Physics 218A Fall 2008

Quasilinear Theory

→ how instabilities / turbulence modify the profiles which drive them

→ Mean field theory → evolves $\langle f \rangle$

→ useful as device for calculating turbulent transport coefficients

i.e. anomalous resistivity

→ special application: turbulent / anomalous resistivity

→ see also: posted supplementary notes
→ Chapter 3 of book manuscript.

→ read Kulsrud: 11.1, 11.2, 14.1, 14.2, 14.7

Quasilinear Theory - Vlasov Plasma

i) Motivation and Overview

D⇒ linear theory determines 'instantaneous stability' of plasma

i.e. $\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int dv \frac{\partial\langle f\rangle/\partial v}{\omega - kv}$

⇒ growth/damping rate $\gamma_k = \gamma_k[\langle f\rangle]$

but $\langle f\rangle$ evolves... If $\langle f\rangle$ evolves slowly:

i.e. "slowly" ⇒ $\frac{1}{\langle f\rangle}\frac{\partial\langle f\rangle}{\partial t} < \gamma_k$

can consider: $\gamma_k = \gamma_k[\langle f(t)\rangle]$ → {evolution driven by instabilities

Physics: mean distribution evolution ....
⇒ driven by relaxation.

⇒⇒ quasilinear theory is concerned with describing and understanding the slow evolution of $\langle f\rangle$...

③ quasilinear theory is "mindless mean field theory", i.e.

$\langle f \rangle = \langle f(v,t) \rangle$ where → $\langle \ \rangle$ eliminates spatial dependence
→ t understood "slow"

so if:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0$$

then Q.L. equation is simply: (upon avg.)

$$\frac{\partial \langle f \rangle}{\partial t} + \frac{\partial}{\partial v}\left\langle \frac{q}{m}\tilde{E}\tilde{f} \right\rangle = 0$$

i.e. generic mean field equation (for $\langle f \rangle$) for mean of conserved order parameter ($\langle f \rangle$)

$$\frac{\partial \langle f \rangle}{\partial t} + \frac{\partial}{\partial v} J_v = 0$$ → phase space continuity equation

$$J_v = \Gamma_v = \left\langle \frac{q}{m} E f \right\rangle = \frac{q}{m}\langle \tilde{E}\tilde{f} \rangle$$

for: $E = \tilde{E}$

$f = \langle f \rangle + \tilde{f}$

elementary closure problem!
i.e. relate $\langle f \rangle$ to $\langle \tilde{E}\tilde{f} \rangle$ → hierarchy!
How close?

simplest example of moment closure.

then Q.L.T. simply takes form:

(i) $\tilde{f} \rightarrow \tilde{f}_{Linear}$ (i.e. linear response of $\tilde{f}$) plug in linear response

i.e. $\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0$ Vlasov Eqn.

$$\Rightarrow -i(\omega - kv)\tilde{f}_{k,\omega} = -\frac{q}{m}\tilde{E}_{k,\omega}\frac{\partial\langle f\rangle}{\partial v}$$

so $$J_v = -\frac{q^2}{m^2}\sum_{k,\omega}|\tilde{E}_{k,\omega}|^2\frac{i}{(\omega - kv)}\frac{\partial\langle f\rangle}{\partial v}$$

and with $\omega = \omega(k)$ only (i.e. spectrum of eigenmodes, only)   i.e. contrast approach to criticality in usual phase transitions (2nd order)
→ here growth of order parameter in broken symmetric phase ... not noise driven

Q.L. equation is:

$$\left.\begin{aligned}\frac{\partial\langle f\rangle}{\partial t} &= \frac{\partial}{\partial v}D\frac{\partial\langle f\rangle}{\partial v}\\ D &= \frac{q^2}{m^2}\sum_k|\tilde{E}_k|^2\frac{i}{\omega - kv}\end{aligned}\right\}\text{"Q.L. equation"}$$

i.e. mindless mean field theory...

with $$\left.\begin{aligned}\epsilon(k,\omega) &= 0\\ \partial_t|E_k|^2 &= 2\gamma_k|E_k|^2\end{aligned}\right\}\rightarrow \text{advance fields.}$$

But,

Surprisingly: Q.L.T. works quite well!

Key issue: why?

N.B.: In contrast to critical phenomena, external noise ignored → instability driven ...

④ Some questions to keep in mind: deterministic

i) why is QL equation a diffusion equation? When is this valid?

↔ nature of "irreversibility" ....

ii) can Q.L. equation be derived from Fokker-Planck theory?

↔ also "irreversibility" related ...

iii) how does Q.L. equation balance the energy-momentum budgets?

iv) when/how does Q.L. theory fail?

↔ related i) ..... What is "Ginzburg Criterion" for Q.L.T. Can such a criterion be formulated?

v) what is dynamics of quasilinear relaxation?

i.e. physics?

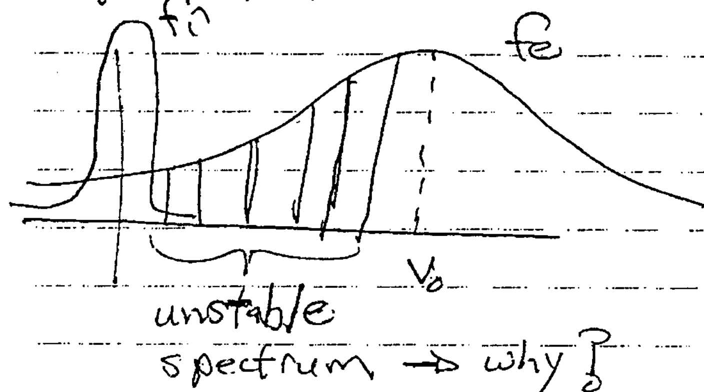
iii) <u>Basic Scales</u> / <u>Regime Definition</u>

① → Generally, Q.L.T. concerned with

i) 'broad' spectrum of:

ii) unstable waves

<u>i.e.</u> for current-driven ion-acoustic (C.D.I-A.) turbulence:



②
→ In <u>finite</u> system, k quantized, i.e.

$k_m = m\pi/L$, etc.

- so, have spectrum of phase velocities

$\omega_m/k_m = \omega(k_m)/k_m = V_{ph\,m}$

- wave-particle resonance occurs when

$V = V_{ph,m}$

then Sir Isaac ⇒

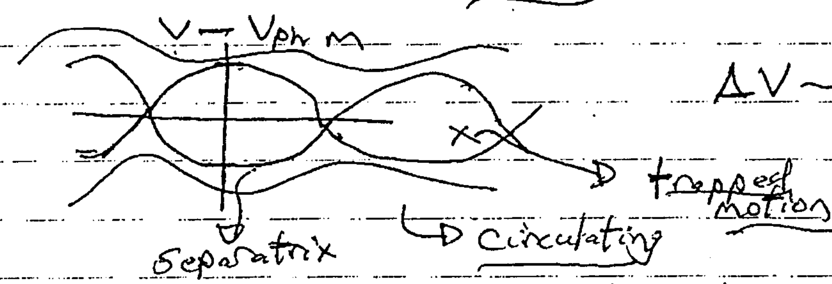
$$m\ddot{x} = \sum_m q E_m \cos(k_m x - \omega_m t)$$ n.b. $\left\{\begin{array}{l}\text{deterministic,}\\ \text{no RPA}\end{array}\right.$

and 1 resonance dominant ⇒

$$m\ddot{x} \cong q E_{m_0} \cos\left(k_m x_0 + (k_m v - \omega_m)t\right)$$

⇒ each resonant velocity defines a phase space island

i.e.

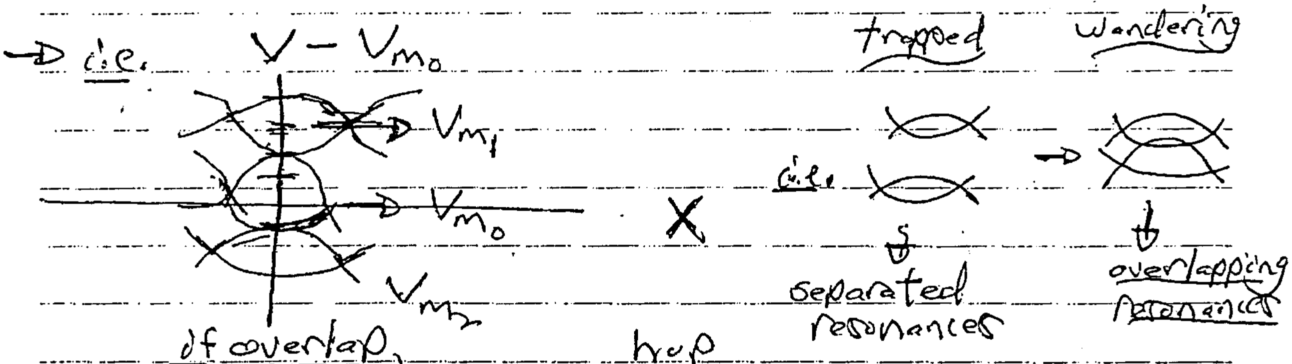


$$\Delta v \sim \left(\frac{q}{m}\phi_0\right)^{1/2}$$

∴

QLT is concerned with the case of:

→ multiple, overlapping resonances → $\left\{\begin{array}{l}\text{separatrix}\\ \text{proximity} \Rightarrow\\ \text{destruction}\end{array}\right.$



∴ particle can wander stochastically from resonance-to-resonance, i.e. hopping

⇒ diffusion in v / $D_v \sim \dfrac{(\Delta v)^2}{\tau_{ac}}$ $\Delta v \to$ resonance width; $\tau_{ac} \to$ pattern time

↳ what is it?

Overlap condition (B.V. Chirikov):

$$\boxed{\frac{1}{2}\left(\Delta V_m + \Delta V_{m+1}\right) \geq V_{ph_{m+1}} - V_{ph_m}.}$$

$\Delta v_T \sim \sqrt{e\tilde{\phi}/m}$

$\therefore \Rightarrow$ particle motion stochastic

$\rightarrow$ fundamental irreversibility $\Rightarrow$ orbit stochasticity (not dissipation, Landau damping ... $\Leftrightarrow$ contrast critical phenomena).

$\rightarrow$ underpinning of diffusion equation.

$\rightarrow$ But, a swindle !? $\rightarrow$ use of un-perturbed orbit in estimate!

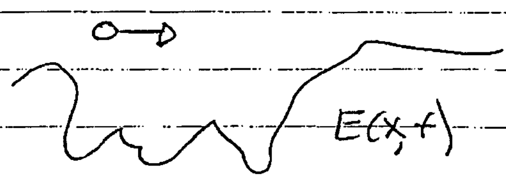
i.e. is $x \to x_0 + vt$ valid?

Consider: linear, un-perturbed orbit?!

have: $E(x,t) = \sum_k E_k \exp[i(kx - \omega_k t)]$

$\therefore$ particle "sees" instantaneous pattern of electric field, from modal superposition

i.e.



∴ relevant comparison is:

$\tau_L \rightarrow$ life time of 'instantaneous' pattern

$\tau_b \rightarrow$ 'bounce time' of particle in pattern.

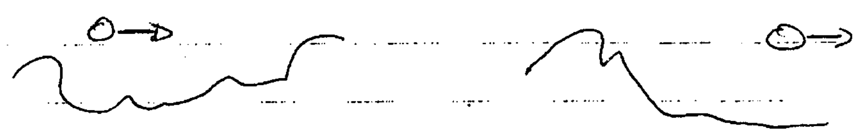
obviously, ① $\tau_L \ll \tau_b \rightarrow$ unperturbed orbit is satisfactory approximation

(pattern changes prior $\leftrightarrow$ bouncing)

② $\tau_L \gg \tau_b \rightarrow$ particle bounces prior pattern changes,

so must consider orbit perturbation. . .

i.e.



$\tau = 0$ $\qquad$ τ_b

vs.

∴ quasilinear theory relevant to evolution when:

① $\rightarrow$ orbits stochastic (Chirikov condition satisfied)

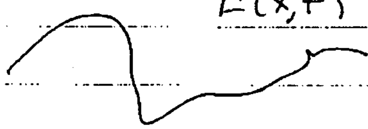
② $\rightarrow$ $\tau_{Life} < \tau_{bounce} \rightarrow$ unperturbed orbits ~ valid.

3)

But, how relate $T_{lifetime}$, T_{bounce} to physical quantities?

Key point: Superposition patterns disperse!

$E(x,t)$

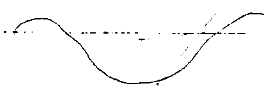


$$\Leftrightarrow \sum_k E_k e^{i(kx-\omega_k t)}$$

$$= \sum_k E_k \exp\left[i\left(k\left[x-\underbrace{\omega_k/k}_{v_{ph}(k)}\,t\right]\right)\right]$$

$\therefore$ $\underbrace{\Delta(\omega_k/k)}_{\text{sets dispersal rate}} \equiv$ spread in phase velocities.

so dispersal rate is $(\text{time})^{-1}$ to disperse by one wavelength



$$1/T_{life} \equiv k\,\Delta(\omega_k/k)$$

$$= k\left(\frac{d\omega_k}{dk}\frac{\Delta k}{k} - \frac{\omega_k}{k^2}\Delta k\right)$$

$$(V_g - V_{ph})$$

$$= \left(\frac{d\omega_k}{dk} - \frac{\omega_k}{k}\right)\Delta k = (V_g(k) - V_{ph}(k))\,\Delta k$$

n.b. $\left\{\begin{array}{l} T_{life} \to \infty \text{ for non-dispersive waves!} \\ \text{Generally; QLT / weak turbulence encounters trouble for } \left\{\begin{array}{l}\text{non dispersive} \\ \text{weakly dispersive}\end{array}\right. \text{waves.}\end{array}\right.$

How systematize?

Consider: $\langle E(x_1,t_1)\, E(x_2,t_2)\rangle_{x,t} = C$

electric field correlation function

$C = C(x_-, \tau)$, for $\left\{\begin{matrix}\text{homogeneous} \\ \text{stationary}\end{matrix}\right\}$ fluctuations

$x_1 = x_+ + x_- \qquad t_1 = t_+ + t_-$
$x_2 = x_+ - x_- \qquad t_2 = t_+ - t_-$

$\langle\ \rangle_{x,t} = \langle\ \rangle_{x_+,t_+}$

so

$$C(x,\tau) = \Big\langle \sum_{k,k'} E_k E_{k'}\, e^{i(k+k')x_+} e^{-i(\omega_k+\omega_{k'})t_+}\, e^{i(k-k')x_-} e^{-i(\omega_k-\omega_{k'})t_-} \Big\rangle_{x_+,t_+}$$

x_+, t_+ average $\Rightarrow k = -k' \quad \omega_k = -\omega_{k'}$

so

$$C(x,\tau) = \sum_k |E_k|^2 e^{ikx} e^{-i\omega_k t_-}$$

Now:

→ assume continuous spectrum

→ for simplicity, take model

$$|E_k|^2 = E_0^2 \Big/ \left[\left(\frac{k-k_0}{\Delta k}\right)^2 + 1\right]$$

→ evaluate on u.p.o.

$$x_- = x_{0-} + V\tau$$

$$\therefore \langle E^2 \rangle = \int dk \frac{E_0^2}{\left[\frac{(k-k_0)^2}{\Delta k^2} + 1\right]} e^{ikx_0} e^{i(kV-\omega_k)\tau}$$

integrating:

phase info. - irrelevant

$$\sim E_0^2 \, e^{ik_0 x_-} e^{-|\Delta k \, x_0|} *$$

$$e^{i(k_0 V - \omega_{k_0})\tau} \, e^{-|\Delta (kV-\omega_k)|\tau}$$

oscillation (→ 0 on resonance)

correlation decay due dispersal and its interplay with resonance.

n.b.: note that spread in doppler-shifted ω is critical now $\Delta(kV - \omega_k) = V\Delta k - V_{gr}\Delta k$

$= |(V - V_{gr})\Delta k|$

$V_{gr} = \frac{d\omega}{dk}$

so

$\langle E^2 \rangle = C(x_-, \tau)$

$= E_0^2 e^{ik_0 x_-} e^{i(k_0 V - \omega_{k_0})\tau} e^{-|\Delta k x_0|}$

$* \exp\left[-|(V - V_{gr})\Delta k|\, \tau \right]$

sets lifetime

$1/\tilde{\tau}_L = |(V - V_{gr}(k))\Delta k| \equiv$ (Autocorrelation Time)$^{-1}$

Note: $\equiv 1/\tilde{\tau}_{ac}$.

- for resonant particles, $V = \omega_k/k$

$1/\tilde{\tau}_L = |(V_{ph} - V_{gr})\Delta k|$ → recovers earlier!

- can think: $|V\Delta k| \to 1/\tilde{\tau}_{ac}^{\text{wave-particle}}$

$|V_{gr}\Delta k| \to 1/\tilde{\tau}_{ac}^{\text{wave}}$.

generally, shorter time dominates, except for non-dispersive waves.

<u>So</u>, can enumerate key time scales

$\tau_{ac} = |\Delta k(v_{ph} - v_{gr})|^{-1}$

$\equiv$ persistance of E pattern ($\langle E^2 \rangle$ autocorrelation) for <u>resonant</u> particles.

γ^{-1} = growth/damping time

$\tau_{Tr} = \left(k\sqrt{q\phi/m}\right)^{-1} \equiv$ trapping time

$\tau_{relax} = \left(\dfrac{1}{\langle f \rangle}\dfrac{\partial \langle f \rangle}{\partial t}\right)^{-1} \equiv$ avg. distribution relaxation time

<u>So</u>

$\tau_{ac} < \tau_{Tr} \rightarrow$ u.p.o. valid

$\tau_{ac}, \gamma^{-1} < \tau_{relax} \rightarrow \langle f \rangle$ closure meaningful.

$\tau_{ac} < \gamma^{-1} < \tau_{relax} \rightarrow$ Q.L.T. valid.

iii.) Energy – Momentum Budgets

→ key Point: There are two ways of implementing the book-keeping and accounting

i.e $\begin{cases} \text{resonant particles} & \text{vs.} \quad \text{'waves'} \\ \underline{\text{or}} & \\ \text{particles} & \underline{\text{vs.}} \quad \text{fields} \end{cases}$

keep in mind: Wave = Field + Non-resonant Particles

i.e for plasma oscillation, $\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$

$$\text{Wave Energy} = W = \frac{\partial}{\partial\omega}(\omega\epsilon)\Big|_{\omega_k} \frac{|E|^2}{8\pi}$$

$$= \omega\frac{\partial\epsilon}{\partial\omega}\Big|_{\omega_k} \frac{|E|^2}{8\pi}$$

$$= 2 \cdot \frac{|E|^2}{8\pi}$$

field non-resonant particle.

(show!)

→ Resonant Particles vs. Waves ?

$$\frac{\partial \langle f \rangle}{\partial t} = -\frac{\partial}{\partial v} \frac{q}{m} \langle \tilde{E} \tilde{f} \rangle$$

$$\frac{\partial}{\partial t} \int dv \frac{mv^2}{2} \langle f \rangle = -\int dv \frac{mv^2}{2} \frac{\partial}{\partial v} \frac{q}{m} \langle \tilde{E} \tilde{f} \rangle$$

$$= \int dv\, \cancel{m} v \frac{q}{\cancel{m}} \langle \tilde{E} \tilde{f} \rangle$$

Plugging in $\tilde{f}_k^{\text{Linear}}$ for $\tilde{f}$:

$$\frac{\partial}{\partial t} \mathcal{E}_{kin} = -i \int dv \frac{v q^2}{m} \sum_k |E_k|^2 \left(\frac{P}{\omega - kv} - i\pi \delta(\omega - kv) \right) \frac{\partial \langle f \rangle}{\partial v}$$

$$\frac{\partial}{\partial t} \mathcal{E}_{kin}^{res} = -\int dv \frac{\pi q^2}{m} \sum_k \frac{\omega}{k|k|} \delta(\omega/k - v) \frac{\partial \langle f \rangle}{\partial v} |E_k|^2$$

resonant only

$$= -\frac{\pi q^2}{m} \sum_k \frac{\omega}{k|k|} \left.\frac{\partial \langle f \rangle}{\partial v}\right|_{\omega/k} |E_k|^2$$

As resonant particles stabilize/destabilize wave, expect resonant particles conserve energy against waves.

→ wave energy evolution :

Recall : $\epsilon = 1 + \frac{\omega_p^2}{k}\int dv \, \frac{\partial\langle f\rangle/\partial v}{\omega - kv}$

$\epsilon^r(\omega_k + i\gamma_k) + i\epsilon^{IM} = 0$

$i\gamma_k = \frac{-i\epsilon^{IM}}{\partial\epsilon^r/\partial\omega}$; $\cancel{i}\gamma_k = -\frac{\cancel{i}\,\epsilon_{IM}}{\partial\epsilon^r/\partial\omega}$

$= -\epsilon_{IM}/\partial\epsilon^r/\partial\omega$

Now, $W \equiv$ Wave Energy Density

$$W = \sum_k \frac{\partial}{\partial\omega}(\omega\epsilon)\frac{|E_k|^2}{8\pi}$$

$$= \sum_k \omega_k \left.\frac{\partial\epsilon^r}{\partial\omega}\right|_{\omega_k} \frac{|E_k|^2}{8\pi}$$

$$\frac{\partial W}{\partial t} = \sum_k 2\gamma_k\,\omega_k \left.\frac{\partial\epsilon^r}{\partial\omega}\right|_{\omega_k}\frac{|E_k|^2}{8\pi} \qquad |E_k|^2 = |E_k^0|^2 e^{2\gamma_k t}$$

$$= \sum_k 2\left(\frac{-\epsilon_{IM}}{\partial\epsilon^r/\partial\omega}\right)\omega_k \frac{\partial\epsilon^r}{\partial\omega}\frac{|E_k|^2}{8\pi}$$

$$= \sum_k -\epsilon_{IM}(k,\omega_k)\,\omega_k\left(\frac{|E_k|^2}{4\pi}\right)$$

$$i\epsilon_{IM} = \frac{\omega_p^2}{k}\frac{\partial\langle f\rangle}{\partial v}\Big|_{\omega/k}\frac{(-i\pi)}{|k|}$$

$(n_0 = 1)$

$$\therefore \frac{\partial W}{\partial t} = \sum_k \pi\frac{q^2}{m}\frac{\omega_k \cancel{4\pi}}{k|k|}\frac{\partial\langle f\rangle}{\partial v}\Big|_{\omega/k}\frac{|E_k|^2}{\cancel{4\pi}}$$

$$= +\frac{\pi q^2}{m}\sum_k \frac{\omega}{k|k|}\frac{\partial\langle f\rangle}{\partial v}\Big|_{\omega/k}|E_k|^2$$

$$\Rightarrow \boxed{\partial_t \mathcal{E}_{kinetic}^{resonant} + \partial_t W = 0}$$

~~$\partial_t \int dv \frac{1}{2}mv^2 f = -q\langle E \tilde{f}\rangle$~~

Note:

- this is essentially a re-write of the Poynting theorem for plasma waves, i.e.

$$\underbrace{\frac{\partial}{\partial t}W}_{\text{wave energy}} + \underbrace{\nabla\cdot\underline{S}}_{\text{divergence of wave energy density flux}} + Q = 0$$

$Q \rightarrow \langle\tilde{E}\cdot\tilde{J}\rangle$ coupling

For homogeneous system: $\underline{\nabla} \cdot \underline{S} = 0$

so $\frac{\partial W}{\partial t} + Q = 0$

Q: $\langle E \cdot J \rangle$ mediated by resonant particles (DC field)

$$\Leftrightarrow \boxed{\frac{\partial W}{\partial t} + \frac{\partial}{\partial t}(RPKED) = 0}$$

RPKED: resonant particle kinetic energy density

Energy Thm I

Waves and Resonant particles conserve energy!

?? What is the fate of RPKED for saturated waves. What must happen ??

→ Now, can observe:

$$W = \underbrace{NRPKED}_{\text{non-resonant particle kinetic energy density}} + \underbrace{FED}_{\text{field energy density}}$$

so, simply re-grouping terms:

$$\frac{\partial}{\partial t}(FED) + \frac{\partial}{\partial t}\underbrace{(RPKED + NRPKED)}_{PKED} = 0$$

So $$\boxed{\frac{\partial}{\partial t}\text{FED} + \frac{\partial}{\partial t}(\text{PKED}) = 0}$$ Energy Thm.

i.e. fields and particles conserve energy.

What is the Physics of all this ? !

$$D = \text{Re}\sum_k \frac{q^2}{m^2}|E_k|^2\left(i/\omega - kv\right)$$

QL diffusion

for general, weakly non-stationary state ....

$$= \sum_k \frac{q^3}{m^2}|E_k|^2\left(|\gamma_k| / \left((\omega-kv)^2+|\gamma_k|^2\right)\right)$$

n.b. causality ⇒ no negative diffusion for damped waves

$$\simeq \sum_k \frac{q^2}{m^2}|E_k|^2\left\{\underbrace{\pi\delta(\omega-kv)}_{\text{resonant diffusion}} + \underbrace{\frac{|\gamma_k|}{\omega_k^2}}_{\text{non-resonant diffusion}}\right\}$$

Resonant Diffusion → irreversible − resonance overlap is underpinning

→ rooted in particle stochasticity

→ resonant diffusion can be obtained from Fokker-Planck calculation (show this)!

→ in principle, can persist in steady state (but how balance energy ....?!)

Non-Resonant Diffusion:

$$D^{NR} = \sum_k \frac{q^2}{m^2} |E_k|^2 \frac{\gamma_k}{\omega_k^2}$$

ponderomotive energy

$$= \frac{1}{2} \partial_t \sum_k |V_k|^2 \qquad \text{where } |V_k|^2 = \frac{q^2}{m^2} \frac{|E_k|^2}{\omega_k^2}$$

→ corresponds to "sloshing" motion energy of particles in wave

i.e. $D^{NR} \sim \partial_t \mathcal{E}_{quiver}$

→ thus reversible, can't be obtained from Fokker-Planck theory → aka "fake diffusion"

→ vanishes in stationary state

Point is that can "count non-resonant" diffusion as:

$$D_{\text{non-resonant}} \left\langle \begin{array}{l} \dfrac{\text{particle sloshing energy}}{\cancel{\text{part of } \underline{\text{wave} \text{ energy density}}}} \\ \text{or} \\ \text{part of total particle} \\ \text{kinetic energy density} \end{array} \right.$$

so <u>two</u> forms of energy conservation!

Note: Physically, the picture of plasma as gas $\begin{cases} \text{- resonant particles} \\ \text{- waves} \end{cases}$ or <u>equivalently</u>

<u>resonant particles</u> + <u>quasi-particles</u>

$$\downarrow \quad \underline{\text{waves}} \left\langle \begin{array}{l} N(k, x, t) \\ \text{WKE, etc.} \end{array} \right.$$

is appealing and will pervade this course.

N.B.: Direct Proof of $\partial_t (PKED + FED) = 0$

From QL equation:

$$\frac{\partial}{\partial t}(PKED) = -\sum_k \int dv\, \frac{\omega_p^2}{k} kv \frac{|E_k|^2}{4\pi} \frac{i}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k} \int \frac{dv}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

so

$$\partial_t (PKED) = -i \sum_k \frac{|E_k|^2}{4\pi} \int dv \frac{\omega_p^2}{k} (\underbrace{kv - \omega}{} + \omega) \times \frac{i}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

→ {cancels denom; product odd in k

$$= -i \sum_k \frac{|E_k|^2}{4\pi} \int dv \frac{\omega_p^2}{k} \frac{\omega}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

using $\epsilon(k,\omega) = 0$

$$= i \sum_k \frac{|E_k|^2}{4\pi} \omega_k$$

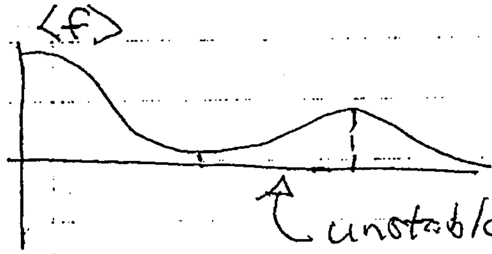
$\omega_k = \omega_k^r + i\gamma_k$

$$= -\sum_k \frac{|E_k|^2}{8\pi} (2\gamma_k)$$

$$= -\partial_t (FED) \quad \checkmark$$

iv.) Applications of Quasilinear Theory

→ Bump on Tail



↑ unstable phase velocities. (bump on tail)

$\omega_k = \omega_{pe}\left(1+\frac{3}{2}k^2\lambda_D^2\right)^{1/2}$

Quasi-linear Equations:

$$\epsilon(k, \omega_k) = 0 \quad \Rightarrow \omega(k), \gamma(k) \text{ from } \langle f \rangle$$

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial \langle f \rangle}{\partial v}$$

$$D = D^R + D^{NR}$$

$$= \sum_k \frac{q^2}{m^2} |E_k|^2 \left\{ \pi\, \delta(\omega_k - kv) + \frac{\gamma_k}{\omega_k^{2}} \right\}$$

$$\frac{\partial}{\partial t}\left(|E_k|^2/8\pi\right) = 2\gamma_k |E_k|^2/8\pi$$

Observe: - resonant diffusion describes dynamics of tail particles

- non-resonant diffusion describes dynamics of bulk Maxwellian

Expect: - tail flattening
with $\Downarrow$ $\Uparrow$
- adjustment of core/bulk profile (i.e. effective "temperature")

Now first consider resonant particles (i.e. on bump):

$$\frac{\partial \langle f\rangle}{\partial t} = \frac{\partial}{\partial v} D^R \frac{\partial \langle f\rangle}{\partial v}$$

$* \langle f\rangle$ and i.b.p. $\Rightarrow$

$\Rightarrow$

$$\frac{\partial}{\partial t}\int_{res} \frac{\langle f\rangle^2}{2} = -\int_{res} dv\, D^R \left(\frac{\partial \langle f\rangle}{\partial v}\right)^2$$

{ generalization $\Rightarrow$ Zeldovich Thm.

stationarity $\Rightarrow$

$$D^R \left(\partial\langle f\rangle/\partial v\right)^2 = 0$$

Now "res" $\rightarrow$ some finite interval of phase velocities

So

stationarity $\Rightarrow$ $D^R = 0$; i.e. fluctuations decay
and damp

or

$\partial \langle f \rangle / \partial v = 0$; plateau forms, removing growth

N.B.: – In 1D $\rightarrow$ plateau
– can generalize

To resolve:

$$D^R = \frac{8\pi^2 q^2}{m^2} \sum_k \frac{|E_k|^2}{8\pi} \delta(\omega - kv)$$

$$\cong \frac{16\pi^2 q^2}{m^2} \int dk \, \mathcal{E}_F(k) \, \delta(\omega - kv)$$

$$D^R = \frac{16\pi^2 q^2}{m^2 v} \mathcal{E}_F(\omega_{pe}/v)$$

So

$$\partial_t D^R = \frac{16\pi^2 q^2}{m^2 v} \left(2\gamma_{\omega_{pe}/v}\right) \mathcal{E}(\omega_{pe}/v)$$

Now, $\gamma_k = -\epsilon_{IM} / \left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega_k}$

$$\gamma_k = \gamma_{\frac{\omega_{pe}}{V}} = \pi V^2 \omega_p \frac{\partial \langle f \rangle}{\partial V}$$

so
$$\frac{\partial D^R}{\partial t} = \frac{16\pi^2 q^2}{m^2 V}\left(2\pi V^2 \omega_p \frac{\partial \langle f \rangle}{\partial V}\right) \mathcal{E}(\omega_p/V)$$

$$= \left(\pi \omega_p V^2 \frac{\partial \langle f \rangle}{\partial V}\right) D^R$$, using D^R defn.

so

$$D^R(V,t) = D^R(V,0) \exp\left[\pi \omega_{pe} V^2 \int_0^t dt' \frac{\partial \langle f \rangle}{\partial V}\right]$$

and :

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial V} D^R \frac{\partial \langle f \rangle}{\partial V}$$

$$= \frac{\partial}{\partial t}\frac{\partial}{\partial V}\left[\frac{D^R}{\pi \omega_{pe} V^2}\right]$$ $\left\{ \begin{array}{l} \text{using } \gamma_k, D \\ \text{definitions} \end{array} \right.$

so

$$\langle f(v,t)\rangle - \langle f(v,0)\rangle = \frac{\partial}{\partial v}\left[\frac{D^R(v,t) - D^R(v,0)}{\pi\omega_p v^2}\right]$$

∴ have:

$$\boxed{\begin{aligned} D^R &= D^R(v,0)\exp\left[\pi\omega_p v^2\int_0^t dt'\,\frac{\partial\langle f\rangle}{\partial v}\right] \\ \langle f(v,t)\rangle &= \langle f(v,0)\rangle + \frac{\partial}{\partial v}\left[\frac{D^R(v,t) - D^R(v,0)}{\pi\omega_p v^2}\right]\end{aligned}}$$

Now, recall seek to know if:

i.) $D^R \to 0 \Rightarrow \partial\langle f\rangle/\partial v\big|_{t\to\infty} < 0$ (fluctuations damp)

ii.) $\partial\langle f\rangle/\partial v \to 0 \Rightarrow$ finite D^R, distribution plateaus.

Now, if $D^R \to 0$,

$$\langle f(v,t)\rangle = \langle f(v,0)\rangle - \frac{\partial}{\partial v}\left[\frac{D^R(v,0)}{\pi\omega_p v^2}\right]$$

$$D^R(0) = \frac{16\pi^2 q^2}{m^2 v}\,\mathcal{E}(\omega_p/v, 0)$$

fluctuation energy

$$\underline{\text{but}} \quad \frac{16\pi^2 q^2}{m^2 V} \frac{\mathcal{E}(0)}{\pi \omega_p V^2} = 2 E_F(0) / (n m v_0^2/2)$$

$\ll 1$, as $n \gg n_b$

$\therefore \quad \langle f(v,t) \rangle \cong \langle f(v,0) \rangle$, to good approx.

but, for resonant velocities,

$\rightarrow$ linear instability $\Rightarrow$ $\partial \langle f \rangle / \partial v > 0$

$\rightarrow$ $D^R \underset{t \to \infty}{\rightarrow} 0$ $\Rightarrow$ $\partial \langle f \rangle / \partial v < 0$

but have (for $D^R \to 0$) $\langle f(t) \rangle = \langle f(0) \rangle$!

$\therefore$ $\boxed{\text{contradiction follows from assumption of } D^R(v,t) \to 0}$

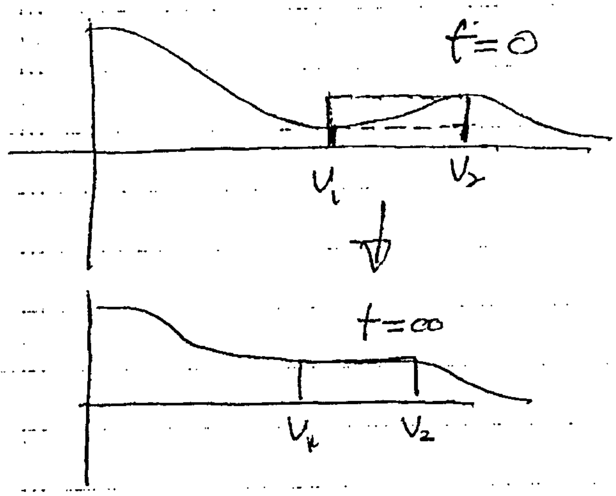
$\therefore$ have established that

$\partial \langle f \rangle / \partial v \Big\}_{res} \rightarrow 0 \quad \Rightarrow$ plateau forms!

For plateau formation, can immediately determine saturation levels from

$$\frac{\partial}{\partial t}(RPKED) + \frac{\partial}{\partial t}(WED) = 0$$

i.e.



$t\to\infty$

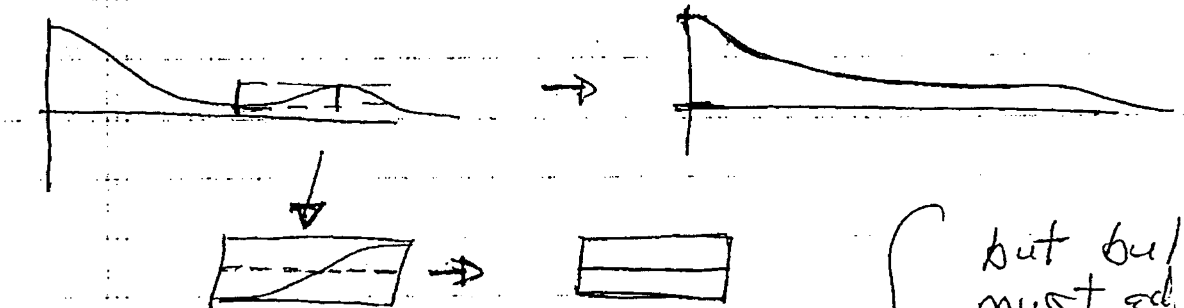
$k = \omega_p/v$

$$\therefore \quad \Delta\left(\int_{v_1}^{v_2} \frac{mv^2}{2}\langle f\rangle\right) = -\Delta\int_{k_1}^{k_2} W_k \; dk$$

but $W_k = 2\,\mathcal{E}(k)$

$$\Rightarrow \quad \Delta\left(\int_{v_1}^{v_2} dv \; \frac{mv^2}{2}\langle f\rangle\right) = -2\Delta\int_{k_1}^{k_2} \mathcal{E}(k)\, dk$$

→ can estimate A (RPKED) analytically, via construction



i.e. beam slows down

{ but bulk must adjust to conserve momentum!

i.e. bulk spreads outward, to conserve momentum as beam slows (bump flattened inward)

Now, for non-resonant particles:

$$\frac{\partial \langle f\rangle}{\partial t} = \frac{\partial}{\partial v} D^{NR} \frac{\partial \langle f\rangle}{\partial v}$$

$$= \frac{\partial}{\partial v} \frac{q^2}{m^2} \sum_k |E_k|^2 \frac{\gamma_k}{(\omega - kv)^2} \frac{\partial \langle f\rangle}{\partial v}$$

$$\cong \frac{8\pi q^2}{m^2} \int dk\, \mathcal{E}(k) \frac{\gamma_k}{\omega_{p0}^2} \frac{\partial^2 \langle f\rangle}{\partial v^2}$$

so, using γ definition:

$$\frac{\partial\langle f\rangle}{\partial t} = \left(\frac{1}{nm}\frac{\partial}{\partial t}\int dk\ \mathcal{E}(k)\right)\frac{\partial^2\langle f\rangle}{\partial v^2}$$

now define $T(t) = \frac{2}{n_e}\int dk\ \mathcal{E}(k,t)$

so $\Rightarrow$

$$\frac{\partial\langle f\rangle}{\partial T} = \frac{1}{2m}\frac{\partial^2\langle f\rangle}{\partial v^2}$$

thus, for initial Maxwellian:

$$\langle f\rangle = \left[m/2\pi\left[T+T(t)-T(0)\right]\right]^{1/2}\exp\left[\frac{-mv^2/2}{\left[T+T(t)-T(0)\right]}\right]$$

Thus, for non-resonant particles

- at saturation

$$T/2 \rightarrow T/2 + \frac{1}{n}\int dk\left[\mathcal{E}(k,\infty)-\mathcal{E}(k,0)\right]$$

i.e. electrons 'heated' by net increase in field energy ....

– can also note:

$$\frac{\partial}{\partial t}(RPKED) + \frac{\partial}{\partial t}(WED) = 0$$

for plasma waves,

$$\frac{\partial}{\partial t}(RPKED) = -2\frac{\partial}{\partial t}(FED)$$

so $\Delta(RPKED) = -2\Delta(FED)$

but

$$\Delta(PKED) = -\Delta(FED)$$

so $\Delta(RPKED) = +2(\Delta(PKED))$

$$\Rightarrow 0 = \Delta(RPKED) + 2\Delta(NRPKED)$$

and

$$\Delta(PKED) - \Delta(RPKED) = -\Delta(FED) - (-2)\Delta(FED)$$

$$\boxed{\Delta(NRPKED) = \Delta(FED)}$$

as shown above!

→ heating is one-sided, to conserve momentum.

A.) Mean Field Theory

i.) Quasilinear Theory

a.) Review of 1D/Basics

Refs:

① "Nonlinear Plasma Theory", by A. A. Galeev and R. Z. Sagdeev
↔ Reviews of Plasma Physics, Vol. 7 (Leontovich)
→ much better than book by same authors with same title

② "Plasma Turbulence", by B. B. Kadomtsev (1966)

③ "Cooperative Effects in Plasmas", by B. B. Kadomtsev
↔ Reviews of Plasma Physics, Vol. 22

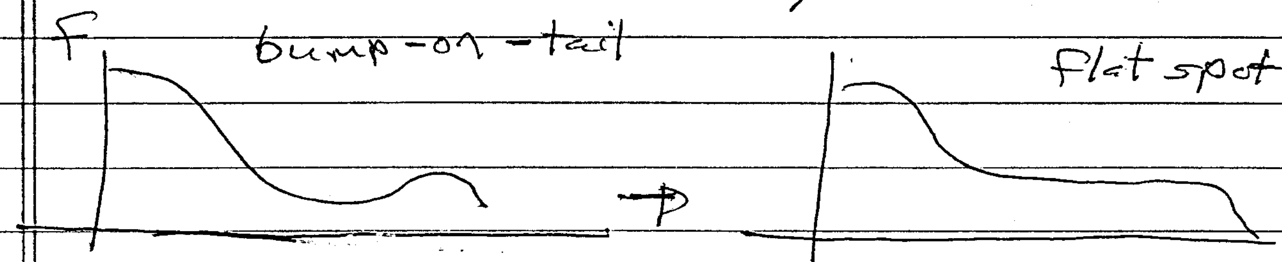
④ "Regular and Chaotic Dynamics", A. J. Lichtenberg and M. A. Lieberman; Springer-Verlag (1991).

⑤ "Hamiltonian Chaos and Fractional Dynamics" G. M. Zaslavsky, Oxford (2005)

(1)

⇒ Seek: How does $\langle f \rangle$ evolve due turbulence instability?

i.e.



aim → nonlinear saturation
→ transport (v, x, ...) ⇒ turbulent transport coefficients
→ relaxation, structure formation

⇒ tend toward uniform, most probable, etc. state?

if ω, k $\gg$ $1/T$, $1/L$

turbulence scales — mean scales

then can employ 2-scale strategy

i.e. $$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0$$

⇒

$$\underbrace{\frac{\partial \tilde{f}}{\partial t} + v\frac{\partial \tilde{f}}{\partial x}}_{\text{linear propagation}} + \frac{q}{m}\tilde{E}\frac{\partial \tilde{f}}{\partial v} = -\frac{q}{m}\tilde{E}\frac{\partial \langle f\rangle}{\partial v}$$

nonlinearity
- mode-mode coupling
- trapping

drive from mean

$\Rightarrow$ fluctuations

and

$$\frac{\partial \langle f\rangle}{\partial t} = -\frac{q}{m}\frac{\partial}{\partial v}\langle \tilde{E}\,\tilde{f}\rangle$$

slow evolution

beats

$\Rightarrow$ mean

- reasonable but closure problem ?
  What to do about $\tilde{f}$

- QLT: ① assume all spectral constituents are eigen modes i.e. all $\omega = \omega(k)$

  ② $\tilde{f}$ treated as linear response

i.e. plug:

$$\tilde{f}_k = \frac{-i\frac{q}{m}\tilde{E}_k\,\partial\langle f\rangle/\partial v}{\omega - kv} \rightarrow \text{linear response}$$

into $\langle f \rangle$ equation, yielding:

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial \langle f \rangle}{\partial v}$$

$$D = \mathrm{Re}\, \frac{q^3}{m^2} \sum_k |E_k|^2 \frac{i}{\omega - kv}$$

$$= \frac{q^3}{m^2} \sum_k |E_k|^2 \left\{ \frac{|\gamma_k|}{(\omega_k - kv)^2 + \gamma_k^2} \right\}$$

Note:

→ $|\gamma_k|$ → causality (i.e. D from spectrum of damped modes)

→ resonant, non-resonant

$$\frac{|\gamma_k|}{(\omega_k - kv)^2 + \gamma_k^2} \rightarrow \pi \delta(\omega_k - kv) + \frac{|\gamma_k|}{\omega_k^2}$$

→ can formulate full set of "quasilinear equations" → describes evolution via mean relaxation i.) $\langle f \rangle \rightarrow$ linear stability, via

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int dv \, \frac{\partial \langle f \rangle / \partial v}{\omega - kv} \Rightarrow \begin{matrix}\omega_{\underline{k}} \\ \gamma_{\underline{k}}\end{matrix}$$

$\downarrow$

ii.) spectral evolution

$$\frac{\partial}{\partial t}|E_{\underline{k}}|^2 = 2\gamma_{\underline{k}}|E_{\underline{k}}|^2 \Rightarrow |E_{\underline{k}}|^2$$

$\downarrow$

iii.) $\langle f \rangle$ modification / evolution

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial \langle f \rangle}{\partial v}$$

$$D = \mathrm{re} \sum_{\underline{k}} \frac{q^2}{m^2}|E_{\underline{k}}|^2 \frac{i}{\omega - kv} \quad = \langle f \rangle$$

$\Rightarrow$ convergence to marginal profile (?, $\rightarrow$ plateau?).

$\rightarrow$ Amazingly QLT works (as description of transport) quite well .... {though <u>not</u> universally}

WHY?

② ⇒ Numbers and Ratios ....

– what is assumed in QLT?

→ linear response adequate – no NL distortion

→ resonant diffusion – Markov process, a la' FPE
→ stochasticity / irreversibility
→ RPA (?)

Exercise:

a.) Derive QL (resonant diffusion) equation from Fokker-Planck theory

b.) use Hamiltonian structure of dynamics to eliminate dynamical friction term (c.f. Lichtenberg and Lieberman)

⇒ 2 dimensionless numbers:

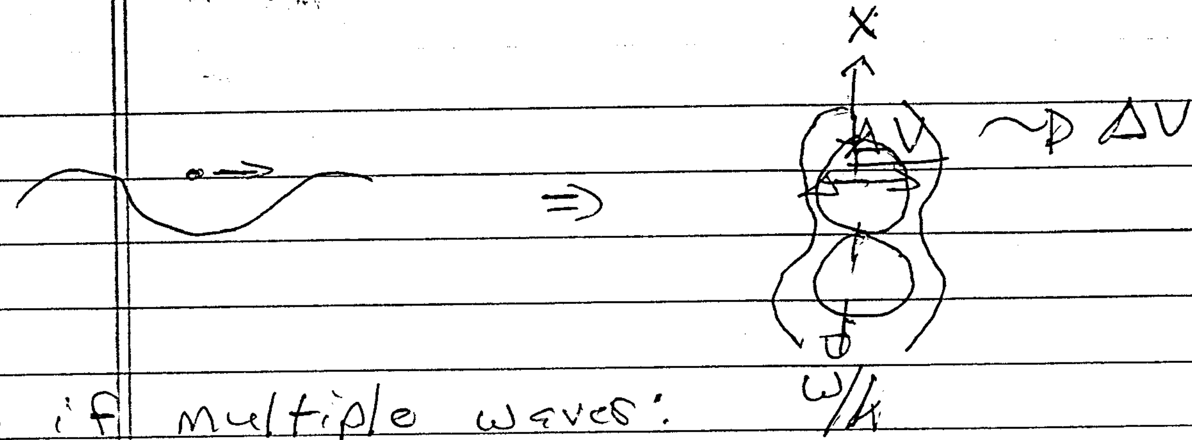
1)

$$S_c = \Delta V \Big/ \left| v_{ph_i} - v_{ph_{i+1}} \right|$$

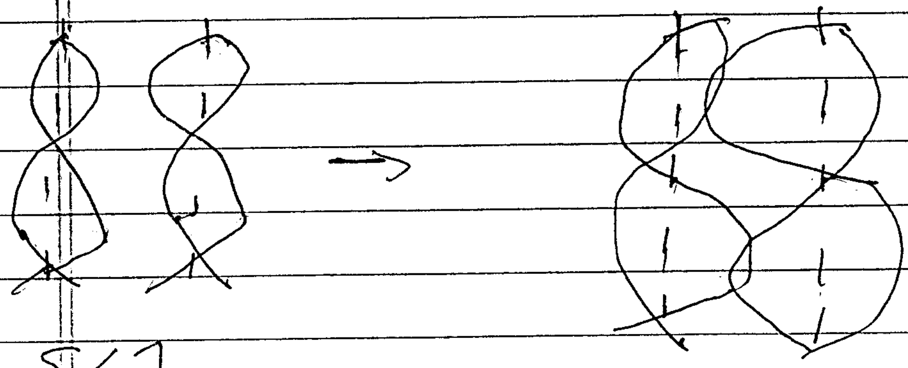
↑ Chirikov #

→ measures stochasticity of particle orbits

i.e. resonances overlap



so if multiple waves:



$\delta < 1$ $\qquad\qquad\qquad\qquad$ $\delta > 1$

⇒ ① independent resonant behavior

⇒ resonance overlap

⇒ diffusive "kicking" from 1 resonance to another.

② Kubo #, Strouhal # → measures memory of flow

i.e.

$$K = \frac{(qE/m)\,\tau_c}{(\Delta V)_c}$$

τ_c → field correlation time

$(\Delta V)_c$ → velocity correlation length for spatial scattering:

$$S = \frac{\tilde{V}\,\underbrace{\tau_c}_{\text{scatterer correlation time}}}{\underbrace{l_c}_{\text{correlation length}}}$$

Note: For $\Delta V_c \sim \Delta V_T \sim \sqrt{2T/m}$

$$K \sim k(\Delta V_T)\,\tau_c$$

$$\sim \underbrace{\omega_b}_{\text{bounce freq.}}\,\tau_c$$

$\Rightarrow$

$k_0 S > 1 \Rightarrow \begin{cases} \text{ordered flow, with memory} \\ \text{persistent scatterer pattern} \end{cases}$

$k_0 S < 1 \Rightarrow \begin{cases} \text{random, short lived scatterers} \\ \text{random flow – suggests RPA} \end{cases}$

usually $k_0 S > 1 \Rightarrow$ linear unperturbed trajectory <u>poor</u> approximation

$\Rightarrow$ need incorporate scattering field.

so, usual wisdom is that QLT valid if:

→ $\sigma_c > 1$ → need stochastic orbits

→ $K < 1$ → need avoid trapping, strong distortion, etc.

but: ?

①
→ is field / scatterer correlation time the relevant time scale ?

②
→ what of $\sigma_c > 1$, $K \sim 1$?

③
→ what of non-resonant piece ?

Regarding ①,

$$D = \sum_k \frac{q^2}{m^2} |E_k|^2 \int_0^\infty e^{i(\omega - kv)\tau} \, d\tau$$

$$|E_k|^2 = E_0^2 \, \Delta k \Big/ \left[(k - k_0)^2 + (\Delta k)^2 \right]$$

⇒

$$D = \sum_k \frac{q^2}{m^2} |E_k|^2 \int_0^\infty d\tau\, e^{i(\omega - kv)\tau}$$

$$= \int dk \frac{q^2}{m^2} \frac{E_0^2 \, \Delta k}{(k-k_0)^2 + (\Delta k)^2} \int_0^\infty d\tau\, e^{i(\omega - kv)\tau}$$

$$\approx \frac{q^2}{m^2} E_0^2 \, (2\pi) \int_0^\infty d\tau\, e^{i(\omega_{k_0} - k_0 v)\tau}$$

$$\exp\left[- \left| \frac{d\omega}{dk} - v \right| \Delta k \, \tau \right]$$

$\rightarrow$ sets correlation decay

$\Rightarrow$ if $D \sim \frac{q^2}{m^2} \langle E^2 \rangle \tau_c$

τ_c $\rightarrow$ wave-particle correlation time.

then:

$$1/\tau_{ac} \sim \left| \frac{d\omega}{dk} - v \right| \Delta k \rightarrow \left\{ \begin{array}{l} \text{wave-particle} \\ \text{decorrelation rate} \end{array} \right.$$

if $v \sim \omega/k$ – resonance

$$1/\tau_{ac_{pkt}} \sim \left| \frac{d\omega}{dk} - \frac{\omega}{k} \right| \Delta k \rightarrow \text{packet dispersal rate}$$

Generally: $\tau_{ac_{w-p}} \neq \tau_{ac_{pkt}} \rightarrow \left\{ \begin{array}{l} \text{differences} \\ \text{more pronounced} \\ \text{in 3D} \end{array} \right.$

So, really need:

→ specify velocity for wave-particle decorrelation being considered.

→ need: $1/\tau_{ac} > \omega_b,\ 1/\tau_{scatt}$

$$\underbrace{\phantom{1/\tau_{scatt}}}_{(k^2 D)^{1/3}}$$

→ $\tau_{ac} \sim \tau_c$ only for resonant particles in 1D.

→ broad spectrum alone is not sufficient to justify QLT → effective dispersion significant!

$$1/\tau_{ac}\Big|_{res} \sim \left|\frac{d\omega}{dk} - \frac{\omega}{k}\right| \Delta k \rightarrow 0; \text{ for non-dispersive waves}$$

→ $K < 1$ criterion most accurate if one takes $\tau_c \sim \tau_{ac,pkt}$

$$K = \frac{q}{m} E\, \tau_{ac,pkt} / \Delta v_T = (k \Delta v_T)\, \tau_{ac} < 1$$

agrees with intuition.

→ "phase randomization" irrelevant ⇒

- can have $\sigma > 1$, $k < 1$ with coherent phases

- QL known to well describe stochastic trajectory divergence in standard map/ magnetic field lines, even for static fields/ fixed phases, cf: Rechester, Rosenbluth, White PRL '80.

- phases fixed in Tsunoda / Malmberg experiments

→ often QLT seems to work reasonably well in limit of $\sigma > 1$, $k \sim 1$
  - unclear why....
  - corrections due granulations needed [PD]
    i.e. $k \sim 1 \Rightarrow$ $\Rightarrow$
    phase space eddys formed....

∴ strong non-stationarity can (boost) applicability of QLT.

Exercise :

→ consider, starting from $B \cdot \nabla \psi = 0$, a heuristic equation for the density of magnetic field lines in a "cylindromak":

$$B_T \frac{\partial N}{\partial z} + \frac{B_\theta(r)}{r} \frac{\partial N}{\partial \theta} + \tilde{B} \cdot \nabla N = 0$$

$$\frac{\partial N}{\partial z} + \frac{B_\theta(r)}{r B_T} \frac{\partial N}{\partial \theta} + \frac{\tilde{B}}{B} \cdot \nabla N = 0$$

from $$\frac{dr}{\tilde{B}_r} = \frac{r\,d\theta}{B_\theta + \tilde{B}_\theta} = \frac{dz}{B_T}$$

→ derive the QL equation for radial diffusion of $\langle N \rangle$

→ what is the QL diffusion coefficient

→ derive expressions for K, S

→ what is the analogue of wave-particle resonance here ..

③ Energetics

- easy to show (c.f. supplementary notes)

RPKED → resonant particle kinetic energy density

$$= \mathcal{E}_R = \int_{res} dv \, \frac{1}{2} m v^2 f$$

WED → wave energy density

$$= \int dk \, \omega_k N_k \qquad N_k = \left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega_k} \frac{|E_k|^2}{8\pi}$$

$$\frac{\partial}{\partial t}\left(\mathcal{E}_R + \int dk \, \omega_k N_k \right) = 0$$

- since WED = FED + NRPKED

FED ↓ field energy density $\sim |E_k|^2 / 8\pi$

NRPKED ↓ non-resonant particle kinetic energy density

then, equivalently can write:

$$\frac{\partial}{\partial t}\left(\mathcal{E}_p + \mathcal{E}_F \right) = 0$$

Exercise: Prove these.

– why important?

– basic consistency requirements

– physical picture ⇒ i.e. "quasi-particle picture"

i.e. represent plasma as:

$+\begin{cases} \text{– resonant particles} \Rightarrow \begin{cases}\text{Vlasov eqn.} + \\ \delta(\omega - kv)\end{cases} \\ \text{– waves} \rightarrow \text{quasiparticles} \Rightarrow \text{WKE} \end{cases}$

$$\frac{\partial N}{\partial t} + (\underline{v}_{gr} + \underline{V}) \cdot \nabla N - \frac{\partial}{\partial x}(\omega + \underline{k} \cdot \underline{V}) \cdot \frac{\partial N}{\partial \underline{k}} = 0$$

⇒ natural, multi-component picture

⇒ useful for accounting for $\begin{cases}\text{momentum} \\ \text{energy}\end{cases}$ transport by waves.

i.e. $\underline{P}_W = \int d\underline{k} \; \underline{k} \, N$

$\mathcal{E}_W = \int dk \; \omega_k \, N$

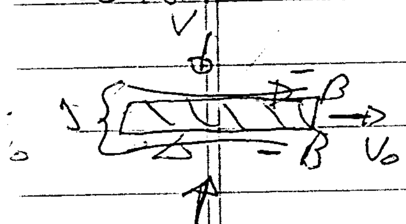
→ $\begin{cases}\text{can compute wave-induced stresses, etc.} \\ \text{formulate radiation hydro, etc.}\end{cases}$

Anomalous Resistivity → {Application of QLT} {Follows / expands on Galeev / Sagdeev Rev. Pl Phys '87.}

→ an instructive and important example of quasilinear theory is anomalous resistivity

→ here try approach classic current-driven ion acoustic instability (CDIA) model of anomalous resistivity via coupled micro-macro dynamics

– consider Sweet-Parker model, i.e.



$$V L = V_{out} \Delta$$

$$V_{out} = V_A$$

$$2 V \frac{B^2}{8\pi} L = \eta J^2 L \Delta$$

cf 218B notes

$\langle E \rangle = \langle \frac{\underline{v} B}{c} \rangle < 0$

(via to flow?)

$$\frac{\Delta}{L} = \frac{V}{V_A} \Rightarrow \boxed{\Delta^2 = \frac{(L)(\eta)}{V_A}} \Rightarrow \text{layer width } \frac{\Delta}{L} \sim 1/\sqrt{R_m}$$

What happens as η decreased ?

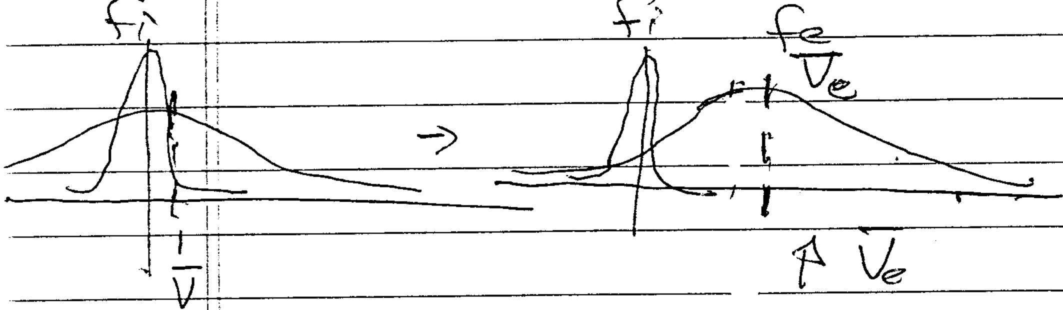
$$-\frac{c}{4\pi\Delta} B = J = n q \bar{V}_e$$

$\bar{V}_e$: electron drift speed

so $\bar{V}_e = cB/4\pi n q \Delta = \frac{d_{skin}^2}{\Delta} \Omega_e$

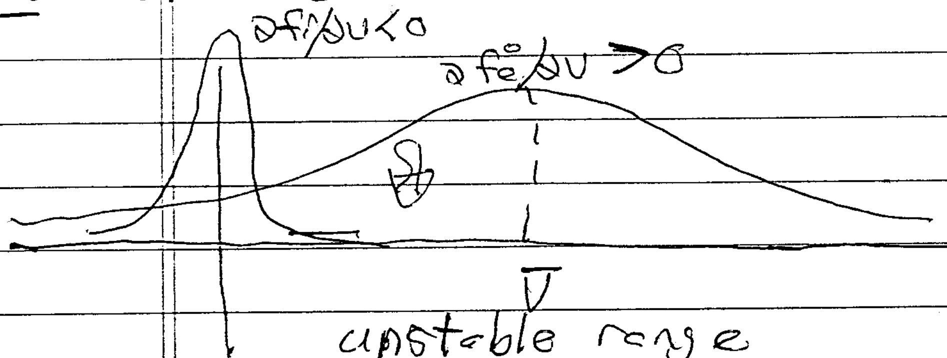
Now, $\bar{V}_e \sim B/\Delta n \Rightarrow \bar{V}_e \uparrow$ as $\left\{ \begin{array}{l} \Delta \downarrow \Rightarrow \text{narrow layer} \\ n \downarrow \to \text{few charge carriers} \\ B \uparrow \Rightarrow \text{stronger field (drive)} \end{array} \right.$

$\Delta \uparrow \Rightarrow$



$\Rightarrow$ decreasing Δ raises $\bar{V}$
$\rightarrow$ up-shifts f_e centroid relative to f_i
$\Rightarrow$ destabilizes CDIA!

i.e. classic scenario of CDIA



$\therefore$ expect CDIA will:
$\rightarrow$ exchange momentum between electrons and waves
so $\rightarrow$ slow down electrons, reduce $\bar{V}_e$
$\rightarrow$ act as "anomalous" "turbulent" resistivity

i.e.
$$\begin{cases} \Delta^2 = \dfrac{L}{V_A}\left(\eta + \underbrace{\eta_A(\bar{V})}_{\text{anomalous resistivity}}\right) \\ \bar{V} = \dfrac{cB}{4\pi}/nq\Delta \end{cases}$$

How calculate:

① Brute Force

- confining oneself to 1D model, ignoring Layer structure, have:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = C(f)$$

here $x \to$ vertical
$v \to$ vertical velocity
vertical $\to$ ⊥ to layer

$M_e V$ * $\Rightarrow$

$$\frac{\partial}{\partial t}\langle P_e\rangle - q\left\langle E\int dv f\right\rangle = -\nu_{e,i}n_0 M_e\bar{V}_e$$

collisional loss to ions

$$\frac{\partial \langle P_e\rangle}{\partial t} - qn_0\langle E\rangle - q\langle\tilde{E}\tilde{n}\rangle = -\nu_{e,i}n_0 M_e\bar{V}_e$$

so

$$\langle E\rangle + \frac{\langle\tilde{E}\tilde{n}\rangle}{n_0} - \frac{1}{n_0 q}\frac{\partial\langle P_e\rangle}{\partial t} = +\frac{\nu_{ei}M_e n_0 q\bar{V}_e}{n_0 q^2}$$

at ⓐ stationary state,

$$\langle E \rangle + \langle \tilde{E} \frac{\tilde{n}}{n_0} \rangle = \eta \langle J \rangle$$

driving field $\sim \langle VB \rangle$

electron acceleration by turbulence ⟹ "anomalous resistivity"

collisional resistivity

to calculate:

$$\langle \tilde{E} \frac{\tilde{n}}{n_0} \rangle = \sum_k +ik\phi_{-k} \frac{\tilde{n}_k^e}{n_0}$$

$$= \int dv \sum_k ik \phi_{-k} \tilde{f}_k^e$$

electron density perturbation

$f_k^e \rightarrow f_k^{e\,L}$

- quasilinear calculation
- stationarity ⟹ resonant transport.

ⓑ Conservation Argument

- as in ⓐ anticipate stationarity ⟹ resonant quasilinear evolution

- recall,

$$\frac{\partial}{\partial t}\left(\mathcal{E}^{RP} + \mathcal{E}^{wave}\right) = 0$$

$$\frac{\partial}{\partial t}\left(P^{RP} + P^{wave}\right) = 0$$

$$\mathcal{E}^{w}_{k} = \omega_k \frac{\partial \epsilon}{\partial \omega}\Big|_{k} \frac{|E_k|^2}{8\pi} \equiv \omega_k \underbrace{N_k}_{\to \text{ \# quanta}}$$

$$P^{w}_{k} = \frac{k}{\omega} \mathcal{E}^{w}_{k} = k N_k$$

as waves (CDIA) electrostatic, can ignore field momentum.

so, for resonant electrons

$$\frac{\partial}{\partial t} P^{RP}_{e} = -\frac{\partial P^{w}}{\partial t} = -\sum_{k} (2\gamma^{e}_{k}) \frac{k}{\omega_k} \mathcal{E}^{w}_{k}$$

γ^{e}_{k} = electron (resonant) growth rate

but

$\frac{\partial}{\partial t} P^{RP}_{electron}$ $\Rightarrow$ slowing down

$\Rightarrow$ macro-representation as effective collision frequency

so $\frac{\partial P^{RP}_{electron}}{\partial t} = -n m_e \underbrace{\nu_{eff}}_{\text{effective collision frequency}} \bar{V}$

slowing down by resonant scattering (resonant particle interaction)

$$\boxed{n m_e \nu_{eff} \bar{V} = \sum_k (2\gamma_k^e) \frac{k}{\omega_k} \mathcal{E}_k^{\omega}}$$

- defines $\bar{V}$
- for macro-micro link
  $\bar{V} = \frac{cB}{4\pi n q \Delta}$

* - n.b. if 2D, 3D theory, i.e. $\perp$ dynamics $\Rightarrow$ non-resonant scattering $\Leftrightarrow$ wave driven momentum flux i.e. $\Pi_{\perp,\parallel} \to \perp$ radiation $\parallel$ momentum. Relation to whistler interpretation of Bellan ?. There need include wave radiation in energy balance.

so now have

$$\boxed{\begin{aligned} n\, m\, \nu_{eff}(B_0, \Delta)\bar{V} &= \sum_k (2\gamma_k^e)\frac{k}{\omega_k}\,\mathcal{E}_k^w \\ \Delta^2 &= \frac{b}{V_A}\left(\eta + \frac{c^2}{\omega_{pe}^2}\nu_{eff}\right) \end{aligned}}$$ ①

⇒ need γ_k^e, $\mathcal{E}_k^w$ and $\langle f^e\rangle$ evolution

at simplest level, proceed via linear/quasilinear theory in 1D
- at more advanced level:
  - consider 1D phase space structures
    → electron/ion clumps, momentum exchange
    → electron scattering off ion hole
  - consider 3D $J_{\parallel}^{*}$ driven instability with electron viscosity

Now, proceed in usual fashion:

γ_k^e → linear theory

$\mathcal{E}_k^w$ → nonlinear saturation

$\langle f_e\rangle$ → QL equation — flattening

For linear theory of CDIA;

$$\nabla^2 \hat{\phi} = -4\pi n_0 |e| \left(\frac{\hat{n}_i}{n_0} - \frac{\hat{n}_e}{n_0} \right)$$

$$\hat{n}_i / n_0 = \frac{k^2 c_s^2}{\omega^2} \frac{|e|\phi}{T}$$

$$\frac{\hat{n}_e}{n_0} = \frac{|e|\hat{\phi}}{T} \left[1 - i\Gamma(k) \right]$$

$\Gamma(k)$,

$$\frac{\partial \tilde{f}}{\partial t} + v \frac{\partial \tilde{f}}{\partial x} = -\frac{|e|}{m_e} \tilde{E} \frac{\partial \langle f \rangle}{\partial v}$$

$$\hat{f} = \frac{|e| \hat{\phi}}{T} \langle f \rangle + g$$

$$\frac{\partial \tilde{g}}{\partial t} + v \frac{\partial \tilde{g}}{\partial x} = -v \frac{\partial}{\partial x} \left(\frac{|e| \hat{\phi} \langle f \rangle}{T} \right) + \frac{|e|}{m_e} \frac{\partial \tilde{\phi}}{\partial x} \frac{\partial \langle f \rangle}{\partial v}$$
$$- \frac{\partial}{\partial t} \left(\frac{|e| \hat{\phi} \langle f \rangle}{T} \right)$$

$$= v \frac{\partial \tilde{\phi}}{\partial x} \frac{|e|}{T} \langle f \rangle + \frac{|e|}{\cancel{m_e}} \frac{\partial \tilde{\phi}}{\partial x} - \frac{(v - \bar{v})}{T/\cancel{m_e}} \langle f \rangle - \frac{\partial}{\partial t} \frac{|e| \hat{\phi} \langle f \rangle}{T_e}$$

$$= -\frac{\partial}{\partial t} \frac{|e| \hat{\phi} \langle f \rangle}{T} + \bar{v} \frac{\partial}{\partial x} \frac{|e| \hat{\phi} \langle f \rangle}{T}$$

$$\Rightarrow g_k = \frac{i(\omega - k\bar{v})}{-i(\omega - kv)} \frac{|e|\hat{\phi}_k}{T} \langle f \rangle$$

$$= -\left(\frac{\omega - k\bar{v}}{\omega - kv}\right) \frac{|e|\hat{\phi}_k}{T} \langle f \rangle$$

$$\omega_k^{2\,(0)} = \frac{k^2 c_s^2}{1 + k^2 \lambda_D^2}$$

$$-i\,\Gamma(k) = \int dv + \frac{(\omega - k\bar{v})}{(\omega - kv)} \langle f \rangle$$

$$\bar{F} = \frac{1}{\sqrt{\pi}} \exp\left[\frac{-(\omega/k - \bar{v})^2}{v_{Th}^2}\right]$$

$$= -(\omega - k\bar{v}) \frac{(+i\pi)}{|k| v_{Th}} \bar{F}\Big|_{\omega/k v_{Th}}$$

$$1 + k^2 \lambda_D^2 = \frac{k^2 c_s^2}{\omega^2} + \frac{(\omega - k\bar{v})}{|k| v_{Th}} (i\pi) \bar{F}\Big|_{\omega/k v_{Th}}$$

$$\omega \to \omega + \delta\omega$$

$$0 = -\frac{2\delta\omega}{\omega} + \frac{(\omega - k\bar{v})}{|k| v_{Th}} (-i\pi) \bar{F}\Big|_{\omega/k v_{Th}}$$

$$\frac{\delta\omega}{\omega} = -\frac{i\pi}{2} \frac{(\omega - k\bar{v})}{|k| v_{Th}} \bar{F}\Big|_{\omega/k v_{Th}}$$

$\delta\omega \Rightarrow i\gamma_k$ growth rate

(a) $$\boxed{\gamma_k \sim -\frac{\pi}{2} \omega_k \frac{(\omega - k\bar{v})}{|k| v_{Th}} \bar{F}\Big|_{\omega/k v_{Th}}}$$ $\Rightarrow$ $\gamma > 0$ for $\bar{v} > c_s$

$\Rightarrow$ critical velocity

for $\langle f \rangle$ evolution,

$$\frac{\partial \langle f \rangle}{\partial t} = +\frac{\partial}{\partial v} \sum_k \frac{+|e|}{m_e} \tilde{E}_{-k} \tilde{g}_k$$

$$= +\frac{\partial}{\partial v} \sum_k \frac{|e|}{m_e} \tilde{E}_{-k} \left(\frac{-(\omega - k\bar{v})}{(\omega - kv)} \frac{|e|\tilde{\phi}_k}{T} \langle f \rangle \right)$$

$$= \frac{\partial}{\partial v} \sum_k \frac{T|e|}{m_e} \frac{+ik\tilde{\phi}_k}{T_e} \left(-(\omega - kv) i\pi \delta(\omega - kv) \right) \frac{|e|\tilde{\phi}_k}{T} \langle f \rangle$$

$$= \frac{\partial}{\partial v} \sum_k (-v_{Th}^2) \left| \frac{|e|\tilde{\phi}_k}{T} \right|^2 k(\omega - k\bar{v}) \, \pi \delta(\omega - kv) \langle f \rangle$$

$$\boxed{\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} \sum_k (-v_{Th}^2) \left| \frac{|e|\tilde{\phi}_k}{T} \right|^2 k(\omega - k\bar{v}) \, \pi \delta(\omega - kv) \langle f \rangle} \quad (8)$$

— mean evolution

Note:

— really only assumed $\langle f \rangle = \langle f((v-\bar{v})^2 / 2v_{Th}^2) \rangle$

so $\frac{\partial \langle f \rangle}{\partial v} = \left(\frac{v - \bar{v}}{v_{Th}^2} \right) \langle f \rangle$

and

$$\langle f \rangle' = - \langle f \rangle$$

→ minimal assumption on structure

- can write as $\bar{v}$ evolution

$$\bar{v} = \int dv\, v \langle f \rangle \Big/ \int dv \langle f \rangle$$

$$\Rightarrow$$

(4) $$\boxed{\frac{\partial \bar{v}}{\partial t} = + \int dv \sum_k k_{T}^{2} \left| \frac{e\tilde{\phi}_k}{T} \right|^2 \frac{k^2}{T} \left(\frac{\omega}{k} - \bar{v} \right) \pi \delta(\omega - kv) \langle f \rangle}$$

$$\omega/k < \bar{v} \Rightarrow \partial \bar{v}/\partial t < 0$$
$$> \bar{v} \Rightarrow \partial \bar{v}/\partial t > 0$$

Remains to determine fluctuation intensity level

Generically, can write:

$$\frac{\partial}{\partial t} \mathcal{E}_k^w = \overset{(a)}{\gamma_k \mathcal{E}_k^w} - \overset{(b)}{\left(\sum_{k'} \omega_k C_1(k, k') \frac{\mathcal{E}_{k'}^w}{nT} \right)} \mathcal{E}_k^w$$
$$- \overset{(c)}{\left(\sum_{k', k''} \omega_k C_2(k, k', k'') \frac{\mathcal{E}_{k'}^w}{nT} \frac{\mathcal{E}_{k''}^w}{nT} \right)} \mathcal{E}_k^w$$

spectral equation constituents:

(a) - linear growth

(b) - quadratic nonlinearity $\rightarrow$ 3 wave coupling / NL ion-wave interaction

(c) - "cubic NL" $\rightarrow$ wave coupling

Now, for ion-acoustic wave:

- 3 wave coupling effects negligible
  ⇒ can't satisfy resonance
- NL wave-particle effects weak ⇒ intrinsically

⇒ consider 4 wave process

$$\boxed{\frac{\partial \mathcal{E}_k^\omega}{\partial t} = \left[\gamma_k - \omega_k B(\omega, k)\left(\frac{\mathcal{E}_k^\omega}{nT}\right)^2\right]\mathcal{E}_k^\omega} \quad (5)$$

- 'cartoon' NL saturation equation

Now, (1)-(5) ⇒ $\begin{cases}\text{coupled, (2)-stationary} \\ \text{micro-macro system}\end{cases}$

⇒ describe anomalous resistivity dynamics and its effect on reconnection

⇒ coupled solution corresponds to solution of the problem

$$ (1) \begin{cases} n m_e \nu_{eff}(B,\Delta)\, \overline{V} = \sum_k 2\gamma_k^e \dfrac{k}{\omega_k} \mathcal{E}_k^w \\ \Delta^2 = \dfrac{L}{V_A}\left(\eta + \dfrac{c^2}{\omega_{pe}^2}\nu_{eff}\right), \quad \overline{V} = cB/4\pi n q \Delta \end{cases} $$

$$ (2) \quad \gamma_k^e = -\frac{\pi}{2}\,\omega_k \frac{(\omega - k\overline{V})}{|k| v_{Th}} \overline{f}\Big|_{\omega/k v_{Th}} $$

$$ (4) \quad \frac{\partial \overline{V}}{\partial t} = \int dv \left(\sum_k v_{Th}^2 \left|\frac{e\tilde{\phi}_k}{T}\right|^2 k^2 \left(\frac{\omega}{k} - \overline{V}\right) \pi c \delta(\omega - kv) \langle f \rangle \right) $$

$$ (5) \quad \frac{\partial \mathcal{E}_k^w}{\partial t} = \left[\gamma_k - \omega_k B(\omega, k)\left(\frac{\mathcal{E}_k^w}{nT}\right)^2\right] \mathcal{E}_k^w $$

Now, stationarity $\Rightarrow$

$$ \mathcal{E}_k^w = nT\,(\gamma_k/\omega_k B)^{1/2} $$

$$ \gamma_k = +\frac{\pi}{2}\left(\frac{\overline{V} - c_s}{|k v_{Th}|}\right) k\omega_k \overline{f}\Big|_{\omega/k v_{Th}} $$

so, for scalings:

$$\nu_{eff} = \frac{1}{nm\bar{v}} \sum_k 2\gamma_k^e \frac{k}{\omega_k} \varepsilon_k^\omega$$

$$\sim \frac{1}{nm\bar{v}} \frac{(\bar{V}-C_s)}{|k|v_{Th}} k\cancel{\omega_k}\, \bar{f}\Big|_{\frac{\omega_k}{kv_{Th}}} \frac{k}{\cancel{\omega_k}}(nT)\left(\frac{\gamma_k}{\omega_k B}\right)$$

$$\sim \frac{(1-C_s/\bar{V})}{\cancel{n}m} \frac{k^2 nT}{|k|v_{Th}} \left(\frac{\gamma_k}{\omega_k B}\right) \bar{f}\Big|_{\frac{\omega_k}{kv_{Th}}}$$

$$\sim (1-C_s/\bar{V})\, \bar{f}\Big|_{\omega/kv_{Th}} \left(\frac{k^2 v_{Th}}{|k|}\right)\left(\frac{\gamma_k}{\omega_k B}\right)$$

$$\sim (1-C_s/\bar{V})\left(\frac{k\cancel{\omega_k}}{|k v_{Th}|}\frac{(\bar{V}-C_s)}{\cancel{\omega_k} B}\bar{f}\right)^{1/2} \bar{f}\,\frac{k^2 v_{Th}}{|k|}$$

$$\sim \left[(\bar{V}-C_s)k\right]^{3/2} \bar{f}^{3/2} \frac{k(v_{Th}/\bar{V})}{|k|\,|k v_{Th}|^{1/2}}$$

$$\boxed{\nu_{eff} \sim \frac{\left[(\bar{V}-C_s)k\right]^{3/2}}{|k v_{Th}|^{1/2}} \frac{v_{Th}}{\bar{V}} \bar{f}^{3/2} \left(\frac{k}{|k|}\right)}$$

Turbulent collision frequency

so have

$$\boxed{\begin{aligned} \Delta^2 &= \frac{L}{V_A}\left(\eta + \frac{c^2}{\omega_{pe}^2}\nu_{eff}\right) \\ \bar{V} &= cB_0 / 4\pi n_0 q \Delta \\ \nu_{eff} &= \frac{\left[(\bar{V}-C_s)k\right]^{3/2}}{|k v_{Th}|^{1/2}} \frac{v_{Th}}{\bar{V}} \bar{F}^{3/2}\left(\frac{k}{|k|}\right) \end{aligned}}$$

with:

$$\boxed{\begin{aligned} \frac{\gamma_k}{\omega_k} &= \frac{\pi}{2}(\bar{V}-C_s)\frac{k}{|k| v_{Th}} \bar{F} \\ \bar{F} &= \frac{1}{\sqrt{\pi}} \exp\left[-(\omega/k - \bar{V})^2 / 2 v_{Th}^2\right] \end{aligned}}$$

→ characterize micro-macro coupling with anomalous resistivity

→ now can envision situation
- finite current, $\Delta \sim (L\eta/V_A)^{1/2}$
  $\nu_{eff} = 0$

so if:

- decrease η ⇒ Δ decreases
- Δ decreases ⇒ $\bar{V}$ increases

- $\bar{V}$ increases $\Rightarrow \gamma_{II} > 0$

- $\gamma_{II} > 0 \Rightarrow \begin{cases} \varepsilon^{\omega}_{4} > 0 \\ \gamma_{eff} > 0 \end{cases} \Rightarrow$

- $\gamma_{eff} > 0 \Rightarrow \begin{cases} \Delta & \text{increases} \\ \bar{V} & \text{decreases} \end{cases}$

$\Rightarrow$ decreasing ν so Δ decreases triggers feedback so Δ increases $\Rightarrow$ self-regulation / ⊕ feedback

$\therefore$ in this model, can expect:

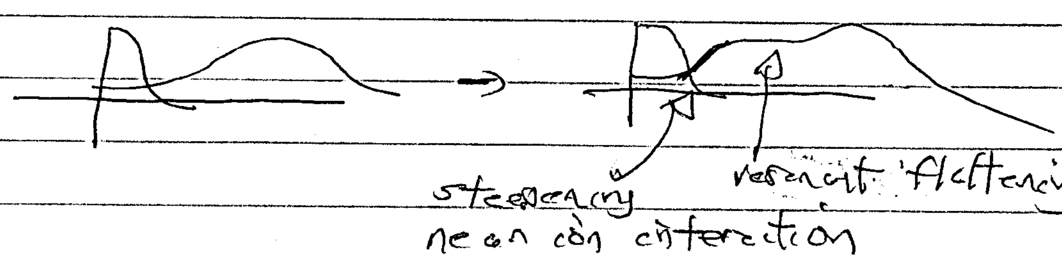
- at low $\nu_{collisional}$, so $\bar{V}\big|_c \sim cB_0/4\pi n_0 q \Delta_c \sim C_s$

$\Rightarrow$ COIA "hovers" near marginal stability

- for stronger drive (i.e. $B_0 \uparrow$)

  $\Rightarrow$ ion interaction important
  $\Rightarrow$ strong ion distortion possibly significant
  $\Rightarrow$ granulation formation important
  $\Rightarrow$ distortion of electron distribution function need be considered

i.e.



⇒ Useful extensions:

- 1D avalanche model → avalanching ⇒ jitter effects on Δ, $\bar{V}$

- nonlinear noise effects via fluctuations

- 2D, 3D ⇒ wave radiation, esp. wave momentum flux $\perp$ layer.

* – granulation effects ⇒ strong distortion hole (cf. later in course)

Comment:

This simple problem is surprisingly poorly understood. Excellent example of:

→ micro-macro feedback

→ self-regulation

→ marginal stability ...