Physica D 48 (1991) 273–294
North-Holland

The dynamics of enstrophy transfer in two-dimensional hydrodynamics*

John Weiss
49 Grandview Road, Arlington, MA 02174, USA

Received 21 September 1990
Accepted 15 October 1990
Communicated by U. Frisch

In this paper the qualitative properties of an inviscid, incompressible, two-dimensional fluid are examined. Starting from the equations of motion we derive a series of equations that govern the behavior of the spatial gradients of the vorticity scalar. The growth of these gradients is related to the transfer of enstrophy (integral of squared vorticity) to the small scales of the fluid motion.

1. Introduction

In this paper the qualitative properties of an inviscid, incompressible two-dimensional fluid are examined. Starting from the equations of motion (Euler's equations) we derive a series of equations that govern the behavior of the spatial gradients of the vorticity scalar. The growth of these gradients is related to the transfer of enstrophy (vorticity) to the small scale motion of the fluid. We find that the gradients of vorticity will tend to grow exponentially fast in a region of fluid, if, in that region, the squared magnitude of the rate of strain exceeds the squared magnitude of the rate of rotation. The rate of rotation can be identified with the vorticity scalar. On the other hand, when the squared vorticity exceeds the squared rate of strain, the vorticity gradients will behave in a periodic manner.

*Note by the editor: This paper has been frequently quoted in the literature on two-dimensional turbulence; it was written in 1981 as a La Jolla Institute preprint and never published in the open literature.

Essentially, when the strain rate exceeds the vorticity, the fluid is in a hyperbolic mode of motion that strongly shears the passively advected vorticity. Conversely, when the vorticity exceeds the strain the fluid is in an elliptical mode of motion that advects the vorticity smoothly. As a consequence the vorticity gradients will tend to concentrate in the regions of hyperbolic motion. That is, between the large scale eddies.

Jack Herring [1, p. 2265] has observed this effect in his study of two-dimensional, anisotropic turbulence, using a closure-based subgrid scale model. Our results confirm this study in the context of the qualitative properties of the equations of motion for the fluid.

We remark that the quantity, $\theta = (\text{strain})^2 - (\text{vorticity})^2$, is found to be, following an observation by Brezis and Bourgoiun [2], related to the second fundamental form of the boundary of the fluid domain, where the boundary is regarded as a smooth manifold, embedded in $\mathbb{R}^2$. This, in turn, implies a global integral constraint on the sign and magnitude of θ. In a sense to be explained below, flows exterior to a smooth, convex boundary are more hyperbolic, while interior flows are constrained toward elliptical modes. A similar behavior has been observed in numerical studies of area-preserving maps of the plane [3, 4]. Particularly interesting, in light of this result, are the remarks in ref. [3] outlining a possible connection between the theory of general area-preserving maps and hydrodynamical systems. This connection is based on the contrasting regions of elliptical and hyperbolic motion that appear, in terms of our results, to be essential to understanding the long time behavior of fluids.

In section 2 we derive from the equations of motion the equations that govern the spatial gradients of the vorticity. The geometrical implications of these equations are examined. In section 3 the results of the numerical simulation of the equations of motion are presented. In section 4 we present the summary and conclusions of the preceeding sections.

2. Gradients of vorticity

The time evolution of an inviscid, incompressible fluid is governed by Euler's equations for the velocity field, $\hat{v}$:

$$\hat{v}_t + \hat{v} \cdot \nabla \hat{v} = \nabla p, \tag{1}$$

where

$$\begin{aligned} &\hat{v} = (u, v)(\hat{x}, t), \\ &\nabla \cdot \hat{v} = 0, \\ &\hat{x} = (x, y) \in \mathrm{D} \in \mathbb{R}^2, \\ &\hat{v} \cdot \hat{m} = 0 \quad \text{for } \hat{x} \in \partial\mathrm{D}, \\ &\hat{m} \perp \partial\mathrm{D}, \\ &t = 0; \quad \hat{v} = \hat{v}_0(\hat{x}). \end{aligned}$$

It is known [5] that there exist smooth, global solutions to (1) where $\hat{v}_0$ and D are smooth. Furthermore, these solutions conserve the total energy:

$$E = \int_{\mathrm{D}} \tfrac{1}{2}(\hat{v} \cdot \hat{v}) = \int_{\mathrm{D}} \tfrac{1}{2}(\hat{v}_0 \cdot \hat{v}_0) \tag{2}$$

and the total enstrophy (squared vorticity):

$$\Omega = \int_{\mathrm{D}} \tfrac{1}{2}(\operatorname{curl} \hat{v} \cdot \operatorname{curl} \hat{v}) = \int_{\mathrm{D}} \tfrac{1}{2}(\operatorname{curl} \hat{v}_0 \cdot \operatorname{curl} \hat{v}_0). \tag{3}$$

It has been conjectured [6] that the existence of these dual invariants implies a transfer of energy toward the large scales of motion across a $k^{-5/3}$ spectral range; and corresponding transfer of enstrophy toward the small scales across a k^{-3} energy spectrum. This conjecture is supported by numerous closure [7, 8] and simulation [9, 10] results. The transfer of energy to the large scales is thought to explain the tendency of numerical simulations to evolve from random velocity fields toward states consisting of a few large-scale regions of like-signed vorticity [10]. On the other hand, the transfer of vorticity (enstrophy) to the small scales of motion is less well understood and the subject of some controversy [11, 12]. Since numerical simulations are limited by finite degree truncations and the use of eddy viscosities that distort the inviscid behavior at small scales, the process of enstrophy transfer has remained somewhat obscure. In this section we present several observations that may help to shed light on the enstrophy transfer process and aid in the interpretation of the numerical simulations.

To begin we note that following a fluid element, the vorticity is conserved. That is, if

$$C = v_x - u_y \tag{4}$$

then

$$C_t + \hat{v} \cdot \nabla C = \frac{\mathrm{d}C}{\mathrm{d}t} = 0. \tag{5}$$

Thus

$$C(\hat{x}(x_0,t),t) = C_0(\hat{x}_0). \tag{6}$$

The vorticity is advected by the velocity field as a passive scalar. However, unlike a passive scalar, the vorticity is dynamically related to (determines) the velocity field, and changes in the distribution of the vorticity imply changes in the advecting velocity field. This intrinsic feedback process can be better understood by introducing the stress tensor, A:

$$A = \nabla \hat{v}, \quad A_{ij} = \frac{\partial v_i}{\partial x_j}. \tag{7}$$

If it is assumed that the fluid is contained in T^2 or $\mathbb{R}^2$ then it can be readily shown from (1) that

$$\frac{\mathrm{d}A}{\mathrm{d}t} + A^2 = \nabla\nabla\Delta^{-1}\operatorname{tr} A^2, \tag{8}$$

where $\mathrm{d}/\mathrm{d}t = \partial_t + \hat{v}\cdot\nabla$ and tr is the trace.

Since

$$\operatorname{tr} A = \nabla\cdot\hat{v} = 0,$$
$$A^2 = \tfrac{1}{2}\operatorname{tr} A^2 I, \tag{9}$$

the anti-symmetric part of (8) is eq. (5). The symmetric part of A is the rate of strain tensor:

$$B = \tfrac{1}{2}(A + A^t) = \begin{pmatrix} u_x & \tfrac{1}{2}(u_y + v_x) \\ \tfrac{1}{2}(v_x + u_y) & v_y \end{pmatrix}. \tag{10}$$

We have found that the above is most conveniently described in the following notation:

$$\theta = u_x - v_y, \quad \psi = v_x + u_y,$$
$$C = v_x - u_y, \quad \lambda = \theta + \mathrm{i}\psi,$$
$$D = \partial_x + \mathrm{i}\partial_y. \tag{11}$$

In this notation, eq. (8) becomes

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = D^2(D\bar{D})^{-1}\tfrac{1}{2}(\lambda\bar{\lambda} - C^2),$$
$$\frac{\mathrm{d}C}{\mathrm{d}t} = 0. \tag{12}$$

Furthermore, since C determines $\hat{v}$, and $\hat{v}$ determines λ, we find

$$\bar{D}\lambda = \Delta(u + \mathrm{i}v), \tag{13}$$
$$\bar{D}\lambda = \mathrm{i}DC. \tag{14}$$

We note that

$$\operatorname{tr} A^2 = \tfrac{1}{2}(\lambda\bar{\lambda} - C^2) \tag{15}$$

is precisely the magnitude of the rate of strain squared minus the rate of rotation squared. We believe this to be a key quantity in understanding two-dimensional hydrodynamics.

In terms of the stream function:

$$\psi = \psi(x,y,t),$$
$$u = \psi_y, \quad v = -\psi_x. \tag{16}$$

We find that

$$\operatorname{tr} A^2 = \tfrac{1}{2}(\lambda\bar{\lambda} - C^2) = \psi_{xy}^2 - \psi_{xx}\psi_{yy} \tag{17}$$

or $\operatorname{tr} A^2$ is equal to the negative of the Gaussian curvature of the stream function.

When $\operatorname{tr} A^2$ is positive the motion is *hyperbolic* in character and when negative, the motion is *elliptical*.

While the structure of eqs. (12) is by no means transparent, combining the Lagrangian derivative, $\mathrm{d}/\mathrm{d}t$, with the Eulerian gradient, D, (in a non-local manner), it does imply that the values of λ following a fluid particle change most rapidly in the regions where $D^2(\operatorname{tr} A^2)$ is large, subject to the smoothing $(D\bar{D})^{-1}$. In a sense the operator $D^2(D\bar{D})^{-1}$ introduces a shape factor in the distribution of $\operatorname{tr} A^2$ that affects the regions where $\mathrm{d}\lambda/\mathrm{d}t$ will tend to be large. We shall return to this later.

Eq. (14) indicates that, in wave space, λ and C have an identical power spectrum, differing purely in phase. By Fourier transforming eq. (14) we find that

$$\hat{\lambda}_{\hat{k}} = \mathrm{i}\frac{\hat{k}}{\hat{k}^*}\hat{C}_{\hat{k}}, \tag{18}$$

where

$$\hat{\lambda}_{\hat{k}} = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\hat{x}\, \mathrm{e}^{ik\cdot\hat{x}} \lambda(\hat{x}),$$

$$\hat{C}_{\hat{k}} = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\hat{x}\, \mathrm{e}^{ik\cdot\hat{x}} C(\hat{x}),$$

$$\hat{k} = k_x + \mathrm{i}k_y,$$

$$k = (k_x, k_y).$$

Letting

$$\hat{\lambda}_{\hat{k}} = |\hat{\lambda}_{\hat{k}}|\mathrm{e}^{\mathrm{i}\alpha},$$

$$\hat{C}_{\hat{k}} = |\hat{C}_{\hat{k}}|\mathrm{e}^{\mathrm{i}\beta},$$

$$\hat{k} = |\hat{k}|\mathrm{e}^{\mathrm{i}\psi},$$

we find

$$|\hat{\lambda}_{\hat{k}}| = |\hat{C}_{\hat{k}}|, \tag{19a}$$

$$\alpha = \pi/2 + 2\psi + \beta. \tag{19b}$$

Furthermore, eq. (14) indicates that the quantity DC (essentially the gradient of the vorticity) is of a certain interest, being simply related to the corresponding derivatives of λ.

By applying the operators, D and $\bar{D}$ to eq. (12) it is found, after some algebraic simplification, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{DC} = B\widehat{DC}, \tag{20}$$

where

$$\widehat{DC} = \begin{pmatrix} DC \\ \bar{D}C \end{pmatrix}$$

and

$$B = \begin{pmatrix} \frac{1}{2}\mathrm{i}C & -\frac{1}{2}\lambda \\ -\frac{1}{2}\bar{\lambda} & -\frac{1}{2}\mathrm{i}C \end{pmatrix}. \tag{21}$$

We remark that for any scalar quantity, w, satisfying the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} w = 0, \tag{22}$$

the quantity

$$\widehat{Dw} = \begin{pmatrix} Dw \\ \bar{D}w \end{pmatrix}$$

will satisfy the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{Dw} = B\widehat{Dw} \tag{23}$$

with B given by (21).

We recall, from eq. (14), that

$$\widehat{DC} = \begin{pmatrix} DC \\ \bar{D}C \end{pmatrix} = -\mathrm{i}\begin{pmatrix} \bar{D}\lambda \\ -D\bar{\lambda} \end{pmatrix}. \tag{24}$$

Thus eq. (20) describes the evolution of complex gradients of the vorticity, complex gradients of any passive scalar, and certain gradients of the strain.

It is immediate that

$$\operatorname{tr} B = 0, \tag{25a}$$

$$B^2 = \tfrac{1}{4}(\lambda\bar{\lambda} - C^2)I = \tfrac{1}{2}\operatorname{tr} A^2\, I. \tag{25b}$$

The eigenvalues of $B, w_{\pm}$, satisfy the equation

$$w^2 = \tfrac{1}{4}(\lambda\bar{\lambda} - C^2). \tag{26}$$

Thus, when

$$\lambda\bar{\lambda} < C^2, \quad w_{\pm} = \pm\tfrac{1}{2}\mathrm{i}\sqrt{C^2 - \lambda\bar{\lambda}}, \tag{27a}$$

$$\lambda\bar{\lambda} > C^2, \quad w_{\pm} = \pm\tfrac{1}{2}\sqrt{\lambda\bar{\lambda} - C^2}. \tag{27b}$$

Since following a particle, C is constant:

$$C = C_0, \tag{28}$$

$$\lambda\bar{\lambda} - C^2 = \lambda\bar{\lambda} - C_0^2. \tag{29}$$

Now, if λ is slowly changing (with respect to $\widehat{DC}$) along a particle path (see eq. (12)) eq. (27) implies for $\widehat{DC}$:

(i) oscillatory behavior when $\operatorname{tr} A^2 < 0$.

(ii) exponential growth when $\operatorname{tr} A^2 > 0$.

The assumption that λ is slowly varying with respect to $\widehat{DC}$ appears reasonable, especially since (24) indicates that the time scale of $\widehat{DC}$ is the same as the gradients of λ, $D\lambda$, and since we are interested primarily in the small scales of motion. However, eq. (12) indicates that variations of λ along a particle path are non-local and conditioned by the distribution of $\operatorname{tr} A^2$. In a sense eq. (12) converts the spatial *intermittency* determined by (27) into a temporal *intermittency* of λ. This, in turn, may modulate the evolution of $\widehat{DC}$ (the spatial *intermittency*).

It is known that the source term in eq. (12) will be largest in those regions where $\frac{1}{2}(\lambda\bar{\lambda} - C^2)$ is most rapidly varying in space. By the above we expect both λ and C to be most rapidly varying when $\lambda\bar{\lambda} - C^2 > 0$. If eq. (23) is differentiated, we find

$$\frac{d^2}{dt^2}\widehat{DC} = \begin{pmatrix} \frac{1}{4}(\lambda\bar{\lambda} - C_0^2) & -\frac{1}{2}\,d\lambda/dt \\ -\frac{1}{2}\,d\bar{\lambda}/dt & \frac{1}{4}(\lambda\bar{\lambda} - C_0^2) \end{pmatrix} \widehat{DC}, \tag{30}$$

where by eq. (12),

$$\frac{d\lambda}{dt} = D^2(D\bar{D})^{-1}\tfrac{1}{2}(\lambda\bar{\lambda} - C_0^2).$$

The local growth rate of $\widehat{DC}$ predicted by (30) is

$$\chi = \pm\left[\tfrac{1}{4}(\lambda\bar{\lambda} - C_0^2) \pm \frac{1}{2}\left(\frac{d\lambda}{dt}\,\frac{d\bar{\lambda}}{dt}\right)^{1/2}\right]^{1/2}. \tag{31}$$

From this it would appear that the variations of λ, as described by $d\lambda/dt$, along a particle path are of some importance. We remark that the operator D^2 in eq. (12) is hyperbolic in its spatial characteristics:

$$D^2 = \partial_x^2 - \partial_y^2 + 2i\,\partial_x\,\partial_y. \tag{32}$$

This would favor largest values of $d\lambda/dt$ in those regions where $\Delta^{-1}\frac{1}{2}(\lambda\bar{\lambda} - C^2)$ is distributed along hyperbolic contours.

West, Lindenberg and Seshardi [15] have demonstrated that when one assumes that λ is a stationary, Gaussian random function with zero mean, then

$$\frac{d}{dt}\begin{pmatrix} \langle DC \rangle \\ \langle \bar{D}C \rangle \end{pmatrix} = \begin{pmatrix} i(\frac{1}{2}C + \operatorname{Im}\Pi) + \operatorname{Re}\Pi & 0 \\ 0 & -i(\frac{1}{2}C + \operatorname{Im}\Pi) + \operatorname{Re}\Pi \end{pmatrix} \times \begin{pmatrix} \langle DC \rangle \\ \langle \bar{D}C \rangle \end{pmatrix} \tag{33}$$

where $\langle\ \ \rangle$ indicates a moment average and

$$\Pi(C_0) = \tfrac{1}{4}\int_{-\infty}^{\infty} \langle \lambda(t)\,\bar{\lambda}(\tau)\rangle\, e^{iC_0(t-\tau)}\,d\tau. \tag{34}$$

Here, the generaly complex quantity Π can be interpreted as the strength of the strain λ, fluctuations at the frequency C_0. In general, when $\operatorname{Re}\Pi > 0$, there occurs an oscillatory, exponential growth of $\langle DC \rangle$ and $\langle \bar{D}C \rangle$.

When the temporal correlations of λ are assumed to be *rapidly varying* with respect to $\langle DC \rangle$, then

$$\langle \lambda(t)\,\lambda(\tau)\rangle \simeq \alpha\delta(t-\tau), \tag{35}$$

where $\alpha > 0$.

Then, eq. (33) simplifies to

$$\frac{d}{dt}\begin{pmatrix} \langle DC \rangle \\ \langle \bar{D}C \rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(iC_0 + \alpha) & 0 \\ 0 & \frac{1}{2}(-iC_0 + \alpha) \end{pmatrix} \times \begin{pmatrix} \langle DC \rangle \\ \langle \bar{D}C \rangle \end{pmatrix}. \tag{36}$$

This predicts an oscillatory, exponential growth of the moments $\langle DC \rangle, \langle \bar{D}C \rangle$. The above does show that the effect of temporal variations of λ is to modulate the otherwise pure exponential growth of DC.

We conclude this section by examining a constraint that relates $\operatorname{tr} A^2$ to the shape of the boundary.

It is a consequence of incompressibility that

$$\operatorname{tr} A^2 = \operatorname{div}(\hat{v} \cdot \nabla \hat{v}). \tag{37}$$

Bourguignon and Brezis [2] have shown that, if the boundary of the fluid domain, ∂D, is defined by

$$\partial D = \{\hat{X} \in \mathbb{R}^n : b(\hat{X}) = 0\}$$

and $\hat{n} = -\nabla b$, the normal at ∂D, then

$$\int_D \operatorname{div}(\hat{v} \cdot \nabla \hat{v}) = \int_{\partial D} (\hat{v} \cdot \nabla \hat{v}) \cdot \hat{n} = \int_{\partial D} \sum_{ij} \frac{\partial^2 b(\hat{X})}{\partial x_i \partial x_j} v_i v_j. \tag{38}$$

The quadratic form

$$\beta(\hat{x}, \hat{v}, \hat{v}) = \sum_{ij} \frac{\partial^2 b}{\partial x_i \partial x_j} v_i v_j \tag{39}$$

is the second fundamental form of ∂D in $\mathbb{R}^n$. By (37),

$$\int_D \tfrac{1}{2}(\lambda\bar{\lambda} - C^2) = \int_{\partial D} \beta(\hat{x}, \hat{v}, \hat{v})\, \mathrm{d}\hat{x}. \tag{40}$$

Several consequences of (40) follow immediately. If $\partial D = \phi$, i.e. $D = \mathbb{R}^2$ or T^2, then

$$\int_D \lambda\bar{\lambda} = \int_D C^2. \tag{41}$$

If the tangent plane to ∂D is exterior to D at every point then $\beta(\hat{x}, \hat{v}, \hat{v})$ is negative definite:

$$-M^2 \hat{v} \cdot \hat{v} < \beta(\hat{x}, \hat{v}, \hat{v}) < -m^2 \hat{v} \cdot \hat{v}, \tag{42}$$

where $\hat{x} \in \partial D$ and M^2, m^2 depend on the curvature of ∂D.

As a consequence

$$-M^2 \int_{\partial D} \hat{v} \cdot \hat{v} < \int_D \tfrac{1}{2}(\lambda\bar{\lambda} - C^2) < -m^2 \int_{\partial D} \hat{v} \cdot \hat{v}. \tag{43}$$

Eq. (43) indicates a constraint toward *elliptical* motion for this type of domain.

If the tangent plane is everywhere contained in D, then $\beta(\hat{x}, \hat{v}, \hat{v})$ is positive definite and

$$m^2 \int_{\partial D} \hat{v} \cdot \hat{v} < \int_D \tfrac{1}{2}(\lambda\bar{\lambda} - C^2) < M^2 \int_{\partial D} \hat{v} \cdot \hat{v}. \tag{44}$$

Exterior domains of this type are constrained towards *hyperbolicity*. Conditions (43) and (44) are, in reality, constraints on $\lambda\bar{\lambda}$, since

$$\int_D C^2 = \int_D C_0^2. \tag{45}$$

The influence of the boundary on the qualitative properties of the motion appears to be quite natural.

3. Numerical simulation

We have studied the evolution of a random initial realization of the stream function. The Euler equations in the stream function–vorticity formulation were solved by the fully dealiased, spectral method of Orszag [13]. The program to implement this method was developed by Professor R. Salmon of Scripps Institute of Oceanography. For this particular simulation a cutoff wavenumber of 32 and an eddy viscosity term $-\sigma\Delta^2 C$, where $\sigma = 1 \times 10^{-6}$, were employed. Starting from an initial stream function with amplitude proportional to $k^2/(1+k^6)$ and a random phase, the method employs a leap-frog time differencing with smoothing every 20 time steps to eliminate the spurious computational mode. Since the energy is normalized to be one and a time step is 0.01, an eddy turnover time consists of nearly 100 time steps. Every 10 time steps the stream function was output onto disk. Thus in the figures a label *Record 17* indicates a time of 1.7.

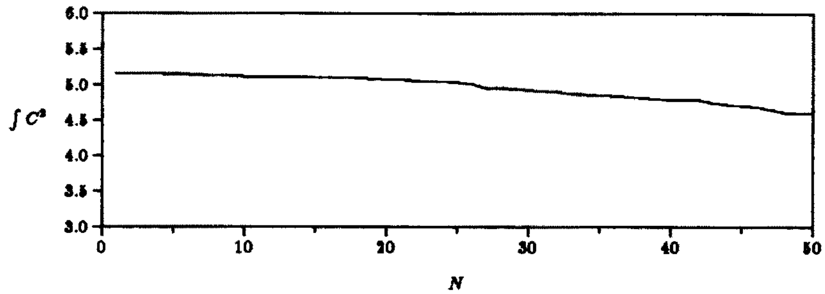

Fig. 1. Total enstrophy $\int_D \operatorname{curl} v \cdot \operatorname{curl} v$. Record number 1 to 50.

From the stream function we have produced contour plots for the vorticity, magnitude of strain, the quantity $\lambda\bar{\lambda} - C^2$, the magnitude of $\mathrm{d}\lambda/\mathrm{d}t$ and the magnitude of the gradients of vorticity. In addition, we have produced graphs of the spectra for the vorticity (= rate of strain) and the gradients of the vorticity.

Since it is necessary to use an eddy–viscosity term to prevent reflection of energy at the cutoff wavenumber, the total enstrophy is decaying with time. See fig. 1. Nevertheless, the qualitative features of the solution are quite interesting.

The random initial gradients of vorticity have evolved by frame 11 into a tightly localized pattern that is related to the central hyperbolic region. There are two hyperbolic regions. The stronger one located at the center of the frame and a weaker one located in upper left corner, with label on the left. These hyperbolic regions may be identified with reference to either the vorticity or $\operatorname{tr} A^2$ contours.

The strain appears to be confined almost solely to the hyperbolic region. The magnitude of $\mathrm{d}\lambda/\mathrm{d}t$ is highly intermittent; being confined to several small regions that appear to migrate from left to right during the evolution.

A curious event occurs in frames 26 through 37. In these frames the vorticity gradients streaming from the hyperbolic centers are observed to interact and intensify. This phenomenon is strikingly similar to the interaction between hyperbolic (unstable) fixed points observed for general area preserving maps [3]. This process is associated with the first peak in the value of the total vorticity gradients; the following decrease being associated with the increased dissipation caused by the transfer of enstrophy to the higher wavenumbers. The further evolution of the system appears to involve the wrapping of vorticity gradients about the hyperbolic centers, by folding and stretching of the fluid in these regions.

Further numerical studies of the phenomena discussed in this paper are presented by M. Brachet et al. in ref. [16].

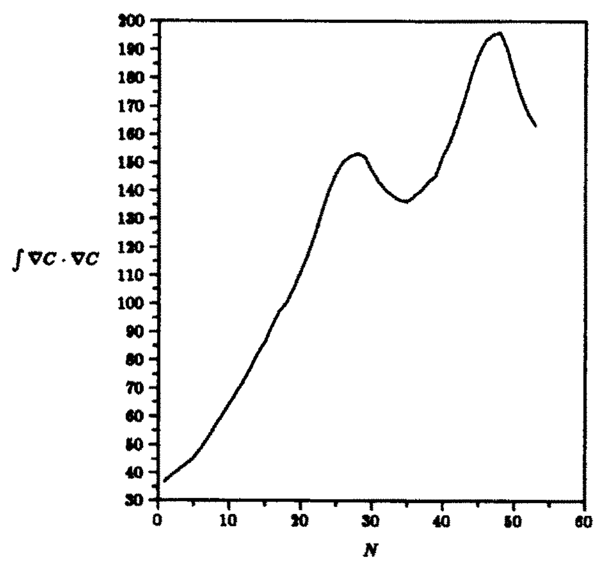

Fig. 2. Total magnitude of vorticity gradients $\int_D \nabla C \cdot \nabla C$. Record number 2 to 54.

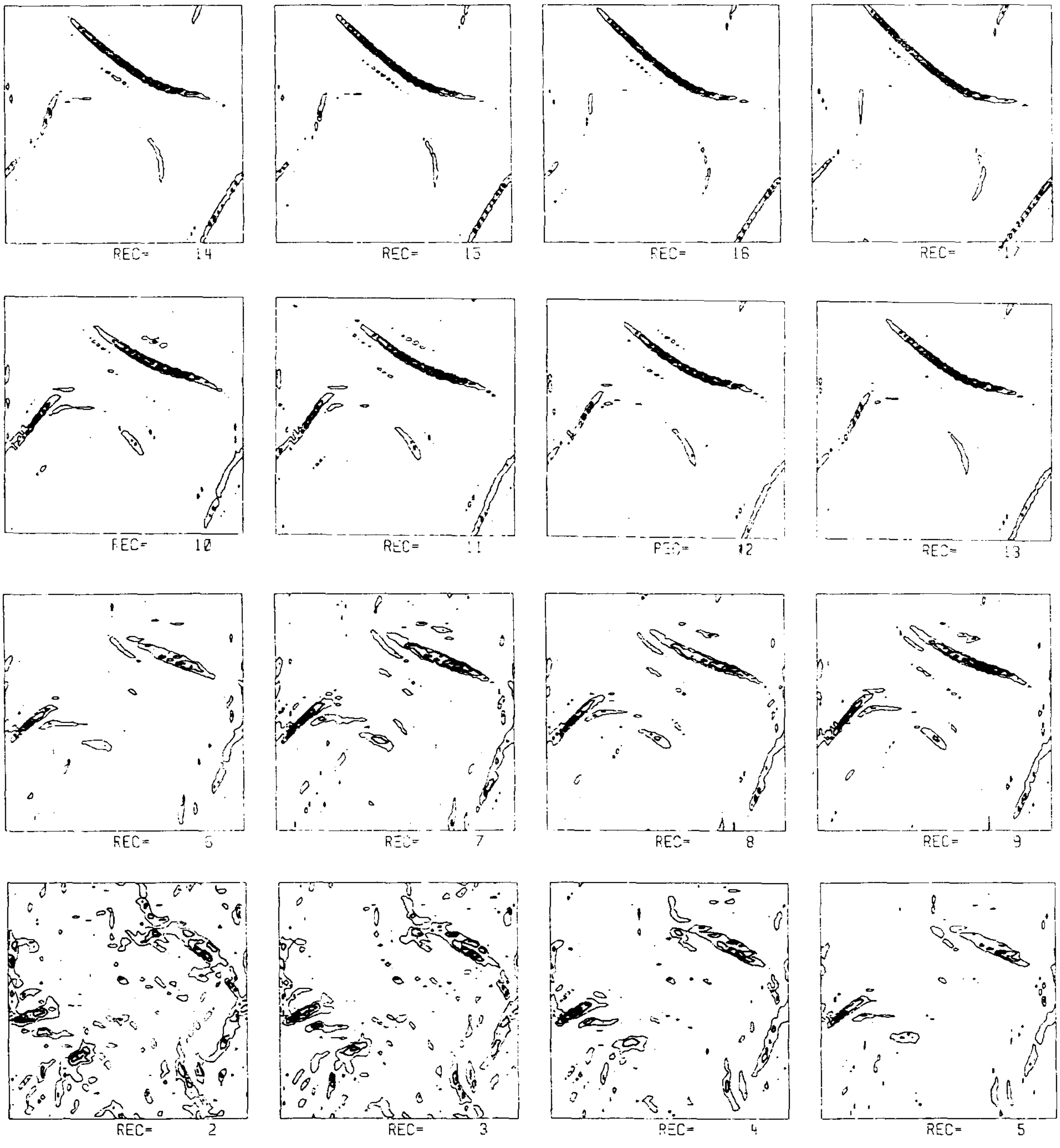

Fig. 3. Contour plots of the magnitude of the vorticity gradients, $\nabla C \cdot \nabla C$. Frames 2 through 53.

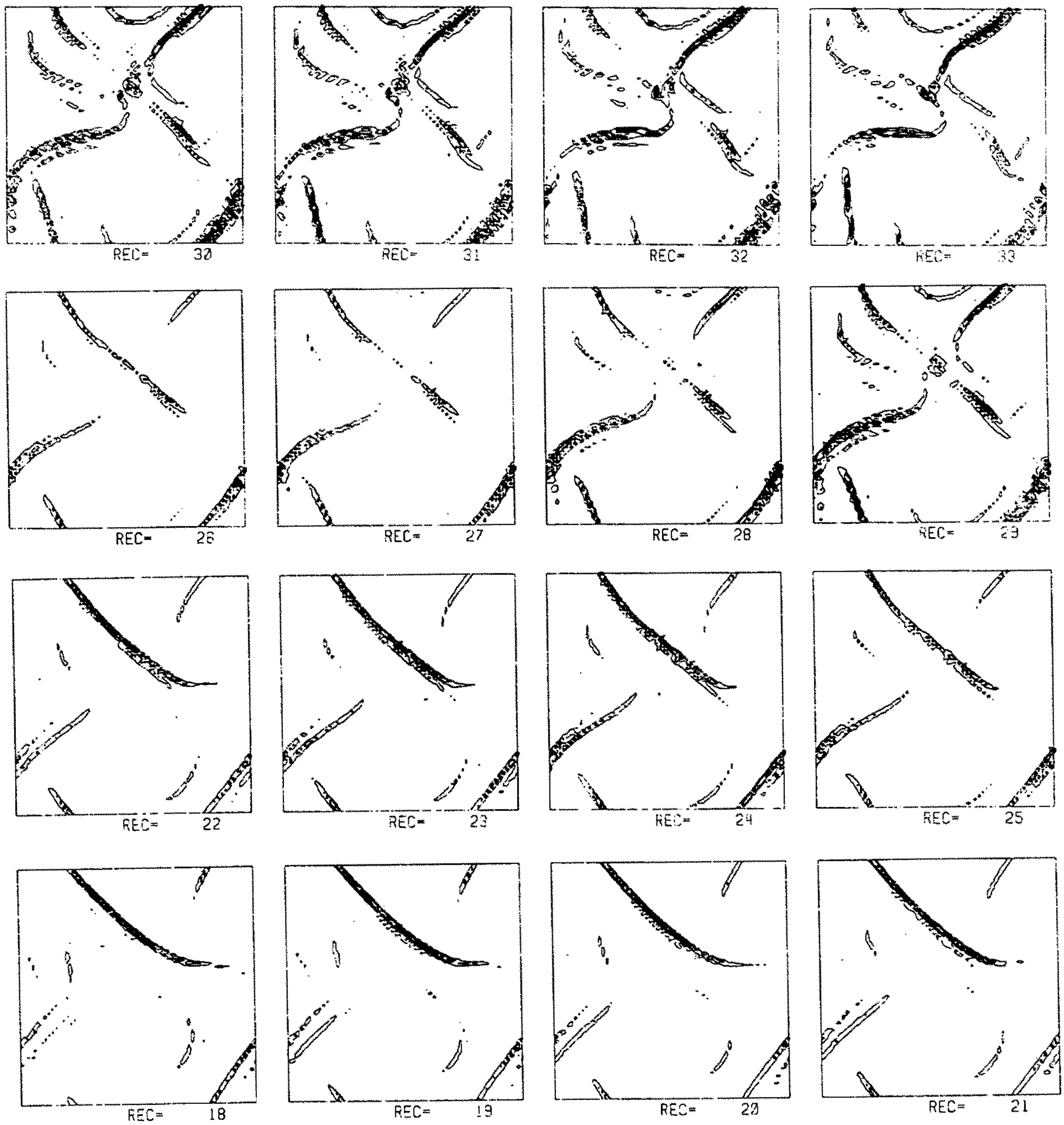

Fig. 3. Continued

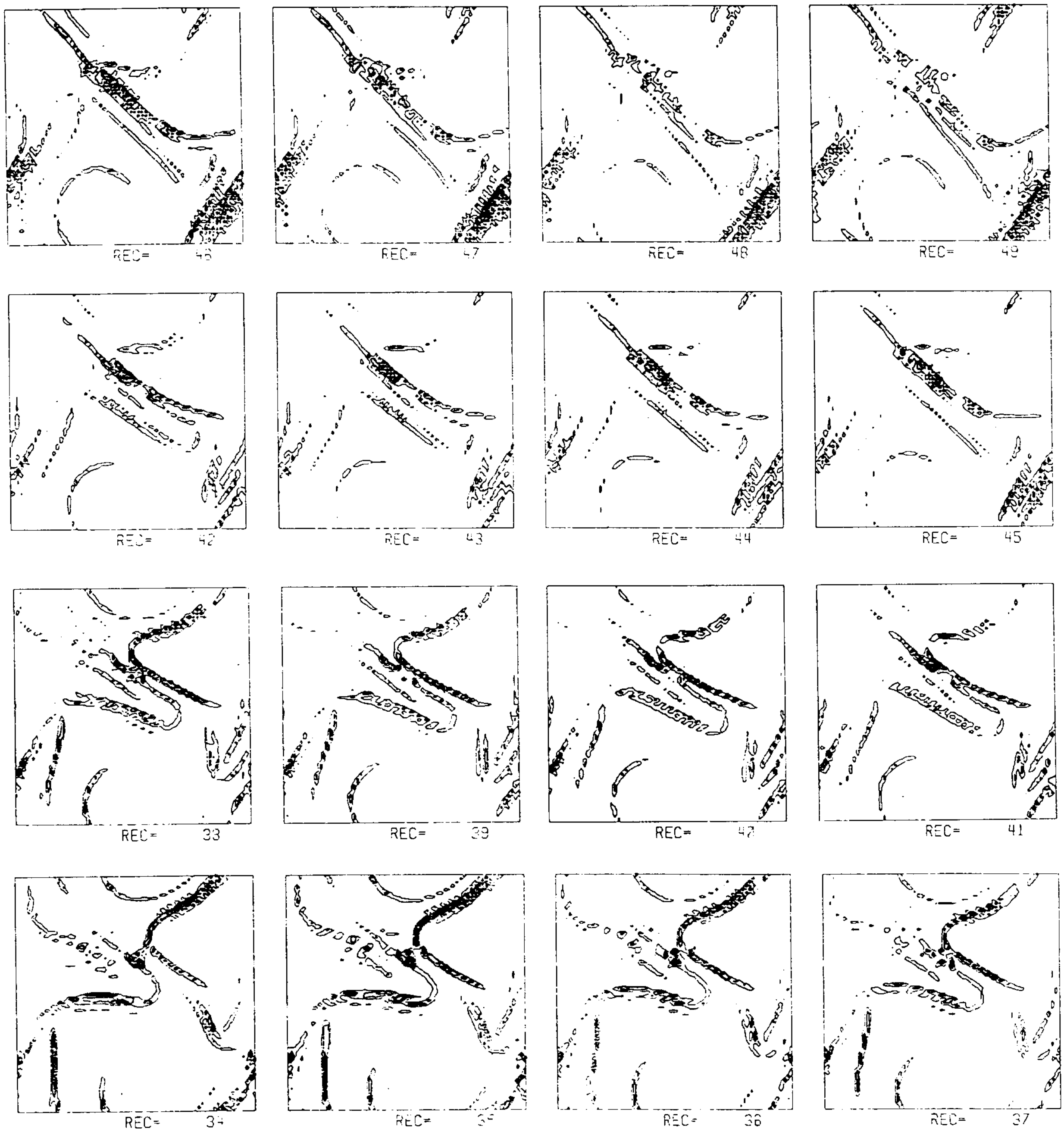

Fig. 3. Continued

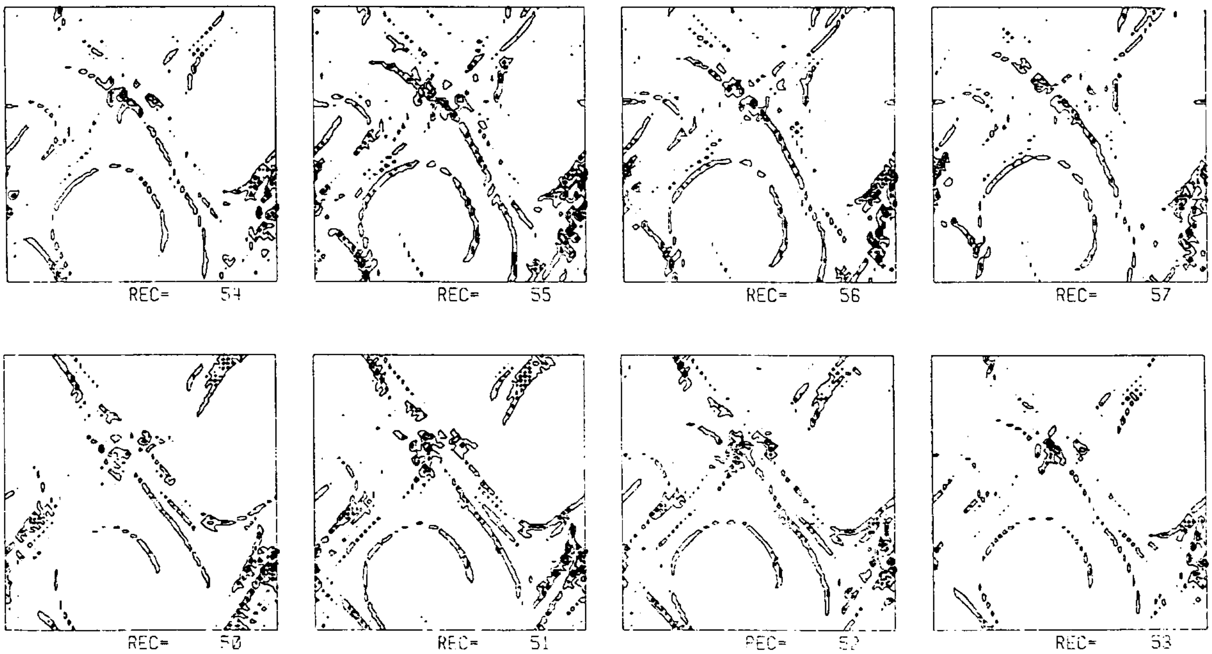

Fig. 3. Continued

4. Summary and conclusions

In section 2 we have presented several equations and relationships that may be useful in understanding the enstrophy transfer process. It is indicated that the transfer of vorticity to small scales occurs in the regions of *hyperbolic* motion; and that the transfer, subject to certain assumptions on the behavior of the strain fluctuations, proceeds exponentially fast. An identity connecting the *shape* of the boundary to the qualitative properties of the flow is established.

The numerical solution of Euler's equations described in section 3 supports, in general, the hypothesis that enstrophy transfer is associated with the stretching and folding of fluid in the hyperbolic regions.

A major unresolved point is the validity of the assumption that the strain is slowly varying along a particle path with respect to the vorticity gradients. Inspection of the numerical solution reveals that the Lagrangian derivative of the strain is highly intermittent in space, and is possibly associated with the turning of the vorticity gradients about the central hyperbolic region. In regard to this, the folding of fluid about the central hyperbolic region that occurs in frames 26 to 37 can be seen, from the spectra for the vorticity gradients, to be the cause of the intensification of the vorticity gradients at the lower wavenumbers. The *bump* in the spectra is subsequently translated to the higher wavenumbers by the stretching process. This sequence of events is somewhat reminiscent of the Smale horseshoe map.

We recall from eq. (12) that

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = D^2(D\bar{D})^{-1}\tfrac{1}{2}(\lambda\bar{\lambda} - C^2). \tag{46}$$

The *shape* operator, $D^2(D\bar{D})^{-1}$, relates $\mathrm{d}\lambda/\mathrm{d}t$ to $\frac{1}{2}(\lambda\bar{\lambda} - C^2)$ in the same manner that λ is related to C in eq. (14). That is,

$$\bar{D}\frac{\mathrm{d}\lambda}{\mathrm{d}t} = D\theta, \quad \theta = \tfrac{1}{2}(\lambda\bar{\lambda} - C^2). \tag{47}$$

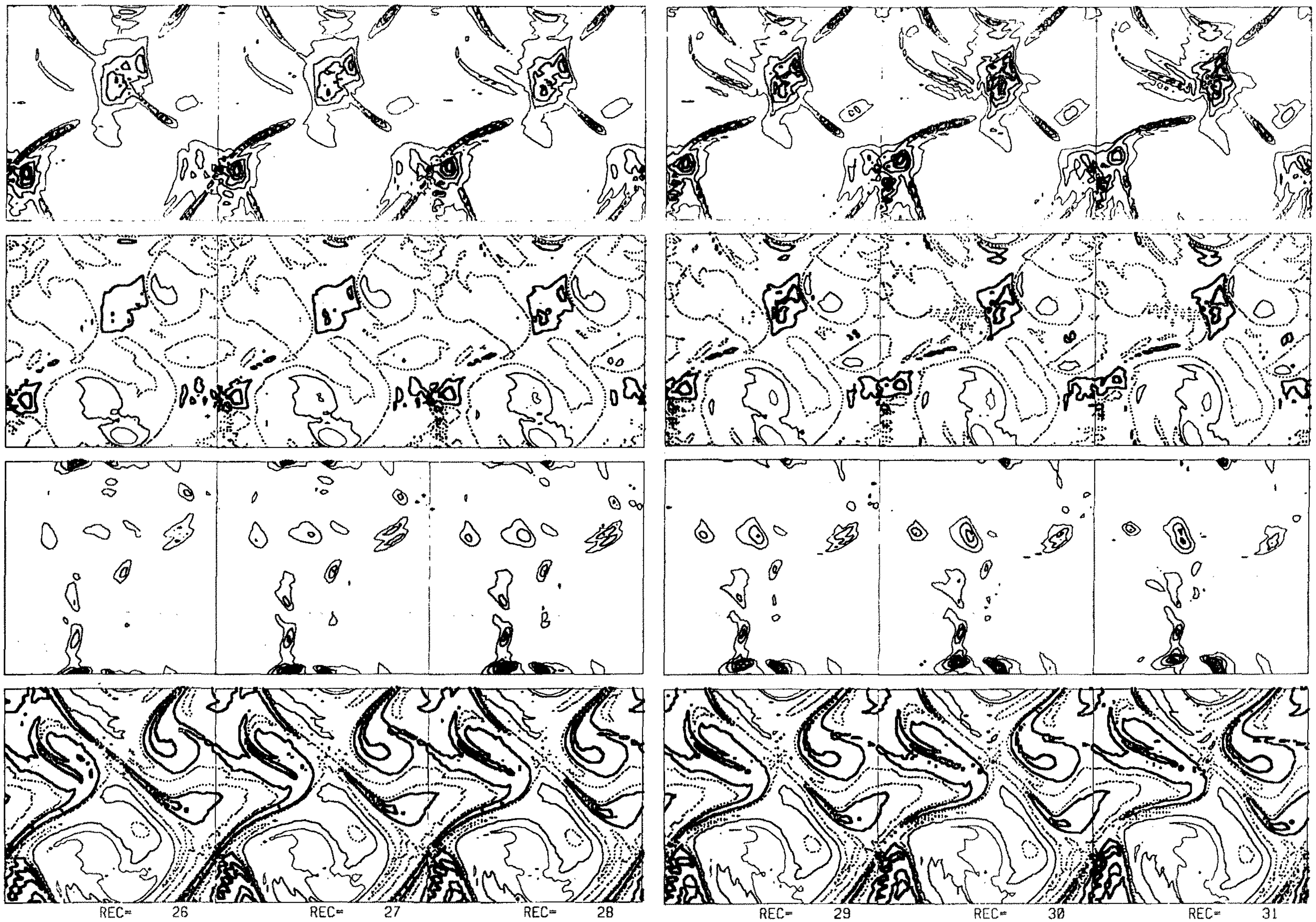

Fig. 4. Contour plots of (from top to bottom of page): (a) magnitude of strain, $\lambda\bar{\lambda}$; (b) trace of $A^2, \lambda\bar{\lambda} - C^2$; (c) magnitude of the Lagrangian derivative of strain, $(\mathrm{d}\lambda/\mathrm{d}t)(\mathrm{d}\bar{\lambda}/\mathrm{d}t)$; (d) vorticity, C.

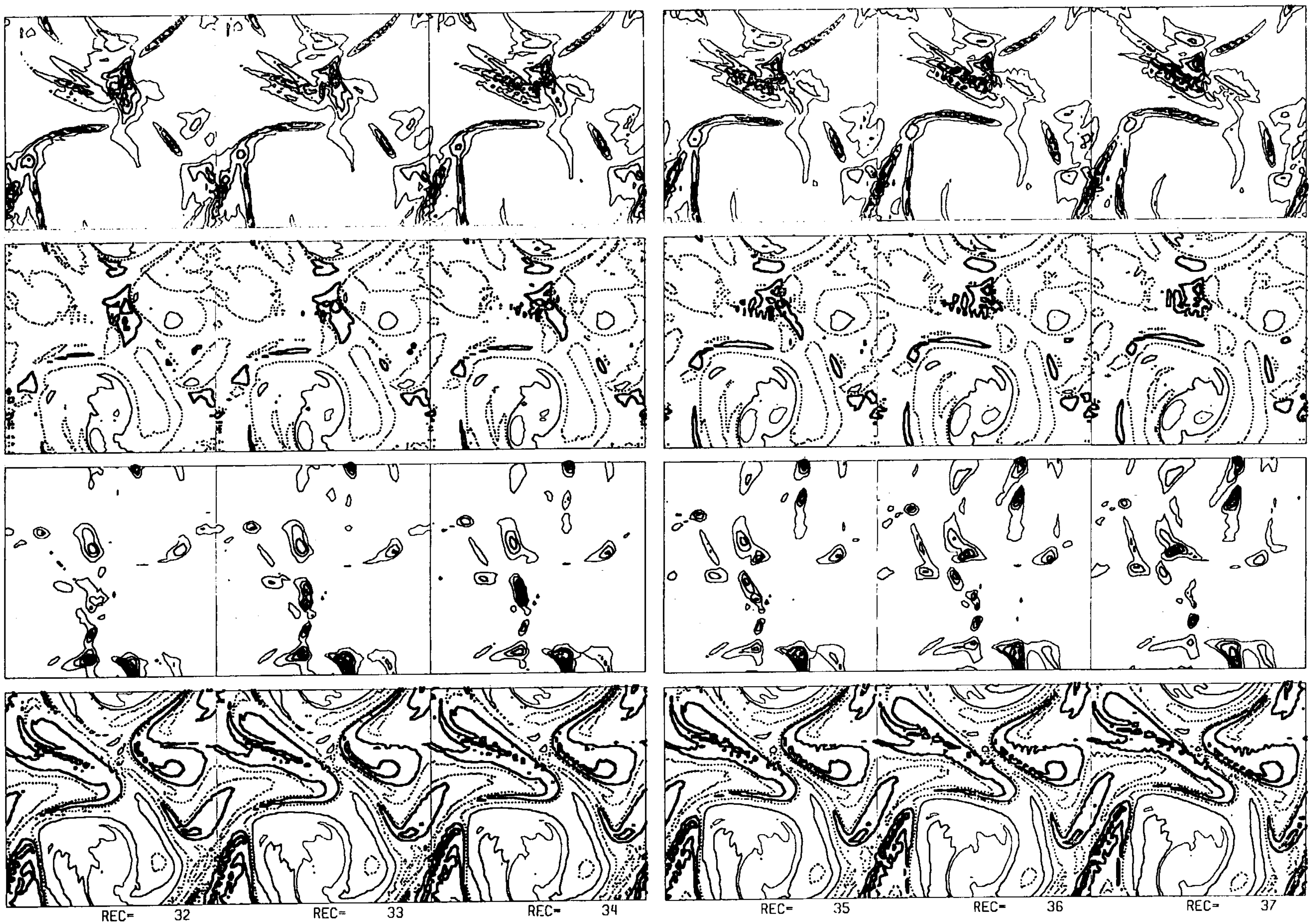

Fig. 4. Continued

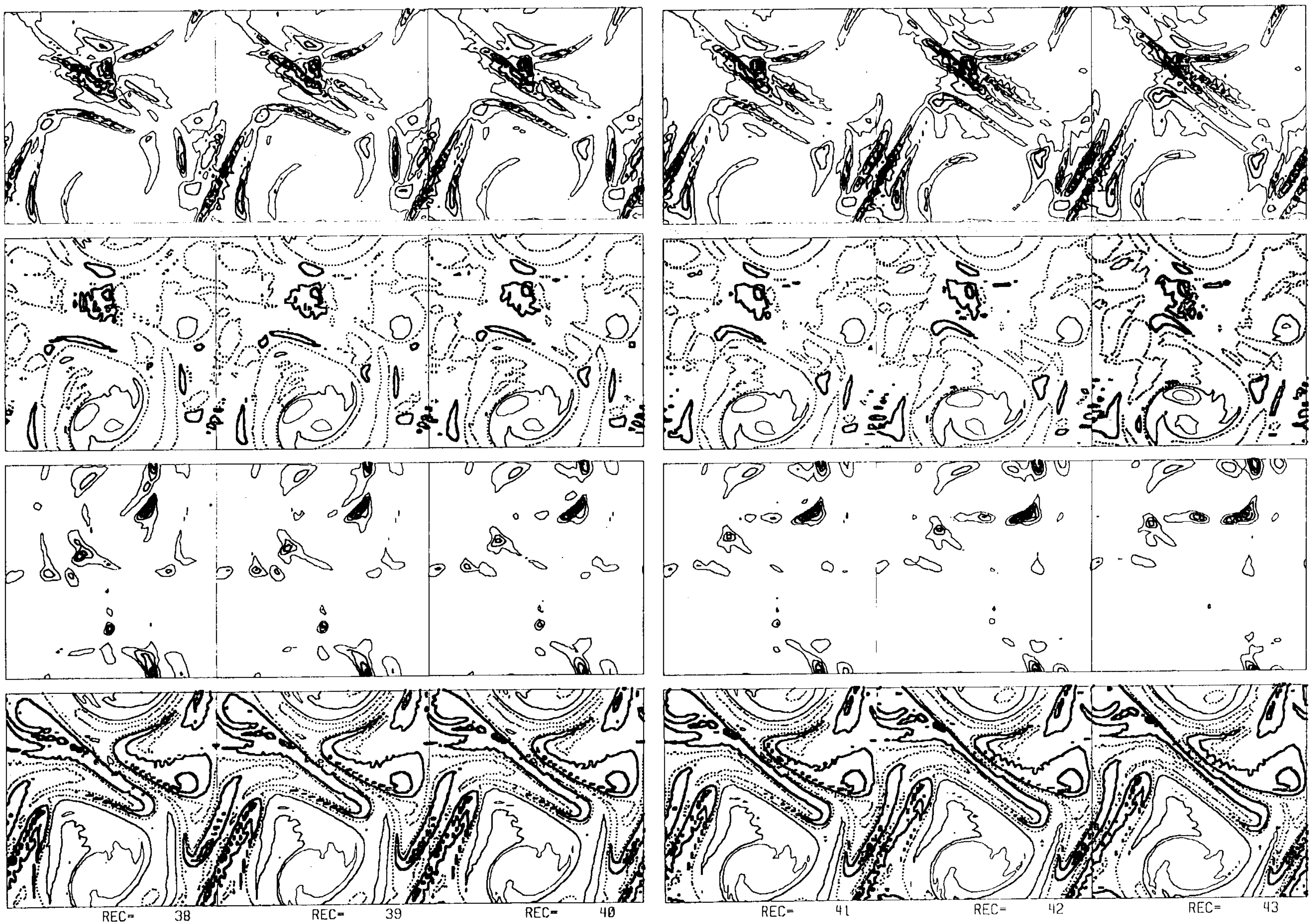

Fig. 4. Continued

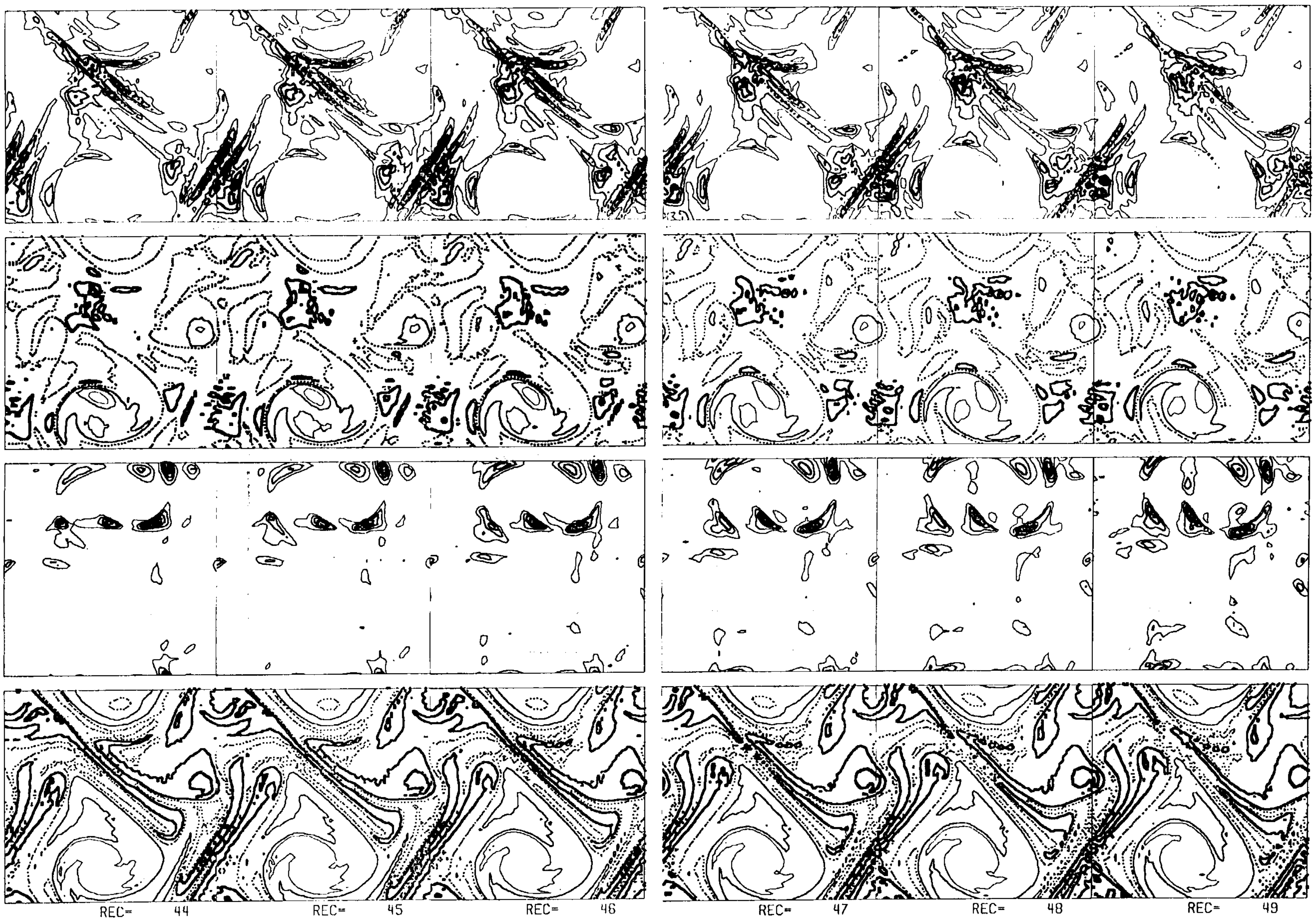

Fig. 4. Continued

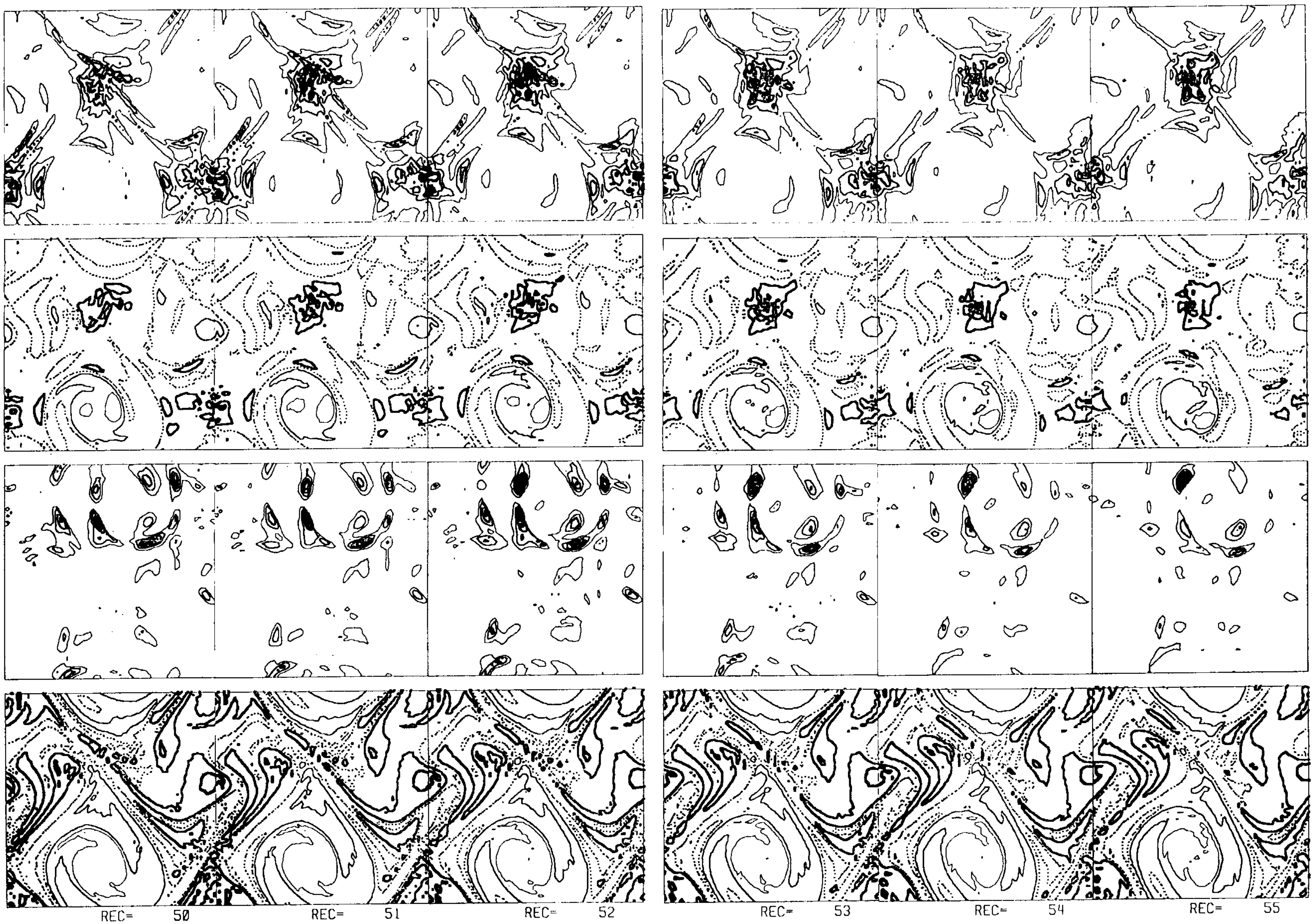

Fig. 4. Continued

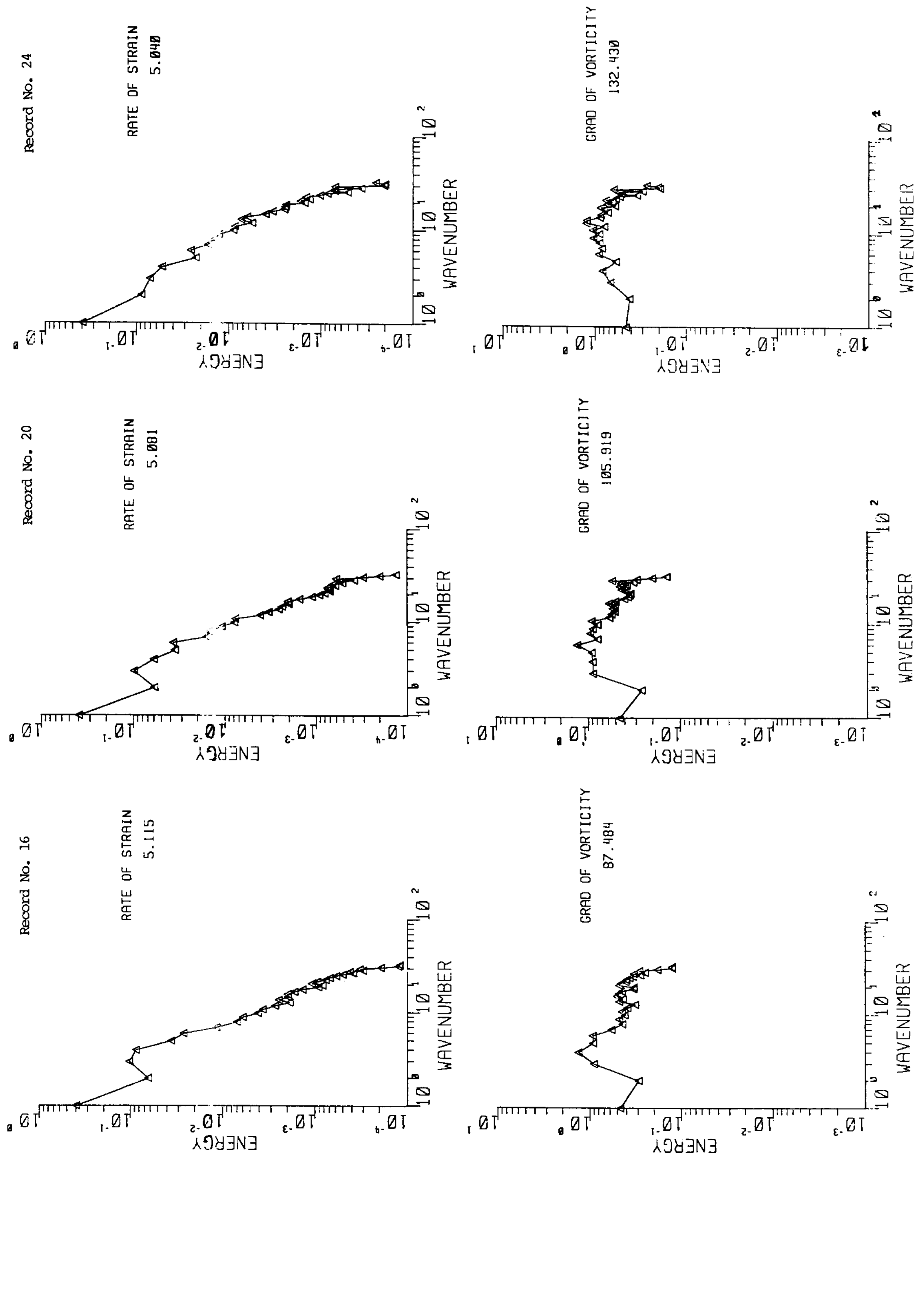

Fig. 5. Spectra of the rate of strain and the gradients of vorticity. Note that the spectra of vorticity is identical to that of the rate of strain.

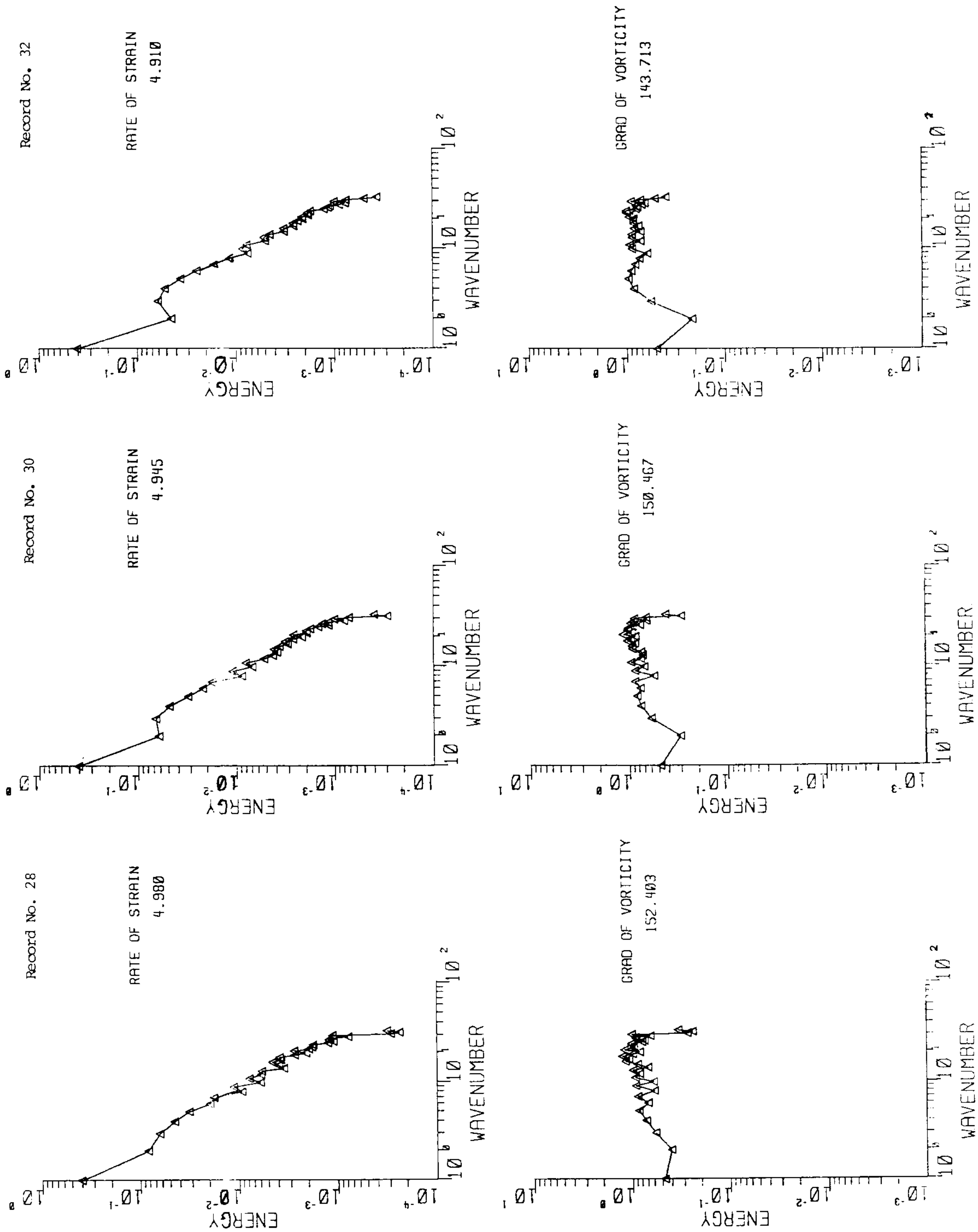

Fig. 5. Continued

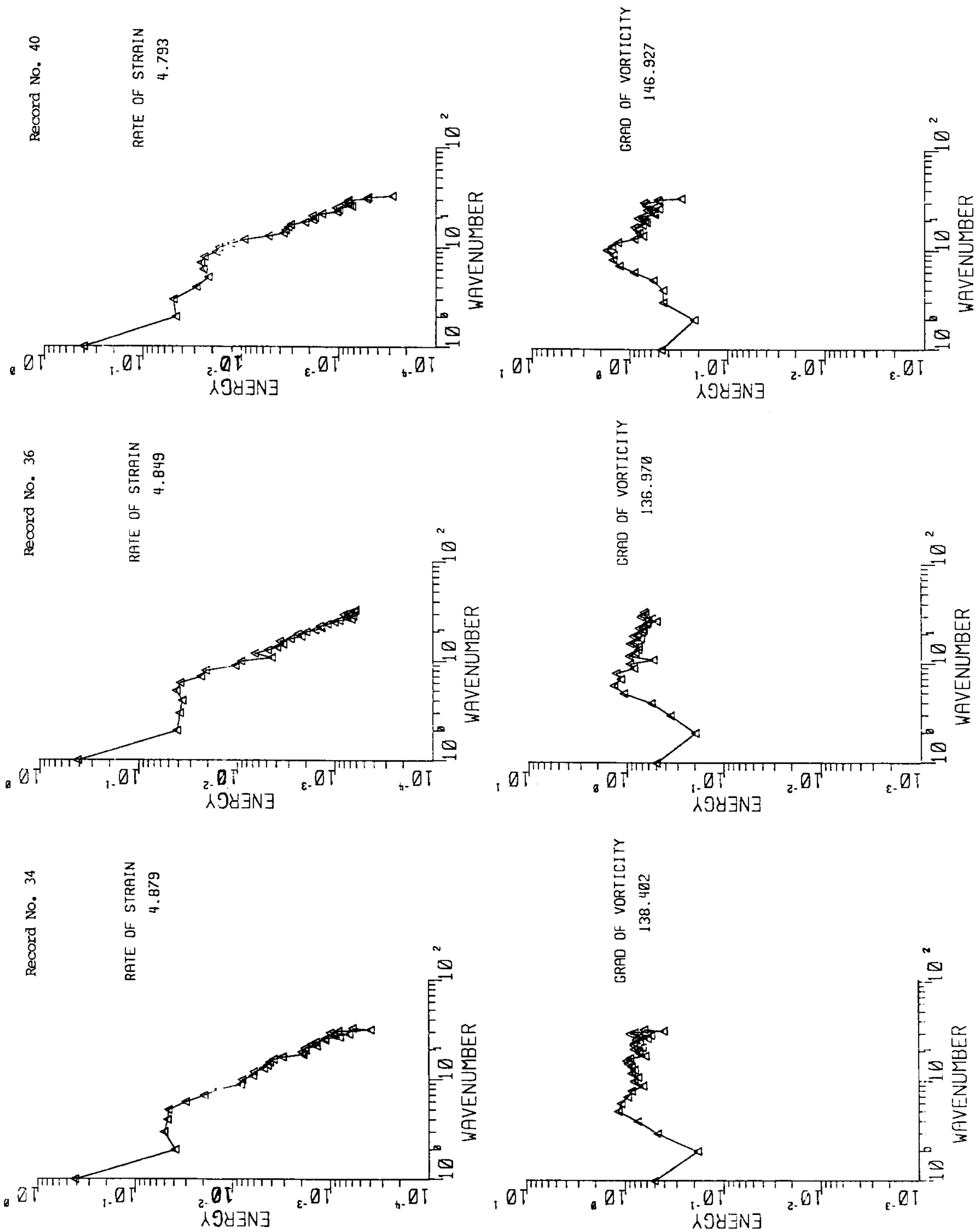

Fig. 5. Continued

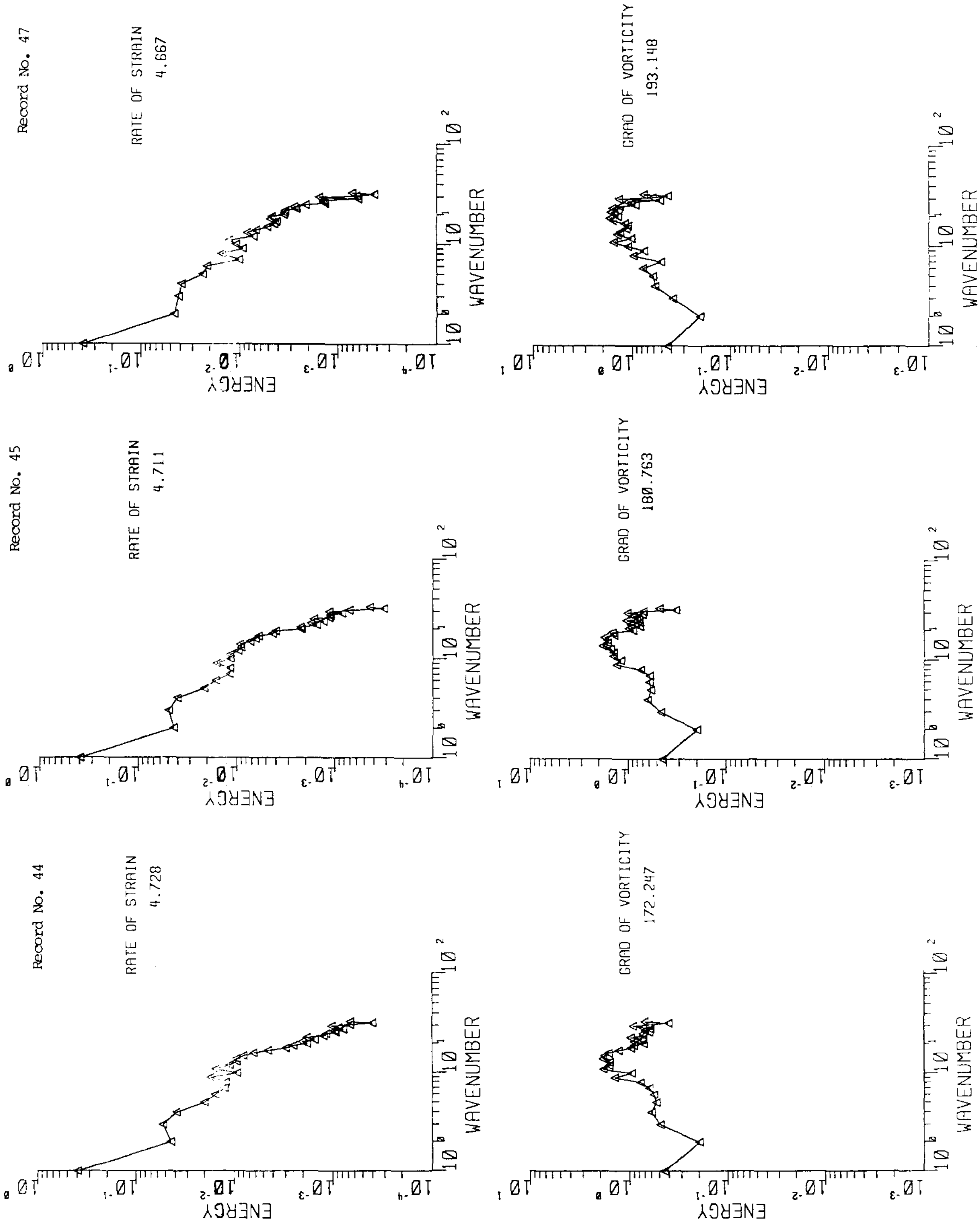

Fig. 5. Continued

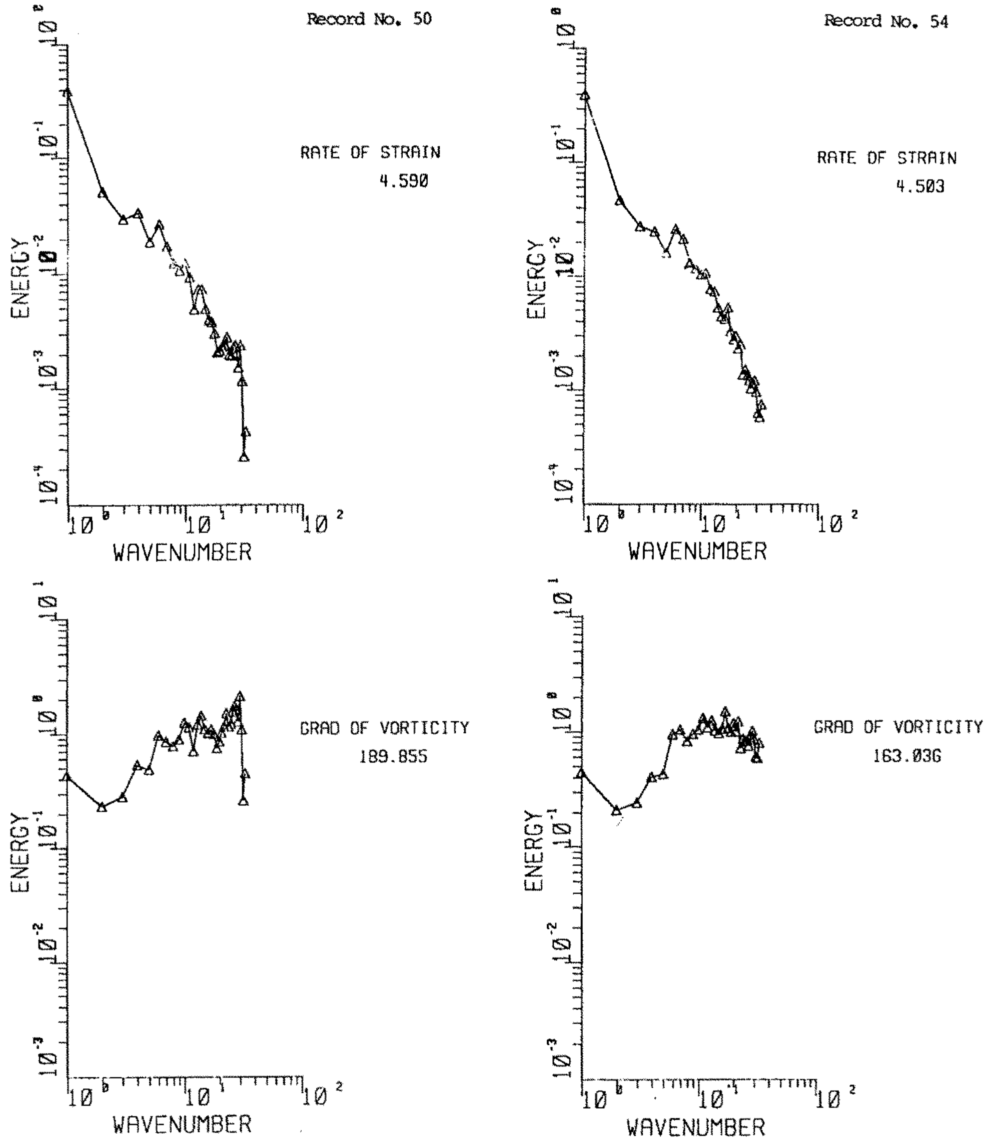

Fig. 5. Continued

At several points in this paper we have remarked that there appear to be similarities between two-dimensional hydrodynamics and the general theory of area-preserving maps. It is useful to remember that hydrodynamics does define an area-preserving map:

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = \hat{v}, \tag{48}$$

where

$$\hat{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \hat{v} = \begin{pmatrix} u \\ v \end{pmatrix}$$

and, when $t = 0$, $\hat{x} = \hat{x}_0$.

The Jacobian J of the map $\hat{x}_0 \to \hat{x}$ is

$$J = \begin{pmatrix} x_{x_0} & y_{x_0} \\ x_{y_0} & y_{y_0} \end{pmatrix}. \tag{49}$$

If

$$A = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \tag{50}$$

and

$$\nabla C = \begin{pmatrix} C_x \\ C_y \end{pmatrix}$$

then

$$\frac{\mathrm{d}J}{\mathrm{d}t} = JA \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} \nabla C = -A \nabla C. \tag{51}$$

From eqs. (51) it follows immediately that

$$\frac{\mathrm{d}}{\mathrm{d}t}(J \nabla C) = 0, \tag{52}$$

and using $J = I$ at $t = 0$

$$J \nabla C = \nabla C_0; \tag{53}$$

or

$$\nabla C = J^{-1} \nabla C_0. \tag{54}$$

Thus, the inverse of the Jacobian is the evolution operator for the gradients of the vorticity. This being the case, the evolution of the vorticity gradients directly reflects the properties of the underlying area-preserving map.

Acknowledgements

I want to thank J. Greene, V. Seshardri and M. Tabor of La Jolla Institute, and R. Salmon of Scripps Institute of Oceanography, for several helpful discussions during the course of this work. I also want to thank Nurit Weiss for help in preparing the numerical studies for this paper.

References

[1] J. Herring, J. Atmos. Sci. 32 (1975) 2254–2271.
[2] J. Bourguigonon and H. Brezis, J. Funct. Anal. 15 (1974) 341–363.
[3] M. Berry and M. Tabor, Ann. Phys. 122 (1979) 26–63.
[4] J.M. Green, J. Math. Phys. 9 (1968) 760–768.
[5] T. Kato, Arch. Ratl. Mech. Anal. 25 (1967) 188.
[6] R. Kraichnan, Phys. Fluids 10 (1967) 1417.
[7] U. Frisch, M. Lesieur and P.L. Sulem, Phys. Rev. Lett. 37 (1967) 895.
[8] A. Pouquet, M. Lesieur, J.C. Andre and C. Basdevant, J. Fluid Mech. 72 (1975) 305.
[9] D. Lilly, J. Fluid Mech. 45 (1971) 395.
[10] B. Fornberg, J. Comp. Phys. 25 (1977).
[11] P. Saffman, Stud. Appl. Math. 50 (1971) 377.
[12] R. Kraichnan, J. Fluid Mech. 67 (1975) 155.
[13] S. Orszag and M. Israeli, Ann. Rev. Fluid Mech. 6 (1974) 281.
[14] R. Fox, J. Math. Phys. 17 (1976) 1148.
[15] B. West, K. Lindenberg and V. Seshardri, Physica A 102 (1980) 470–488.
[16] M. Brachet, M. Meneguzzi, H. Politano and P.L. Sulem, J. Fluid Mech. 194 (1988) 333–349.