microscopic Hamiltonian is time-reversal invariant the factored equations remain even in time and don't reduce to equilibrium. Furthermore the dimensionless expansion parameter, which is something like $U_3^2/\overline{U}_2^3$ may not be small. This is the case, for example, in a fluid when the Reynolds number is high (the viscosity small) and in a kinetic or Boltzmann-like equation at low frequencies, when the collision term dominates the equation. Under such conditions we must find a new expansion parameter, or a different way of truncating. A similar difficulty almost always arises when the fluctuations tend to oscillate,[21] even when $U_3^2 \ll \overline{U}_2^3$. The point is that the parameter which actually occurs when we invert truncated equations is $U_3^2[(\partial/\partial t) - \overline{U}_2]^{-3}$ and this parameter will be large near any resonance predicted by the mean field. This is a basic difficulty with all truncation procedures. They do not treat secular properties satisfactorily and, hence, are ineffective for discussing properties like frequency shifts and lifetimes that come about via collision broadening.

"Mass renormalization" is a procedure that overcomes this difficulty, at least in part. What mass renormalization does is to express the theory not in terms of the dimensionless parameter $U_3^2[(\partial/\partial t) - \overline{U}_2]^{-3}$ but in terms of $U_3^2[\langle (\delta\psi)^2 \rangle]^3$, everywhere in a unique self-consistent equation for $\langle (\delta\psi)^2 \rangle^{-1} - [(\partial/\partial t) - \overline{U}_2]$ in terms of the parameter $U_3^2[\langle (\delta\psi)^2 \rangle]^3$. While the error in $\langle (\delta\psi)^2 \rangle$ would be large and $\langle (\delta\psi)^2 \rangle$ would be large without mass renormalization because $U_3^2[(\partial/\partial t) - \overline{U}_2]^{-3}$ can be infinite, the renormalized equation that gives $[(\partial/\partial t) - \overline{U}_2]$ as $\langle (\delta\psi)^2 \rangle^{-1}$ plus corrections in powers of $U_3^2[\langle (\delta\psi)^2 \rangle]^3$ saturates[22] and does not diverge. The error is never arbitrarily large even though it is not necessarily a very good approximation when terms are omitted. In many ways it may be likened to a Padé-approximate scheme[23] for functions. (As the analogy suggests, there is no compelling reason for expecting convergence in addition to order by order finiteness.)

A second renormalization, "charge renormalization," carries the procedure the final step. Basically charge renormalization is a method for determining the non-Gaussian part of $\langle (\delta\psi)^4 \rangle$, that is, $\langle (\delta\psi)^4 \rangle - 3\langle (\delta\psi)^2 \rangle^2$, as it appears in the equation for $\langle (\delta\psi)^2 \rangle^{-1}$ {in the combination $U_3^2\langle (\delta\psi)^2 \rangle[\langle (\delta\psi)^4 \rangle - 3\langle (\delta\psi)^2 \rangle]$} exactly and uniquely in terms of the intrinsic dimensionless parameter of the interacting system $\langle (\delta\psi)^3 \rangle^2/\langle (\delta\psi)^2 \rangle^3$. This exact closed equation completes the system. Naturally, the exact and unique equation closing the set is extremely complicated. It does, however, lend itself to systematic approximations which do not demand a

small U_3. The words "mass" and "charge" normalization refer to the fact that in electrodynamics the bare mass is $\overline{U}_2$ and the measured mass is a space-time average of $\langle(\delta\psi)^2\rangle^{-1}$. Likewise U_3^2 is essentially the square of the bare charge, and a space-time mean over coordinate arguments of $\delta\psi$ in the combination $\langle(\delta\psi)^3\rangle^2/\langle(\delta\psi)^2\rangle^3$ the square of the fine-structure constant. The masses and charges we are discussing are, of course, functions, not numbers, since both U_2 and U_3 and their replacements are nonlocal.[24]

Clearly the point of the rearrangement is to take advantage of the fact that the non-Gaussian character of the slow fluctuations that are actually present are often weaker than, and even relatively independent of, the deviations that would be present if the interactions were unmodified.[25] Under these circumstances we can hope to get approximate results for the fluctuations of interest in terms of a self-consistently determined small skewness although U_3^2/U_2^3 is large for some arguments.

Renormalization has another related advantage. Because the expressions that occur in the calculation are measurable we can relate some we wish to calculate to others that are measured, not calculated. For example, although in principle the equilibrium properties of a classical fluid must emerge from the dynamical equations we shall derive, it may be convenient to insert measured instantaneous equilibrium properties or instantaneous equilibrium properties calculated by other means.[26] Three other illustrations deserve mention: In discussing the coupling of hydrodynamic modes, we must use true vertices, which in the long-wavelength limit, are related to thermodynamic derivatives.[27] In discussing phase transitions, and in properly improving upon the direct-interaction approximation[28] for turbulence, charge renormalization plays an essential role.[29] Finally, in a rather trivial fashion it is necessary for the elimination of the effects of infinite hard-core potentials. (The classic example in electrodynamics is the expression of low-energy scattering processes[30] in terms of the measured charge of the electron, the bare charge being eliminated.)

Now that we have summarized why such equations are interesting and what ideas are involved, it seems appropriate to say a word about the equations themselves. It will turn out, in detail, that the equations relate $\langle\psi(1)\rangle$ and two functions of two space-time points, the desired fluctuations $\langle\delta\psi(1)\delta\psi(2)\rangle$ and the response function that gives the change induced in $\langle\delta\psi(1)\rangle$ by an infinitesimal external disturbance $U_1(2)$. The latter two functions satisfy nonlinear equations in which three functions of three space-time points occur. These functions

describe (i) $\langle\delta\psi(1)\delta\psi(2)\delta\psi(3)\rangle$, (ii) the linear change in the fluctuation produced by an external disturbance, and (iii) the second-order change produced in the field $\langle\psi(1)\rangle$ by infinitesimal changes in $U_1(2)$ and $U_1(3)$.

It is because the combination is a rather complicated and unsymmetrical one when expressed in this form, that difficulties arise in the direct resummation and reorganization of perturbation theory,[31] making it so opaque even to fourth order in the coupling U_3. In fact the renormalization proposed by Wyld[9] and by Lee[32] does not work to higher orders since, as the above comments suggest, three renormalized vertices, only one of which is nonvanishing to lowest order, are required. By contrast, phrased in terms of our matrix operator, the procedure can be readily understood, to arbitrary order, in spatially inhomogeneous systems, and reexpressed in terms of two closed equations. The two renormalized propagators and three renormalized vertex functions appear naturally as the nonvanishing parts of one matrix propagator and one matrix vertex function and the combinatoric and recurrent "double-counting" problems with "bare" graphs are eliminated.[33]

The equations are now correct for arbitrary inhomogeneous spatial distributions. With regard to nonstationary behavior in time, it would appear they are correct for pure states and for impure states in which the initial field and two-field correlation functions are given, but all higher correlations are random consistent with the prescribed initial values for these two functions.

III. DERIVATION OF BASIC EQUATIONS

Let us now return to our classical problem. We have an algebra of observables $\psi(1)$, $\psi(2)$, ... and states which are linear functionals on them. Less "mathematically" the states give us the values of correlations of the field ψ at different points. We shall not, in general, be concerned only with pure states—states in which all properties that could be specified, are specified—but with mixed states in which we have only statistical information. Presumably initial conditions determine a pure state and thus the pure states can be parametrized in terms of specified functions of the space variable at a given time. Likewise, mixed states can be described by giving, at a specified time, expectation values for all the moments of the classical field, i.e., for all instantaneous correlations. A class of mixed states are those for which $\langle\psi(1)\rangle$ and $\langle\psi(1)\psi(2)\rangle$ are specified at a fixed time. We shall always use what is known as the Heisenberg picture. In this picture, the values measured at different times are described by the values of the field for different time arguments. The state does

not change with time. Let us define time-ordered products of operators that depend on the time in the usual way, i.e., $[A(t_1)A(t_2)\cdots A(t_n)]_+$ is the product in which the A's are ordered from right to left in order of increasing time. For our classical ψ this makes no difference since the random variables ψ commute. Let us, however, next extend the algebra to include all time-ordered products of the operator ψ and $\hat{\psi}$, where we take $\hat{\psi}$ to satisfy

$$[\psi(i\vec{r}t), \hat{\psi}(i'\vec{r}'t)] = \delta(\vec{r} - \vec{r}')\delta(i - i') , \qquad (3.1a)$$

$$\frac{\partial\hat{\psi}(1)}{\partial t} + U_2(21)\hat{\psi}(2) + 2U_3(231)\hat{\psi}(2)\psi(3) = 0 . \qquad (3.1b)$$

Let each linear functional on the physical quantities $\psi(1)\cdots\psi(n)$ be extended to time-ordered products of ψ and $\hat{\psi}$ in such a way that (i) it is unchanged when the product contains no $\hat{\psi}$, (ii) its values on states containing $\hat{\psi}$'s are consistent with Eqs. (3.1a) and (3.1b) and with (i), (iii) its value is zero whenever the left-most factor of the product is a $\hat{\psi}$. The last restriction is possible because Eq. (3.1b) contains no term $\hat{U}_1(1)$. For some purposes, however, it will be useful to consider what would happen if $U_1(1)$ were not equal to zero. In that case all elements of the matrix correlation function would contain four different quantities $\langle(\delta\psi(i\vec{r}t)\delta\hat{\psi}(i'\vec{r}'t'))_+\rangle$, $\langle(\delta\psi(i\vec{r}t)\delta\psi(i'\vec{r}'t'))_+\rangle$, $\langle(\delta\hat{\psi}(i\vec{r}t)\delta\psi(i'\vec{r}'t'))_+\rangle$, and $\langle(\delta\psi(i\vec{r}t)\delta\hat{\psi}(i'\vec{r}'t'))_+\rangle$ and condition (iii) could only be imposed, say, at times after $\hat{U}_1(i\vec{r}t)$ vanished.

The asymmetry of the restrictions is connected with our desire to treat problems in which the system is stationary until we apply our external force or describe our initial conditions, whereas we do not require absolute equilibrium beyond any fixed finite time in the future. In stationary problems, the asymmetry is not there. We could equally work with the convention $\langle(\psi\hat{\psi})\rangle = 0$.

It is possible to deduce a number of relations between the abstract definitions introduced above and various averaging processes over initial configurations, with and without interaction but we shall not discuss these here.

The consistency of the above requirements is not obvious. We shall speak about it further below, and describe in a following paper the equivalence, to all orders in perturbation theory, of a variety of conditions. We shall not, however, prove their
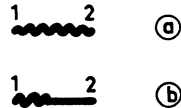
uniqueness in any conclusive nonperturbative fashion.

On the basis of the requirements imposed above we see immediately that the matrix of the four quantities defined above really contains only two independent functions.

One of them is the function $\langle(\delta\psi(1)\delta\hat{\psi}(2))_+\rangle$, which to conform with Wyld, we will designate later by a thick wavy line [Fig. 1(a)]. The second is $\langle(\delta\psi(1)\delta\hat{\psi}(2))_+\rangle$, which we may write alternatively as

$$\langle(\delta\psi(\vec{r}_1 t_1)\delta\hat{\psi}(\vec{r}_2 t_2))_+\rangle = \eta(t_1 - t_2)\langle\delta\psi(\vec{r}_1 t_1)\delta\hat{\psi}(\vec{r}_2 t_2)\rangle$$

$$= \langle(\delta\hat{\psi}(\vec{r}_2 t_2)\delta\psi(\vec{r}_1 t_1))_+\rangle , \qquad (3.2)$$

and which we shall designate by a thick line [Fig. 1(b)]. It corresponds to the thick straight line of Wyld but we shall reserve the notation of a thick straight line for the matrix propagator containing all four elements, at least in this section. The quantity in Eq. (3.2) describes the response at the point 1 to an infinitesimal impulse at the point 2.

In the absence of anharmonicity, it is the quantity $\langle\psi(1)\hat{\psi}(1')\rangle$ which gives the free response, i.e., in the Navier-Stokes fluid

$$\langle(\psi(1)\hat{\psi}(1'))_+\rangle_0 = \left(\frac{\partial}{\partial t_1} - \nu\nabla_1^2\right)^{-1}\delta(11')$$

$$= \int\frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot(\vec{r}_1-\vec{r}_1')-\nu k^2(t_1-t_1')}\eta(t_1 - t_1')$$

$$= \frac{\eta(t_1 - t_1')}{(4\pi\nu|t_1 - t_1'|)^{3/2}}\exp\left(\frac{-(\vec{r}_1 - \vec{r}_1')^2}{4\nu(t_1 - t_1')}\right). \qquad (3.3)$$

Likewise in the fluid of particles

$$\langle(\psi(1)\hat{\psi}(1'))_+\rangle_0 = \left(\frac{\partial}{\partial t_1} - \frac{\vec{p}_1\cdot\vec{\nabla}_1}{m}\right)^{-1}\delta(11')$$

$$= \eta(t_1 - t_1')\delta(\vec{p}_1 - \vec{p}_1')$$

$$\times\delta((\vec{r}_1 - \vec{r}_1') - (t_1 - t_1')(\vec{p}_1/m)) \qquad (3.4)$$

describes free propagation.

The remaining quantity $\langle(\delta\hat{\psi}(1)\delta\hat{\psi}(2))_+\rangle$ vanishes. As we have now stated several times, the bookkeeping will be greatly expedited by the introduction of a two-component matrix $\Phi(\alpha i\vec{r}t)$, where $\Phi(+i\vec{r}t) = \psi(i\vec{r}t)$ and $\Phi(-i\vec{r}t) = \hat{\psi}(i\vec{r}t)$, and by searching for the correlations of Φ. Furthermore, it will be useful to generate our equations in terms of a "non-Hermitian Hamiltonian," i.e., we may write

$$\dot{\Phi}(1) = [\Phi(1), \mathcal{K}] \qquad (3.5)$$

and

$$\mathcal{K} = \int\gamma_1(1)\Phi(1)\,d1 + \frac{1}{2!}\int\gamma_2(12)\Phi(1)\Phi(2)\,d1\,d2$$

$$+ \frac{1}{3!}\int\gamma_3(123)\Phi(1)\Phi(2)\Phi(3)\,d1\,d2\,d3 , \qquad (3.6)$$

FIG. 1.   Propagators $\langle(\delta\psi(1)\delta\hat{\psi}(2))_+\rangle$ and $\langle(\delta\psi(1)\delta\psi(2))_+\rangle$.

and because ψ and $\hat{\psi}$ have canonical commutation relations and the given potentials $U_i(1 \cdots i)$ are all instantaneous, i.e., contain $\delta(t_1 - t_2) \cdots \delta(t_1 - t_i)$; the operator $\mathcal{K}$ can be made symmetrical in the arguments 1, 2, and 3. The commutators of Φ satisfy

$$[\Phi(\pm i\vec{r}t), \ \Phi(\pm i'\vec{r}'t)] = 0 \ ,$$

$$[\Phi(\pm i\vec{r}t), \ \Phi(\mp i'\vec{r}'t)] = \pm \ \delta(ii')\delta(\vec{r} - \vec{r}') \ , \quad (3.7)$$

or

$$[\Phi(\alpha i \vec{r}t), \ \Phi(\alpha' i'\vec{r}'t)] = (i\sigma_2)(\alpha\alpha')\delta(ii')\delta(\vec{r} - \vec{r}') \ ,$$

where $i\sigma_2$ is the usual Pauli matrix $i\sigma_2(\alpha\alpha')$, whose four elements are $i\sigma_2(\pm \mp) = \pm 1$, $i\sigma_2(\pm \pm) = 0$. [Symmetrizing introduces an effective coupling of the form $\hat{U}(1)\psi(1)$ which violates condition (iii) but this correction, which amounts to a translation, can be eliminated and we shall assume that is done.]

The equation of motion for the operator Φ is now precisely the same equation that arises in the quantum theory of fields, i.e.,

$$-i\sigma_2\dot{\Phi}(1) = \gamma_1(1) + \gamma_2(12)\Phi(2) + \tfrac{1}{2}\gamma_3(123)\Phi(2)\Phi(3) \ ,$$

$$(3.8)$$

where we have introduced a summation convention for repeated indices, and the techniques for finding the correlations proceed in a similar fashion.[34]

The first step is to relate the mean values in a given state to the values in the same state when $\gamma_1(1)$ is replaced by $\gamma_1(1) + \eta(1)$, i.e., to consider the generating functional

$$S = \{\exp[\int_{t_i}^{t_f} \Phi(3)\eta(3)]\}_+$$

and to relate the functions

$$G_1^\eta(1) = \langle(S\Phi(1))_+\rangle/\langle S\rangle = \delta \ln \langle S\rangle/\delta\eta(1),$$

$$G_2^\eta(12) = \langle(S\Phi(1)\Phi(2))_+\rangle/\langle S\rangle - G_1^\eta(1)G_1^\eta(2) \quad (3.9)$$

$$= \delta^2 \ln\langle S\rangle/\delta\eta(1)\delta\eta(2)$$

to one another. These functions reduce, when $\eta = 0$, to the desired correlation functions.

Taking into account the definitions of S and G and the equations of motion, we may write

$$-i\sigma_2\dot{G}_1^\eta(1) = \gamma_1(1) + \eta(1) + \gamma_2(12)G_1^\eta(2)$$

$$+ \tfrac{1}{2}\gamma_3(123)[G_2^\eta(23) + G_1^\eta(2)G_1^\eta(3)] \ . \quad (3.10)$$

The equation for the function $G^\eta(12)$ is obtained by differentiating with respect to η,

$$-i\sigma_2\dot{G}_2^\eta(11') = \delta(11') + [\gamma_2(12) + \gamma_3(123)G_1^\eta(3)]G_2^\eta(21')$$

$$+ \tfrac{1}{2}\gamma_3(123) \frac{\delta}{\delta\eta(3)} G_2^\eta(21') \ , \quad (3.11)$$

or using the fact that

$$G_2^{-1\eta}(25)\delta G_2^\eta(51') = - [\delta G_2^{-1\eta}(25)]G_2^\eta(51')$$

and using the chain rule of differentiation to write

$$\frac{\delta}{\delta\eta(3)} = G_2^\eta(34) \frac{\delta}{\delta G_1^\eta(4)} \ ,$$

$$-i\sigma_2\dot{G}_2(11') - [\gamma_2(12) + \gamma_3(123)G_1(3)]G_2 (21') \quad (3.12)$$

$$- \Sigma(12)G_2(21') = \delta(11') \ .$$

In this equation we have suppressed the η which always occurs implicitly and introduced

$$\Sigma(11') = \tfrac{1}{2}\gamma_3(123)G_2(25)G_2(34)\left(- \frac{\delta G_2^{-1}(51')}{\delta G_1(4)}\right) \ .$$

$$(3.13)$$

The quantity

$$\Gamma_3(541') = - \frac{\delta G_2^{-1}(51')}{\delta G_1(4)} = - \frac{\delta\eta(1')}{\delta G_1(4)\delta G_1(5)}$$

$$= \frac{\delta[\ln\langle S\rangle - \eta(3)G_1(3)]}{\delta G_1(4)\delta G_1(5)\delta G_1(1')} \quad (3.14)$$

symmetric in these indices plays the role of the renormalized charge. The equation

$$\Sigma(11') = \tfrac{1}{2}\gamma_3(123)G_2(25)G_2(34)\Gamma_3(451') \quad (3.15)$$

is the equation which determines the quantity Σ to all orders, as a power series in γ_3 and G_2. The equation can be described diagrammatically and the nth-order terms characterized by their graphical properties. In particular we have to the lowest order $\Gamma_3(451') = \gamma_3(451')$ and Fig. 2(a), where thick lines stand for propagators and dots for the bare vertices γ_3. Repeated indices, which are integrated over, are not indicated. The second term in the series is obtained by iteration of the exact equation[35]

FIG. 2. Approximations to the self-energy $\Sigma(11')$ (a), (b); and its reexpression in terms of a vertex function (c), (d).

$$\Gamma_3(541') = \gamma_3(451') + \frac{\delta\Sigma(51')}{\delta G_1(4)}$$

$$= \gamma_3(451') + \frac{\delta\Sigma(51')}{\delta G_2(67)}\frac{\delta G_2(67)}{\delta G_1(4)}$$

$$= \gamma_3(451') + \frac{\delta\Sigma(51')}{\delta G_2(67)} G_2(68)\Gamma_3(894)G_2(97) .$$

$$(3.16)$$

Since $\delta\Sigma(51')/\delta G_2(67) = \gamma_3(568)G_2(89)\gamma_3(971')$ to the next order, we derive to the next order [Fig. 2(b)]. We see immediately, by induction, that since higher-order terms in Σ are obtained from lower-order ones by breaking open single lines and introducing pairs, that the graphs are all of a form which cannot be cut into two disjoint parts by cutting a single line.

Of course, the equation written above, before iteration, in which the vertex designated by $\Gamma_3(123)$ is represented by a thick dot; from which three lines merge [Fig. 2(c)] is exact. The same is true for the equation to which it is coupled [Fig. 2(d)],

$$\Gamma_3 = \gamma_3 + I_4 G_2 G_2 \Gamma_3 , \qquad (3.17)$$

in which the interaction kernel $I_4 \equiv \delta\Sigma(12)/\delta G_2(56)$ is represented by a box.

Because the last quantity is the derivative of Σ with respect to G_2 the equations represent the complete description of the interacting system in terms of four equations, two for Σ and Γ_3 in terms of γ_3 and G_2, and two others which close the system by determining G_1 and G_2 in terms of γ_1, γ_2, and γ_3. Specifically, with the abbreviation $[G_2^0]^{-1}(12) = -i\sigma_2 (\partial/\partial t_1)\delta(t_1 - t_2) - \gamma_2(12)$ we have[1]

$$\Gamma_3 = \gamma_3 + (\delta\Sigma/\delta G_2)G_2 G_2 \Gamma_3 ,$$

$$\Sigma = \tfrac{1}{2}\gamma_3 G_2 G_2 \Gamma_3 ,$$

$$\qquad\qquad (3.18)$$

$$G_2^{-1} = [G_2^0]^{-1} - \gamma_3 G_1 - \Sigma ,$$

$$[G_2^0]^{-1}G_1 - \tfrac{1}{2}\gamma_3 G_1^2 - \tfrac{1}{2}\gamma_3 G_2 = \gamma_1 .$$

We remark in passing that, by extending the number of components of Φ and generalizing $[G_2^0]^{-1}$ to the situation in which there is no time derivative, the same four equations are valid when there are many-body forces[36] as well. The difference between the equations for G_1 and G_2 and the equation for Γ_3 is fundamental.

The degree to which the equations can be approximately solved depends primarily on the degree to which some small parameter can be found for approximately treating the first two of these equations. The standard technique, however, is to use the first two to generate a power-series expansion for γ_3 in powers of Γ_3. In terms of that power series the last two equations determine G_1 and G_2.

We have already commented on the first approximation

$$\gamma_3 = \Gamma_3 ,$$

$$[G_2^0]^{-1} = G_2^{-1} + \gamma_3 G_1 + \tfrac{1}{2}\Gamma_3 G_2 G_2 \Gamma_3 , \qquad (3.19)$$

$$[G_2^0]^{-1}G_1 = \tfrac{1}{2}\gamma_3 G_1^2 + \tfrac{1}{2}\gamma_3 G_2 + \gamma_1 .$$

The second approximation would involve, in addition to the last equation [cf. Figs. 3(a) and 3(b)],

$$\gamma_3 = \Gamma_3 - \Gamma_3 G_2 \Gamma_3 G_2 \Gamma_3 G_2 ,$$

$$[G_2^0]^{-1}(11') - \gamma_3 G_1 - G_2^{-1}(11') \qquad (3.20)$$

$$= \tfrac{1}{2}[\Gamma_3 G_2 G_2 \Gamma_3](11') - \tfrac{1}{2}[\Gamma_3 G_2 G_2 \Gamma_3 G_2 \Gamma_3 G_2 G_2 \Gamma_3](11') .$$

There are a number of other statements one can make about the structure of the diagrams in nth order. A variational principle[37] can be presented; the fact that the equations really are power series in one parameter $(\Gamma_3 G_2^{3/2})$ can be formulated in a more useful form[38] (the renormalization group); and theorems on the properties of vertices Γ_3 relating to conserved quantities[39] (Ward identities) can be presented.

In the body of this paper, however, we would like to make our equations, more, rather than less concrete by discussing the two systems introduced above.

IV. SIMPLE APPROXIMATIONS AND REDUCTION TO PREVIOUS SYSTEMS OF EQUATIONS

As a first illustration of how these equations behave we examine the problem in which they refer to the Navier–Stokes equations. Because we are interested in them when a system is in stationary turbulent configurations, it will be useful to suppose, as did Wyld, that the system is stirred by a random Gaussian force, translationally invariant in time and space. In such a system it is not difficult to show[40] that this random force may be represented by an additional term $U_2(+ i\vec{r}t; -i'\vec{r}'t')$ $= \langle f^{ext}(i\vec{r}t)f^{ext}(i'\vec{r}'t')\rangle \equiv X$ or $\gamma_2(-i\vec{r}t; -i'\vec{r}'t') = X$, all other components vanishing. The spatial homogeneity implies that G_1 vanishes, and consequently, we have

FIG. 3. Self-consistent equations for the vertex and self-energy.
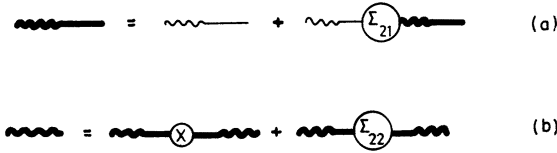
FIG. 4. Dyson equations with external driving X.

$$G_2^{-1} = \begin{pmatrix} 0 & \dfrac{\partial}{-\partial t}\delta - v\nabla^2\delta - \Sigma_{12} \\ \dfrac{\partial}{\partial t}\delta - v\nabla^2\delta - \Sigma_{12} & -X - \Sigma_{22} \end{pmatrix}. \qquad (4.1)$$

Taking into account the spatial symmetry, we have

$$G_2(k\omega) = \begin{pmatrix} X(k\omega) + \Sigma_{22}(k\omega) & i\omega + vk^2 - \Sigma_{12}(k\omega) \\ -i\omega + vk^2 - \Sigma_{12}^*(k\omega) & 0 \end{pmatrix} D^{-1}$$

and
$$(4.2)$$

$$D(k\omega) = \left| i\omega + vk^2 - \Sigma_{12}(k\omega) \right|^2 ,$$

so that the elements G_{ij} of the matrix G_2 are given by

$$G_{12}(k\omega) = \left[-i\omega + vk^2 - \Sigma_{21}(k\omega) \right]^{-1}$$

$$= G_{21}(k - \omega) = G_{21}(k\omega)^* , \qquad (4.3)$$

$$G_{11}(k\omega) = G_{12}(k\omega)[X(k\omega) + \Sigma_{22}(k\omega)]G_{21}(k\omega) .$$

Into these equations we must insert $X(k\omega)$, $\Sigma_{21}(k\omega)$, and $\Sigma_{22}(k\omega)$. Diagrammatically we may write the first as in Fig. 4(a), with the light line standing for $(-i\omega + vk^2)^{-1}$. We represent the second in Fig. 4(b). Into these equations we must insert Σ_{21} and Σ_{22}. In matrix form, both are represented by $\frac{1}{2}\gamma_3 G_2 G_2 \Gamma_3$. To first order the only nonvanishing element of γ_3 is the element U_3 that involves two ψ and one $\hat\psi$ [Fig. 5(a)]. Since the Fourier transform of a product is the convolution of the Fourier transforms the product involves a convolution of the unknown function.
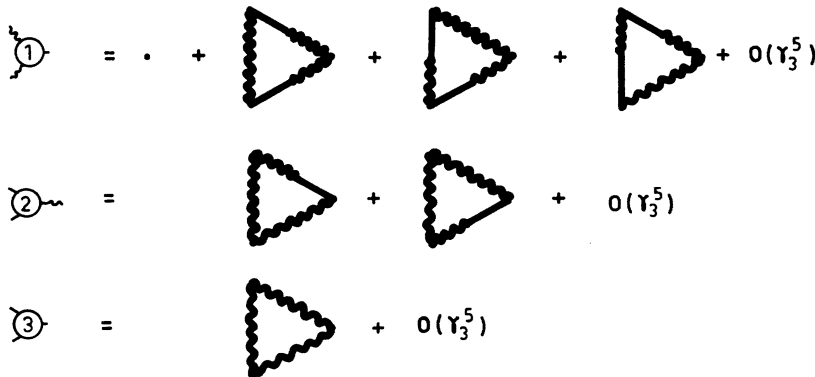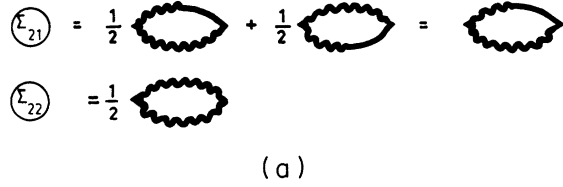

(a)

(b)

FIG. 5. Simplest approximation to the self-energy and an exact equation using vertices.

To the next order we may either expand in the true or the bare vertex. We have argued that an expansion in terms of the true vertex is preferable. However for comparison purposes let us merely first note that the exact equation is given in Fig. 5(b).

A term involving three wiggly lines entering a three-point vertex is the unique term contributing to the connected part of $\langle (\hat\psi\,\hat\psi\,\hat\psi)_+ \rangle$ and, therefore, must vanish. This has been checked to each order, recalling that G_{12} is retarded, but since it follows from our general construction we shall not reproduce the perturbative proof. The remaining three equations for the nonvanishing elements of Γ_3 derived from Eq. (3.16) with the self-energy [Fig. 5(a)] are shown in Fig. 6. If we also replaced each curly line by a "bare curly" line (Fig. 7), and eliminated the heavy line in favor of a light line we would obtain the 44 fourth-order diagrams enumerated by Wyld with the proper weightings. We ask the reader to believe the verification.

We see that beyond fourth order in γ_3 the renormalization is not as he surmised. Three different vertex functions having the significance mentioned

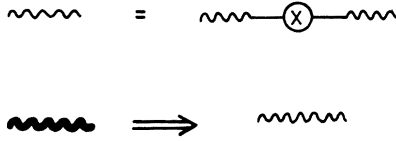FIG. 6. First approximation to the vertex functions.

FIG. 7. Change in notation which yields the diagrammatic equations of Wyld.

in the Introduction (in principle there could be four) are required in order to have a closed "rigorous" theory. Two of them satisfy homogeneous equations. Wyld introduced only $\Gamma_3^{(1)}$.

Likewise, with a label to distinguish magnetic field and velocity, and the same proliferation procedure, we could produce all 300 diagrams of Lee. As Wyld noted, to second order these equations are just Kraichnan's direct interaction approximation (Fig. 8). Their successes and failures have been the subject of much discussion and we have little to add. His equations would be rigorous were it not for vertex renormalization effects, or to put it differently the whole problem of strong turbulence is contained in a proper treatment of the vertex renormalization.

It is interesting to note that without random external forces, the equation for the fluctuations of the classical field are homogeneous. Thus, the equilibrium fluctuations in a quiescent fluid are those required by self-consistency or detailed balance. This is in line with the picture that the Boltzmann-like equations suggest, even though these equations have faults. The behavior of the equations for the classical fluid, when it is weakly interacting, is therefore far from trivial. Of course, this is not a great surprise since for a weakly interacting system, close to equilibrium, the phase-space fluctuation function, discussed in Eq. (2.3), has been shown to satisfy a Fokker-Planck equation which gives interesting corrections to the Boltzmann equation, modifying the velocity of sound, etc.

It is instructive to reduce these equations to more familiar ones by introducing the quantities

$$S_R(11') = \eta(t_1 - t_1')\langle(\delta\psi(1)\delta\psi(1'))\rangle \ ,$$

$$G_{12}(11') = G_R(11') = \langle(\delta\psi(1)\delta\hat\psi(1'))_+\rangle \ , \qquad (4.4)$$

$$\langle\delta\psi(1)\delta\psi(1')\rangle = S_R(11') + S_R(1'1) \equiv S(11') \ .$$

We have the equations

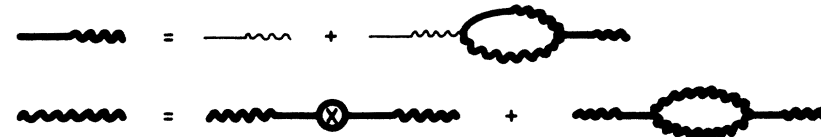$$S_R(11') = \eta(11')G_R(1\bar1)\Sigma_{22}(\bar1\bar1')G_R(1'\bar1') = \eta S(11')$$

and

$$G_R^{-1}(11') = [G_R^0]^{-1}(11') - \Sigma_{12}(11') \ . \qquad (4.5)$$

Applying $[G_R^0]^{-1}$ to the equation for S_R, denoting by $\tilde S(11') \equiv S(11')\delta(t_1 - t_1')$, and noting that $G_R^{-1}(11')$ vanishes when $t_1 < t_1'$, we obtain

$$G_R^{-1}(1\bar1)S_R(\bar11') = \tilde S(11') + \eta(t_1 - t_1')[\Sigma_{22}(1\bar1)G_R(1'\bar1)]$$

$$+ \eta(t_1 - t_1')[\Sigma_{12}(1\bar1)S_R(1'\bar1)] \ .$$

When both the density and potential are weak, (4.4) and (4.5) should reduce to the equation studied by Forster and Martin[41] in connection with a weakly interacting fluid. It was shown in that paper that because the fluctuation dissipation theorem determined $\tilde S$ in terms of the frequency spectrum of S and the equilibrium value of $\langle\psi\rangle$, that the scheme determined all properties to the required order in the potential. At present, the reduction has been verified insofar as collisional terms are concerned but only incompletely for second-order effects.[42]

If we neglected the last two terms we would obtain

$$S_R(11') = G_R(1\bar1)\tilde S(\bar11') \ ,$$

and hence a single equation for S_R (or G_R) in terms of itself and the instantaneous value of $S(11')$, i.e., $\tilde S(11') = S(11')\delta(t_1 - t_1')$.

In particular, we would obtain to second order in Γ

$$G_R^{-1} = [G_R^0]^{-1} - \Gamma_3^{(1)}G_R G_R \tilde S \Gamma_3^{(1)} - \tfrac12 \Gamma_3^{(2)}G_R G_R \Gamma_3^{(1)} \ ,$$

$$(4.6)$$

where $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are the renormalized vertices containing (1) and (2) incoming straight lines we introduced above.

Without further[42] assumptions we have been unable to justify, and it appears impossible to reduce, the theory near equilibrium to the form assumed by Kawasaki, namely, a form involving only S_R (or G_R) and the instantaneous value of S. It is nonetheless worthwhile to observe that the equations we have written clarify and permit one to attempt to justify the assumptions implicitly made by Kawasaki[43] (i.e., presumably the assumptions hidden in the assertion of separated time scales and Gaussian equilibrium correlations). If we were able to show that the nonhydrodynamic elements of G_R were regular, and that the contributions from the vertex he omits were also regular, we could introduce the regular quantities.

FIG. 8. Direct interaction approximation.

$$G_{R_{\text{reg}}}^{-1}(i\vec{r}t;\,j\vec{r}'t')=[G_R^0]^{-1}(i\vec{r}t;\,j\vec{r}'t')$$

$$-\sum_{\substack{m,k \\ \text{not both} \\ \text{hydrodynamic}}}\Gamma_3^{(1)}(i\vec{r}t;\,m\vec{\vec{r}}\,\bar{t};\,k\vec{\vec{r}}\,\bar{\bar{t}})$$

$$\times G_{R_{\text{reg}}}G_{R_{\text{reg}}}\tilde{S}\Gamma_3^{(1)}\ ,$$

and then write

$$G_R^{-1}=G_{R_{\text{reg}}}^{-1}-\sum_{\substack{k,m \\ \text{hydrodynamic}}}\Gamma_3^{(1)}G_RG_R\tilde{S}\Gamma_3^{(1)}$$

$$+\cdots\ (\text{reg. terms}).$$

As we understand it, it is this plausible equation with which Kawasaki actually works, asserting plausibly that in it, the vertices are given by instantaneous thermodynamic functions at the long wavelengths at which singular terms arise. Needless to say, a justification of Kawasaki's strikingly successful predictions, and an estimate of the errors involved from a microscopic starting point represents a formidable task. The same is true for the discussion of long-time correlations[44] away from the critical point. It is obvious from the above equations that they will be there unless there are compensations and cancellations. It is not obvious that such cancellations do not occur.

Note that far from equilibrium, even away from the transition point, the coupled nonlinear equations for $G_R(11')$ and $S_R(11')$ do not depend only on time differences. They give solutions which presumably are appropriate for a system with specified initial values for the two-point correlation functions and averaged values for the unspecified higher-order correlation functions. The solution of the equations is probably quite hopeless in this case. Indeed when Σ is set equal to zero, the equation reduces to the nonlinear Vlasov equation which is complicated enough to defy any general discussion. When this mean force term is not dominant there appears to be little worth saying.

At the other extreme, for equilibrium fluids,

the equations for S_R and G_R are reminiscent of some of the earliest approximations attempted for fluids. In many ways G_R is the function which plays the role of self-diffusion, that is to say, it describes the effect of infinitesimally increasing the probability of finding a particle at $\vec{r}\vec{p}t$. The changed probability at a later time, a momentum specified version of the self-diffusion function occurs in our equations in much the same fashion as in the Vineyard convolution approximation.[45]

APPENDIX A

In this appendix we would like to return to our exact equations and indicate some additional features that may not be apparent to the reader who has not been involved with field theoretic problems. First, let us make some simple observations. The diagrams that we generated for the fluctuations were symmetrical but the equations are not apparently so. We have written

$$\Sigma(11')=\tfrac{1}{2}\gamma_3(123)G_2(24)G_2(35)\Gamma_3(451')\qquad(\text{A1})$$

$$=\tfrac{1}{2}\Gamma_3(123)[G_2(24)G_2(35)-G_2(26)G_2(37)$$

$$\times I_4(67;89)G_2(84)G_2(95)]\Gamma_3(451')\ ,\qquad(\text{A2})$$

which is simply related to the "doubly connected" part of the four-point correlation function. To be more specific the four-point correlation function can be written in terms of products of lower-order correlation functions and a connected part as

$$=[(\delta/\delta\eta(1))+\langle\Phi(1)\rangle][(\delta/\delta\eta(2))+\langle\Phi(2)\rangle][(\delta/\delta\eta(3))+\langle\Phi(3)\rangle]\langle\Phi(4)\rangle$$

$$=[(\delta/\delta\eta(1))+\langle\Phi(1)\rangle][(\delta/\delta\eta(2))+\langle\Phi(2)\rangle][G_2(34)+G_1(3)G_1(4)]$$

$$=[(\delta/\delta\eta(1))+\langle\Phi(1)\rangle][G_3(234)+G_1(2)G_2(34)+G_1(3)G_2(24)+G_1(4)G_2(23)$$

$$+G_1(2)G_1(3)G_1(4)]$$

$$=G_4(1234)+G_3(234)G_1(1)+G_3(134)G_1(2)+G_3(124)G_1(3)+G_3(123)G_1(4)$$

$$+G_2(12)G_2(34)+G_2(13)G_2(24)+G_2(14)G_2(23)+G_1(1)G_1(2)G_1(3)G_1(4)\ .\qquad(\text{A3})$$

The connected part, or cumulant, $G_4(1234)$, is also given by

$$G_2(1\bar{1}) \frac{\delta}{\delta G_1(\bar{1})} G_2(2\bar{2}) \frac{\delta}{\delta G_1(\bar{2})} G_2(34)$$

$$= G_2(1\bar{1})G_2(2\bar{2})G_2(3\bar{3})G_2(4\bar{4}) \left[\Gamma_4(\bar{1}\bar{2}\bar{3}\bar{4}) + \Gamma_3(\bar{2}\bar{4}\bar{5})G_2(5\bar{6})\Gamma_3(\bar{6}\bar{3}\bar{1}) + \Gamma_3(\bar{2}\bar{3}\bar{5})G_2(5\bar{6})\Gamma_3(\bar{6}\bar{1}\bar{4}) + \Gamma_3(\bar{2}\bar{1}\bar{5})G_2(5\bar{6})\Gamma_3(\bar{6}\bar{3}\bar{4}) \right] ,$$

(A4)

$$\Gamma_i(12\cdots i) \equiv -\frac{\delta}{\delta G_1(1)} \cdots \frac{\delta}{\delta G_1(i-1)} \eta(i) = -\frac{\delta}{\delta G_1(1)} \cdots \frac{\delta}{\delta G_1(i-2)} G_2^{-1}(i-1,i) .$$

From the equality

$$\frac{\delta G_2(12)}{\delta G_1(3)\delta G_1(4)}\bigg|_{\gamma_2} = 2\frac{\delta \Sigma(34)}{\delta \gamma_2(12)}\bigg|_{G_1} = 2\frac{\delta[-G_2^{-1}(34) - \gamma_2(34)]}{\delta\gamma_2(12)}\bigg|_{G_1} ,$$

(A5)

it follows that

$$G_4(1234) = 2[\delta G_2(12)/\delta\gamma_2(34)]_{G_1} - G_2(14)G_2(23) - G_2(13)G_2(24) + G_2(1\bar{1})G_2(2\bar{2})\Gamma_3(\bar{1}\bar{2}\bar{5})G_2(5\bar{6})\Gamma_3(\bar{6}\bar{3}\bar{4})G_2(3\bar{3})G_2(4\bar{4}) ,$$

(A6)

or in graphical language that the set of graphs involving 1, 2, 3, 4 that cannot be divided into two parts by cutting a single line,

$$\tfrac{1}{2} G_2(1\bar{1})G_2(2\bar{2})D_4(\bar{1}\bar{2};\bar{3}\bar{4})G_2(4\bar{4})G_2(3\bar{3})$$

$$= \tfrac{1}{2} G_2(1\bar{1})G_2(2\bar{2})[\Gamma_4(\bar{1}\bar{2}\bar{3}\bar{4}) + \Gamma_3(\bar{1}\bar{3}\bar{5})G_2(5\bar{6})\Gamma_3(\bar{2}\bar{4}\bar{6})$$

$$+ \Gamma_3(\bar{1}\bar{4}\bar{5})G_2(5\bar{6})\Gamma_3(\bar{3}\bar{6}\bar{2})] G_2(4\bar{4})G_2(3\bar{3}) ,$$

is equal to $G_4(1234)$ minus the last term in (A6) and to

$$= 2G_2(1\bar{1}) \frac{\delta\Sigma(\bar{1}\bar{2})}{\delta\gamma_2(34)}\bigg|_{G_1} G_2(\bar{2}2)$$

$$= 2G_2(1\bar{1})G_2(\bar{2}2) \frac{\delta\Sigma(\bar{1}\bar{2})}{\delta G_2(\bar{3}\bar{4})}\bigg|_{G_1} \frac{\delta G_2(\bar{3}\bar{4})}{\delta\gamma_2(34)}\bigg|_{G_1}$$

$$= G_2(1\bar{1})G_2(2\bar{2})[I_4(1_{\text{sym}} - G_2G_2I_4)^{-1}](\bar{1}\bar{2};\bar{3}\bar{4})$$

$$\times G_2(\bar{3}3)G_2(\bar{4}4) .$$

In this short-hand notation in which $1_{\text{sym}}(12;34)$ $= \tfrac{1}{2}[\delta(13)\delta(24) + \delta(14)\delta(23)]$, and, as before, $[\delta\Sigma(12)/\delta G_2(34)]_{G_1} = I_4(12;34)$,

$$(1_{\text{sym}} + \tfrac{1}{2} G_2G_2D_4)(12;34) = (1_{\text{sym}} - G_2G_2I_4)^{-1}(12;34) .$$

We recognize that the combination which occurs

in (A2) therefore may also be expressed as

$$\Sigma(11') = \tfrac{1}{2}\gamma_3(123)G_2(2\bar{2})G_2(3\bar{3})\gamma_3(\bar{3}\bar{2}1')$$

$$+ \tfrac{1}{4}\gamma_3(123)G_2(2\bar{2})G_2(3\bar{3})D_4(\bar{2}\bar{3};\bar{4}\bar{5})$$

$$\times G_2(\bar{4}4)G_2(\bar{5}5)\gamma_3(451')$$

$$= \tfrac{1}{2}\gamma_3(123)[\delta G_2(23)/\delta\gamma_2(45)]_{G_1}\tfrac{1}{2}\gamma_3(451') .$$

Hence, the expression for the self-energy is basically the doubly connected part of the four-point function and likewise the expression for D_4 generated as a power series in $\Gamma_3(G_2G_2G_2)^{1/2}$ by iteration of the power series for $G_2G_2I_4$ in terms of $\Gamma_3(G_2G_2G_2)^{1/2}$ is an expression for the true four-point function in terms of the true (a renormalized) three-point function.

APPENDIX B

We have argued that there is an equation that gives $\Gamma_3(G_2G_2G_2)^{1/2} \equiv \bar{\Gamma}_3$ in terms of $\gamma_3(G_2G_2G_2)^{1/2} = \bar{\gamma}_3$ by dimensional reasoning, that is to say, because the essential ingredient of any graph is a vertex and half of the propagator legs. To see this analytically and to generate a *single equation* which accomplishes the charge renormalization we proceed as follows:

$$\delta\{S - \gamma_1(1)G_1(1) - \tfrac{1}{2}\gamma_2(12)[G_2(12) + G_1(1)G_1(2)] - \tfrac{1}{8}\gamma_3(123)[G_3(123) + 3G_2(12)G_1(3)] + G_1(1)G_1(2)G_1(3)]\}$$

$$= -\gamma_1(1)\delta G_1(1) - \tfrac{1}{2}\gamma_2(12)\delta G_2(12) - \gamma_2(12)G_1(1)\delta G_1(2) - \tfrac{1}{8}\gamma_3(123)\delta G_3(123)$$

$$- \tfrac{1}{2}\gamma_3(123)G_1(3)\delta G_2(12) - \tfrac{1}{2}\gamma_3(123)G_2(23)\delta G_1(1) - \tfrac{1}{2}\gamma_3(123)G_1(2)G_1(3)\delta G_1(1)$$

$$= -\tfrac{1}{2}\delta\text{Tr}\left\{i\sigma^{(2)} \frac{\partial}{\partial t_1}[G_2(11') + G_1(1)G_1(1')] + \ln G_2\right\} + \tfrac{1}{4}\gamma_3(123)G_2(2\bar{2})G_2(3\bar{3})\Gamma_3(\bar{2}\bar{3}1')\delta G_2(11') - \tfrac{1}{6}\gamma_3(123)\delta G_3(123)$$

$$= -\tfrac{1}{2}\delta\text{Tr}\{\cdots\} + \tfrac{1}{4}G_2^{-1/2}\bar{\gamma}_3\bar{\Gamma}_3 G_2^{-1/2}\delta G_2 - \tfrac{1}{6}G_2^{-1/2}G_2^{-1/2}G_2^{-1/2}\bar{\gamma}_3\delta G_2^{1/2}G_2^{1/2}G_2^{1/2}\bar{\Gamma}_3$$

$$= -\tfrac{1}{2}\delta\text{Tr}\{\cdots\} - \tfrac{1}{6}\bar{\gamma}_3(123)\delta\bar{\Gamma}_3(123) .$$
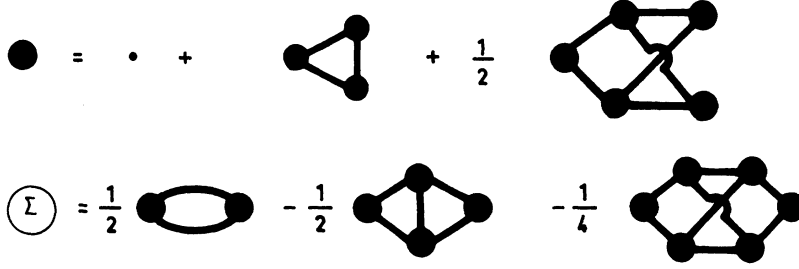
(B1)

FIG. 9. Self-consistent approximation to the vertex and self-energy.

It follows that $\bar{\gamma}_3$ is a function only of $\bar{\Gamma}_3$ so that the theory depends on a single variable as we asserted. We may use this fact to obtain an equation for $\bar{\gamma}_3$ or $\bar{I}_4 \equiv (G_2^{1/2})^2 I_4 (G_2^{1/2})^2$ in terms of $\bar{\Gamma}_3$. One way to do this is to observe that

$$I_4 \equiv \delta\Sigma/\delta G_2 = \tfrac{1}{2}\gamma_3(\delta/\delta G_2)G_2 G_2 \Gamma_3$$

$$= \gamma_3 G_2 \Gamma_3 + \tfrac{1}{2}\gamma_3 G_2 G_2(\delta\Gamma_3/\delta G_2) \quad,$$

$$\bar{I}_4 = \bar{\Gamma}_3^2 - \bar{\Gamma}_3\bar{I}_4\bar{\Gamma}_3 - \tfrac{3}{4}\bar{\gamma}_3\bar{\Gamma}_3 + \tfrac{3}{4}\bar{\gamma}_3(\delta\bar{\Gamma}_3/\delta\bar{\gamma}_3)\bar{\gamma}_3 \quad,$$

$$\bar{I}_4 = \bar{\Gamma}_3^2 - \bar{\Gamma}_3\bar{I}_4\bar{\Gamma}_3 + \tfrac{3}{4}\bar{\Gamma}_3(1 - \bar{I}_4)[(\delta\ln\bar{\Gamma}_3/\delta\ln\bar{\gamma}_3) - 1]\bar{\Gamma}_3 .$$

Since $\bar{\gamma}_3 = (1 - \bar{I}_4)\bar{\Gamma}_3$, this equation determines $\bar{I}_4$ as a function of $\bar{\Gamma}_3$ beginning with (Fig. 9)

$$\bar{I}_4 = \bar{\Gamma}_3^2 + \tfrac{1}{2}\bar{\Gamma}_3^4 + O(\bar{\Gamma}_3^6) \quad,$$

$$\bar{\gamma}_3 = \bar{\Gamma}_3 - \bar{\Gamma}_3^3 - \tfrac{1}{2}\bar{\Gamma}_3^5 - O(\bar{\Gamma}_3^7) \quad. \tag{B3}$$

An alternative way of writing I_4 in terms of Γ_3 is given by

$$\bar{I}_4 = \bar{\Gamma}_3^2 - \bar{\Gamma}_3\bar{I}_4\bar{\Gamma}_3 - \tfrac{3}{4}\bar{\Gamma}_3(1 - \bar{I}_4)\,\frac{\delta\ln(1 - \bar{I}_4)}{\delta\ln\bar{\Gamma}_3(1 - \bar{I}_4)}\,\bar{\Gamma}_3 \quad, \tag{B4}$$

and an alternative form for $\bar{\Gamma}_3$ in terms of $\bar{\gamma}_3$ is

$$\bar{\Gamma}_3 = \bar{\gamma}_3 + \tfrac{1}{4}\bar{\gamma}_3\bar{\Gamma}_3^2 + \tfrac{3}{4}\bar{\gamma}_3^2(\delta\bar{\Gamma}_3/\delta\bar{\gamma}_3)\bar{\Gamma}_3 \quad. \tag{B5}$$

The other equations are also given a natural normalization. The equation

$$G_2^{1/2}[G_2^0]^{-1}G_2^{1/2} = 1 + \tfrac{1}{2}\bar{\Gamma}_3(1 - \bar{I}_4)\bar{\Gamma}_3$$

determines $G_2^{1/2}[G_2^0]^{-1}G_2^{1/2}$ as a function of $\bar{\Gamma}_3$. Also we have

$$[G_2^0]^{-1}G_1 = \gamma_1 + \tfrac{1}{2}\gamma_3 G_1 G_1 + \tfrac{1}{2}\gamma_3 G_2$$

or with $G_2^{-1/2}G_1 \equiv \bar{G}_1$ and $G_2^{1/2}\gamma_1 \equiv \bar{\gamma}_1$

$$(1 + \tfrac{1}{2}\bar{\gamma}_3\bar{\Gamma}_3)\bar{G}_1 = \tfrac{1}{2}\bar{\gamma}_3 + \tfrac{1}{2}\bar{\gamma}_3\bar{G}_1\bar{G}_1$$

so that $\bar{G}_1 = G_2^{-1/2}G_1$ is determined in terms of $\bar{\Gamma}_3$ by the equation

$$[1 + \tfrac{1}{2}\bar{\Gamma}_3(1 - \bar{I}_4)\bar{\Gamma}_3]\bar{G}_1 = \bar{\gamma}_1 + \tfrac{1}{2}[\bar{\Gamma}_3(1 - \bar{I}_4)][1 + \bar{G}_1\bar{G}_1] \quad.$$

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[1]J. Schwinger, Proc. Natl. Acad Sci. 37, 452 (1951); Proc. Natl. Acad. Sci. 37, 455 (1951). The method is discussed, for example, by N. N. Bogolyubov and D. V. Shirkov [Introduction to the Theory of Quantized Fields (Interscience, New York, 1959)] and by P. Roman [Introduction to Quantum Field Theory (Wiley, New York, 1968)].

[2]R. P. Feynman, Phys. Rev. 76, 749 (1949); Phys. Rev. 76, 769 (1949); F. J. Dyson, Phys. Rev. 75, 486 (1949); Phys. Rev. 75, 1736 (1949). The method is discussed in the books listed in Ref. 1 and also by J. D. Bjorken and S. D. Drell [Relativistic Quantum Fields (McGraw-Hill, New York, 1965)].

[3]H. W. Wyld, Jr. and D. Pines, Phys. Rev. 127, 1851 (1962); D. Pines and J. R. Schrieffer, Phys. Rev. 125, 804 (1962); D. F. Dubois, in Lectures in Theoretical Physics, edited by W. E. Brittin and A. O. Barut (Gordon and Breach, New York, 1967), Vol. 9C, p. 469.

[4]L. P. Kadanoff and G. Baym, Quantum Statistical Mechanics (Benjamin, New York, 1962); G. M. Eliashberg, Zh. Eksp. Teor. Fiz. 41, 1241 (1961) [Sov. Phys.—JETP 14, 886 (1962)]; and most directly, G. Mazenko, Phys. Rev. A 3, 2121 (1971).

[5]L. P. Kadanoff and J. Swift, Phys. Rev. 166, 89 (1968).

[6](a) K. Kawasaki, Phys. Rev. 150, 291 (1966); (b) Ann. Phys. (N. Y.) 61, 1 (1970), and intervening articles.

[7]P. Resibois and M. de Leener, Phys. Rev. 178, 806 (1969); Phys. Rev. 178, 819 (1969); and other articles.

[8]R. H. Kraichnan, J. Fluid Mech. 5, 497 (1959); J. Math. Phys. 2, 124 (1961); J. Math Phys. 3, 205 (1962); Phys. Fluids 7, 1723 (1964); in Dynamics of Fluids and Plasmas, edited by S. I. Pai (Academic, New York, 1966).

[9]H. W. Wyld, Jr., Ann. Phys. (N.Y.) 14, 143 (1961).

[10]S. F. Edwards, J. Fluid Mech. 18, 239 (1964); S. F. Edwards and D. Sherrington, Proc. Roy. Soc. 90, 3 (1967); S. F. Edwards and W. D. McComb, J. Phys. A 2, 157 (1969). These papers seem closer to the "quasiparticle" procedures of R. Balian and C. De Dominicis [Physica 30, 1927 (1964)] than to the "Green's-function" methods, and we shall not discuss them further.

[11]Although Kraichnan did not discuss the fully renormal-

ized theory, he was apparently aware of the flaws in Wyld's procedure and the need for three vertices in fourth order. It seems that Kraichnan's rules for calculating the renormalized vertices to a given order generate the quantities which are given exact nonperturbative definitions here. We are grateful to Dr. Kriachnan for providing us with old unpublished notes on the fourth-order vertex, on the basis of which he apparently came to conclusions similar to those reported here.

[12]After our work was completed we became aware of the work cited in Ref. 10 in which a similar, but not identical, operator appears to be used. The chronology may be an indirect argument for its "naturalness".

[13]The notion of a free-energy functional for dynamics is normally based on phenomenological considerations like those in J. Frenkel, *Kinetic Theory of Liquids* (Dover, New York, 1955), Chap. 7. In the present case it is possible to cast the equations in terms of a stationary functional, but we have not seen if and how the two are related. The use of functional-integral formulations of coupled nonlinear problems is discussed, for example, in R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965). Functional-integral formulations of the turbulence problem are usually associated with E. Hopf [J. Ratl. Mech Anal. $\underline{1}$, 87 (1952)]. Because of the additional operator, the functional integral associated with the equations derived below would be slightly different. It is generated from the functional differential equation straightforwardly. See, for example, P. C. Martin and J. Schwinger, Phys. Rev. $\underline{115}$, 1352 (1959).

[14]The description of response in classical systems in terms of Poisson brackets is discussed in various places. One place where a discussion in the spirit of the present work may be found is P. C. Martin, in *Many Body Physics*, edited by C. deWitt and R. Balian (Gordon and Breach, New York, 1968); another is R. Kubo, Rep. Prog. Phys. $\underline{23}$, 255 (1966).

[15]This example is only illustrative. The energy is unbounded and the system is unstable. If we were not prepared to sacrifice truth for simplicity at this point we would have chosen the Duffing, Rayleigh, or van der Pol oscillator; Lord Rayleigh, *Theory of Sound* (Dover, New York, 1945), Vol. I; B. van der Pol, Phil. Mag. $\underline{3}$, 65 (1927); or G. Duffing, *Erzwungenes Schwingungen bei Verandlicher Eigenfrequenz* (Vieweg und Sohn, Baunschweig, 1918). Their equations have been discussed in a statistical way recently by H. Risken [Fortschr. Phys. $\underline{16}$, 261(1968)] and by J. B. Morton and S. Corrsin [J. Stat. Phys. $\underline{2}$ 153 (1970)]. To include them by the techniques discussed here, we should introduce a second dynamical variable that plays the role of the square of the oscillator displacement. In terms of this variable the coupling is cubic. This procedure is employed, for example, in the discussion of A. Migdal {Zh. Eksp. Teor. Fiz. $\underline{55}$, 1964 (1968) [Sov. Phys.—JETP $\underline{28}$, 1036 (1969)]}. The other procedure would be to keep four-point interactions, as, for example, in C. De Dominicis and P. C. Martin, J. Math. Phys. $\underline{5}$, 14 (1964); $\underline{5}$, 31 (1964). In high-energy-physics language, it is convenient sometimes to introduce an intermediate boson instead of working with a four-fermion interaction. Neither of those alternatives, in our opinion, merits elaboration here.

[16]This description of classical systems has been uti-

lized by many authors in recent years. We cite, for illustrative purposes, P. C. Martin, Ref. 14, although there are no doubt earlier references in which this viewpoint is stressed.

[17]See, for example, Ref. 14, or L. P. Kadanoff and P. C. Martin, Ann. Phys. (N.Y.) $\underline{24}$, 419 (1963).

[18]Cumulants no doubt have a large mathematical literature. One of the first tabulations in the physics literature seems to be that of E. Meeron [J. Chem. Phys. $\underline{27}$, 67 (1957)]. They are also discussed in the works of Kubo, Kraichnan, Martin and Schwinger, and Martin and De Dominicis.

[19]The relation of the Hartree-Fock method of cumulant discard is noted in Ref. 13 among other places. When there are superfluids and the mean value of the field does not vanish, all terms in the statistical discussion must be retained. This leads to the approximation of M. Girardeau and L. Arnowitt [Phys. Rev. $\underline{113}$, 755 (1959)].

[20]The corresponding assumption in turbulence theory, i.e., the quasinormal approximation is usually associated with the names of M. Millionstchikov [C. R. (Dokl.) Acad. Sci. USSR $\underline{32}$, 615 (1941)]; I. Proudman and W. H. Reid [Phil. Trans. Roy. Soc. London A $\underline{247}$, 163 (1954)], and T. Tatsumi [Proc. Roy. Soc. London A $\underline{239}$, 16 (1957)].

[21]A typical problem of this type concerns the frequency shift in the energy between atomic levels due to a coupling U_3 which involves the two atomic states and the electromagnetic field. The correction to the photon absorption spectrum is large at the unperturbed frequency (where it is changed from a δ-function peak to a small value) and in the neighborhood of the "true" shifted frequency.

[22]That is to say if in lowest order $G^{-1} = (z - E_a)$ and in next order $G^{-1} = (z - E_a) - \lambda^2/(z - E_b)$, the ratio $\Delta G/G$ to second order is $\lambda^2 (z - E_b)^{-1}(z - E_a)^{-1}$, which is large near E_a and near E_b. [Near E_a, it describes the fact that the resonance is shifted to $E_a + \lambda^2/(E_b - E_a)$ when $\lambda^2/(E_a - E_b)^2 \ll 1$, as is discussed in Ref. 21, but it does so in an ineffective way.] In the other limit it is even a less controlled approximation. By contrast, the equations $G^{-1} = z - E_a - \lambda^2 H$ and $H^{-1} = z - E_b - \lambda^2 G$, which describe the problem in the second form, give $G = (z - E_a)^{-1} 2(1 + t)^{-1}$, $t \equiv [1 - 4\lambda^2/(z - E_a)(z - E_b)]^{1/2}$, and $\Delta G/G = (1 - t)/(1 + t)$, which is bounded. In the second form, the absorption which is given by $\text{Im} G(z)$ is altered from $\pi \delta(x - E_a)$ to a function spread between E_a and $E_a + 4\lambda^2/(E_b - E_a)$ when $\lambda^2/(E_b - E_a)^2 \ll 1$ [with a mean value given by $E_a + \lambda^2/(E_b - E_a)$]. On the other hand, even when $E_a = E_b$ and z is approximately E_a, $\Delta G/G$ is bounded and of this order for $|(z - E_a)| \lesssim \lambda$.

[23]R. H. Kraichnan [in *The Padé Approximant in Theoretical Physics*, edited by G. Baker and J. Gammel (Academic, New York, 1970)] has discussed turbulence approximations using Padé techniques, and many other examples are given in the book. The approximants under discussion here are "matrices", not numbers.

[24]Charge and mass renormalization are discussed in the classic context in Refs. 1 and 2. They are discussed in a context closer to the present one by De Dominicis and Martin, Ref. 15.

[25]The degree on non-Gaussian behavior is characterized by various aspects of the three- and four-point cumulants, to which names like "kurtosis" and "skewness" have been given. Estimates for them can be found for example, in the article by Gibson and Masiello [in *Sta-*

tistical Models and Turbulence, edited by M. Rosenblatt and C. van Atta (Springer-Verlag, Berlin, 1972)]. They are not very small, but, unlike the Reynolds number, they are somewhat less than unity.

[26]For example, by subtracting or differentiating, one may eliminate all dependence on the bare coupling constant (or Reynolds' number) in a differential equation. The solution to this equation may be found using as a boundary condition, the observed value of the coupling. Presumably, if there is a range of wave numbers for which the behavior is relatively insensitive to the bare coupling, it is possible to get an approximate solution in this region, even though at very high wave numbers the relative insensitivity of the renormalized coupling will manifest itself in an extreme sensitivity of the high-wave-number coupling to small changes in its low-wave-number value.

[27]Thus the "susceptibilities" and derivatives used in Refs. 5 and 6 are the ones that result from the "mode coupling" and are measured—not the bare couplings that occur in the equations of motion. This point is discussed from a point of view closer to our own in the article of J. A. Hertz [Int. J. Magnetism $\underline{1}$, 253 (1971)].

[28]This is the name given to the natural lowest-order nonlinear approximation by R. H. Kraichnan [J. Fluid Mech. $\underline{5}$, 497 (1959)].

[29]Since the only corrections to the direct-interaction approximation (d.i.a.) Ref. 28 are vertex corrections, the nature of the approximations of T. Nakano (University of Illinois, report of work prior to publication) in which, on the one hand, the d.i.a. is supposed to be corrected, and on the other, "vertex corrections are not taken into account," are difficult to comprehend and seem *ad hoc*.

[30]One proves, for example, that the charge measured in Thomson and Compton scattering experiments and by Coulomb's law are the same even though this charge does not appear in the "Hamiltonian." See, for example, Refs. 1 and 2.

[31]It is by carrying out this resummation that Wyld arrives at his "renormalized" equations. See M. J. Beran [*Statistical Continuum Theories* (Interscience, New York, 1968)] for a resume. Kraichnan also derives his equations in essentially this fashion.

[32]L. L. Lee, Ann. Phys. (N.Y.) $\underline{32}$, 292 (1965). Note that we are referring to a difficulty other than the problem in Wyld (Ref. 31) set straight by Lee.

[33]Note that with a proper renormalization the "counting" difficulties which were noted by Morton and Corrsin (Ref. 15) and which forced them to use a "hybrid" expansion with both real and bare vertices, do not arise.

[34]The techniques are described in many sources, among them, Refs. 1 and 2, and, in addition, Kadanoff and Baym (Ref. 3), Martin and Schwinger (Ref. 13), De Dominicis and Martin (Ref. 15), etc.

[35]That all the dependence on G_1 in Σ occurs through the dependence on G_2 can be checked iteratively or formally. As a result, more explicitly, $(\delta \Sigma / \delta G_1)_{\gamma_2} = (\delta \Sigma / \delta G_2)_{G_1} \times (\delta G_2 / \delta G_1)_{\gamma_2}$.

[36]The comments made in Ref. 15 concerning the need *only* for three-point interactions with an extended field

may now be recalled. There appears little advantage, practically or esthetically, for working instead with different types of vertices with four points (as also discussed in De Dominicis and Martin, Ref. 15) or with arbitrary numbers of points discussed by F. Englert and C. De Dominicis [Nuovo Cimento $\underline{53A}$, 1021 (1968)]. All the additional topological theorems must be restatements of those for the three-point function, as the similarity in proof suggests.

[37]The construction of the variational principle in terms of the correlation functions from the equations relating . them is discussed in Ref. 15. Crucial ideas were introduced by J. Luttinger and J. Ward [Phys. Rev. $\underline{118}$, 1417 (1960)] and by G. Baym [Phys. Rev. $\underline{127}$, 1391 (1962)].

[38]The renormalization group is discussed in Bogolyubov and Shirkov, Ref. 1. There has been a recent resurgence of interest in view of the work of K. Wilson and others on phase transitions [K. Wilson, Phys. Rev. B $\underline{4}$, 3174 (1971); Phys. Rev. B $\underline{4}$, 3184 (1971); F. J. Wegner (Brown University, report of work prior to publication)].

[39]Ward identities are discussed, for example, by C. G. Callan, S. Coleman, and R. Jackiw [Ann. Phys. (N.Y.) $\underline{59}$, 42 (1970)].

[40]The question of a random force will be dealt with at greater length in a succeeding paper, where turbulence is considered more specifically.

[41]D. Forster and P. C. Martin, Phys. Rev. A $\underline{2}$, 1575 (1970).

[42]While we believe the remaining problems are technical, Kraichnan believes these difficulties are fundamental in the application to equilibrium fluids.

[43]See Ref. 6(b), Sec. 4. We have not been able to prove that the diagrams that give the response to an impulse (i.e., G) are consistent with definition (4.3) which is also given for this function. If they are both correct, a nontrivial theorem must be proven, and we tend to doubt it. This disbelief also applies to the statement that the fluctuation-dissipation theorem implies their equality—at least as we understand that theorem.

[44]This point has received a great deal of attention recently. See, for example, J. R. Dorfman and E. G. D. Cohen, Phys. Rev. Letters $\underline{25}$, 1257 (1970); H. H. Ernst, E. H. Hauge, and J. M. J. van Leeuwen, Phys. Rev. Letters $\underline{25}$, 1254 (1970); Y. Pomeau, Phys. Rev. A $\underline{3}$, 1174 (1971); K. Kawasaki, Phys. Lett. A $\underline{32}$, 379 (1970); $\underline{34A}$, 12 (1971); R. Zwanzig, Proceedings of the IUPAP Conference on Statistical Mechanics, Chicago, 1971 (unpublished). Indeed, as stressed by Resibois, these theories are all similar in spirit to the work of Kadanoff and Swift (Ref. 5) from which the same results can be obtained by techniques they used for the critical point. To our knowledge no one has really shown that these cancellations do not occur for transport coefficients, and no computer experiments indicate such nonexponential tails. For self-diffusion, which is not a transport process, it is silly to worry about a possible cancellation in view of the computer calculations of B. Alderand and T. Wainwright [Phys. Rev. A $\underline{1}$, 18 (1970)].

[45]G. H. Vineyard, Phys. Rev. $\underline{110}$, 999 (1958).