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Chapter 4

Shock Waves

Here we shall follow closely the pellucid discussion in chapter 2 of the book by G. Whitham, beginning with the simplest possible PDE, describing *chiral* (unidirectional motion) waves,

$$\rho_t + c_0\,\rho_x = 0\ . \tag{4.1}$$

The solution to this equation is an arbitrary right-moving wave (assuming $c_0 > 0$), with profile

$$\rho(x,t) = f(x - c_0 t)\ , \tag{4.2}$$

where the initial conditions on Eqn. 4.1 are $\rho(x, t = 0) = f(x)$. This is even simpler than the Helmholtz equation, $\rho_{tt} = c_0^2\,\rho_{xx}$, which is nonchiral, *i.e.* waves move in either direction. Anyway, nothing to see here, so move along.

4.1 Nonlinear Chiral Wave Equation

The simplest nonlinear PDE is a generalization of Eqn. 4.1,

$$\rho_t + c(\rho)\,\rho_x = 0\ . \tag{4.3}$$

This equation arises in a number of contexts. One example comes from the theory of vehicular traffic flow along a single lane roadway. Starting from the continuity equation,

$$\rho_t + j_x = 0\ , \tag{4.4}$$

one posits a constitutive relation $j = J(\rho)$, in which case $c(\rho) = J'(\rho)$. If the individual vehicles move with a velocity $v = v(\rho)$, then

$$J(\rho) = \rho\, v(\rho) \quad \Rightarrow \quad c(\rho) = v(\rho) + \rho\, v'(\rho)\ . \tag{4.5}$$

It is natural to assume a form $v(\rho) = c_0\,(1 - a\rho)$, so that at low densities one has $v \approx c_0$, with $v(\rho)$ decreasing monotonically to $v = 0$ at a critical density $\rho = a^{-1}$, presumably corresponding to bumper-to-bumper traffic. The current $J(\rho)$ then takes the form of an inverted parabola. Note the difference between the individual vehicle velocity $v(\rho)$ and what turns out to be the group velocity of a traffic wave, $c(\rho)$. For $v(\rho) = c_0\,(1 - a\rho)$, one has $c(\rho) = c_0\,(1 - 2a\rho)$, which is *negative* for $\rho \in \big[\frac{1}{2}a^{-1}, a^{-1}\big]$. For vehicular traffic, we have $c'(\rho) = J''(\rho) < 0$ but in general $J(\rho)$ and thus $c(\rho)$ can be taken to be arbitrary.

Another example comes from the study of chromatography, which refers to the spatial separation of components in a mixture which is forced to flow through an immobile absorbing 'bed'. Let $\rho(x,t)$ denote the density of the desired component in the fluid phase and $n(x,t)$ be its density in the solid phase. Then continuity requires

$$n_t + \rho_t + V\rho_x = 0\ , \tag{4.6}$$

where V is the velocity of the flow, which is assumed constant. The net rate at which the component is deposited from the fluid onto the solid is given by an equation of the form

$$n_t = F(n,\rho)\ . \tag{4.7}$$

In equilibrium, we then have $F(n,\rho) = 0$, which may in principle be inverted to yield $n = n_{\rm eq}(\rho)$. If we assume that the local deposition processes run to equilibrium on fast time scales, then we may substitute $n(x,t) \approx n_{\rm eq}\big(\rho(x,t)\big)$ into Eqn. 4.6 and obtain

$$\rho_t + c(\rho)\,\rho_x = 0 \quad , \quad c(\rho) = \frac{V}{1 + n'_{\rm eq}(\rho)}\ . \tag{4.8}$$

We solve Eqn. 4.3 using the *method of characteristics*. Suppose we have the solution $\rho = \rho(x,t)$. Consider then the family of curves obeying the ODE

$$\frac{dx}{dt} = c\big(\rho(x,t)\big)\ . \tag{4.9}$$

This is a family of curves, rather than a single curve, because it is parameterized by the initial condition $x(0) \equiv \zeta$. Now along any one of these curves we must have

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x}\,\frac{dx}{dt} = \frac{\partial\rho}{\partial t} + c(\rho)\,\frac{\partial\rho}{\partial x} = 0\ . \tag{4.10}$$

Thus, $\rho(x,t)$ is a constant along each of these curves, which are called *characteristics*. For Eqn. 4.3, the family of characteristics is a set of straight lines[1],

$$x_\zeta(t) = \zeta + c(\rho)\,t\ . \tag{4.11}$$

The initial conditions for the function $\rho(x,t)$ are

$$\rho(x=\zeta, t=0) = f(\zeta)\ , \tag{4.12}$$

where $f(\zeta)$ is arbitrary. Thus, in the (x,t) plane, if the characteristic curve $x(t)$ intersects the line $t=0$ at $x=\zeta$, then its slope is constant and equal to $c\big(f(\zeta)\big)$. We then define

$$g(\zeta) \equiv c\big(f(\zeta)\big)\ . \tag{4.13}$$

[1]The existence of straight line characteristics is a special feature of the equation $\rho_t + c(\rho)\,\rho_x = 0$. For more general hyperbolic first order PDEs to which the method of characteristics may be applied, the characteristics are curves. See the discussion in the Appendix.

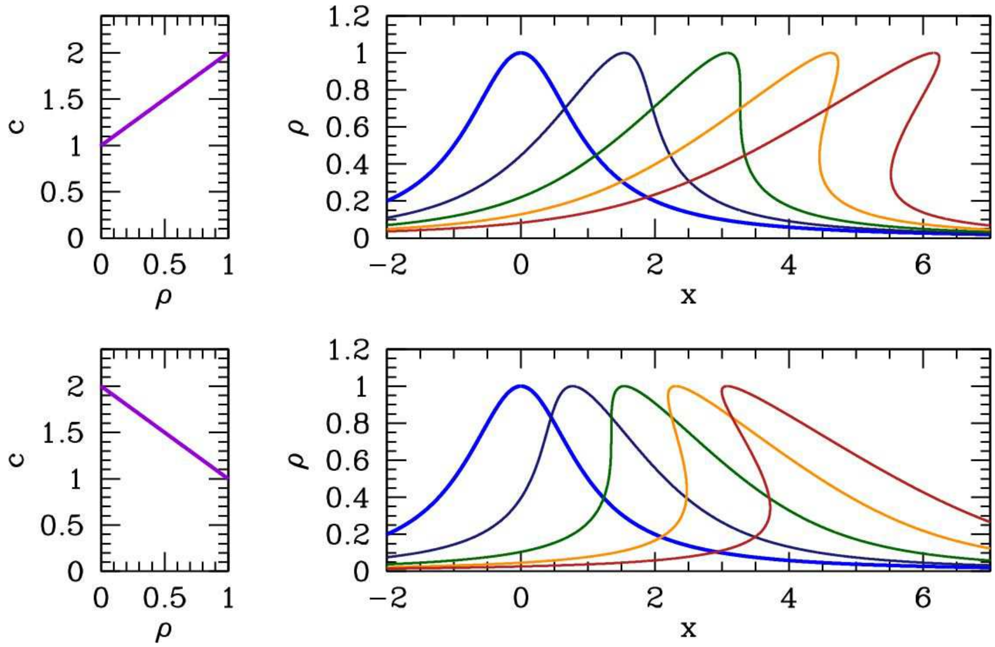

Figure 4.1: Forward and backward breaking waves for the nonlinear chiral wave equation $\rho_t + c(\rho)\,\rho_x = 0$, with $c(\rho) = 1 + \rho$ (top panels) and $c(\rho) = 2 - \rho$ (bottom panels). The initial conditions are $\rho(x, t = 0) = 1/(1+x^2)$, corresponding to a break time of $t_{\rm B} = \frac{16}{3\sqrt{3}}$. Successive $\rho(x,t)$ curves are plotted for $t = 0$ (thick blue), $t = \frac{1}{2}t_{\rm B}$ (dark blue), $t = t_{\rm B}$ (dark green), $t = \frac{3}{2}t_{\rm B}$ (orange), and $t = 2t_{\rm B}$ (dark red).

This is a known function, computed from $c(\rho)$ and $f(\zeta) = \rho(x = \zeta, t = 0)$. The equation of the characteristic $x_\zeta(t)$ is then

$$x_\zeta(t) = \zeta + g(\zeta)\,t \; . \tag{4.14}$$

Do not confuse the subscript in $x_\zeta(t)$ for a derivative!

To find $\rho(x,t)$, we follow this prescription:

(i) Given any point in the (x,t) plane, we find the characteristic $x_\zeta(t)$ on which it lies. This means we invert the equation $x = \zeta + g(\zeta)\,t$ to find $\zeta(x,t)$.

(ii) The value of $\rho(x,t)$ is then $\rho = f\big(\zeta(x,t)\big)$.

(iii) Equivalently, at fixed t, for $\zeta \in (-\infty, +\infty)$ plot $\rho = f(\zeta)$ *vs.* $x = \zeta + g(\zeta)t$. This obviates the inversion in (i).

(iv) This procedure yields a unique value for $\rho(x,t)$ provided the characteristics do not cross, *i.e.* provided that there is a unique ζ such that $x = \zeta + g(\zeta)\,t$. If the characteristics do cross, then $\rho(x,t)$ is either *multi-valued*, or else the method has otherwise broken down. As we shall see, the crossing of characteristics, under the conditions of single-valuedness for $\rho(x,t)$, means that a *shock* has developed, and that $\rho(x,t)$ is *discontinuous*.

We can verify that this procedure yields a solution to the original PDE of Eqn. 4.3 in the following manner. Suppose we invert

$$x = \zeta + g(\zeta)\, t \qquad \Longrightarrow \qquad \zeta = \zeta(x,t) \ . \tag{4.15}$$

We then have

$$\rho(x,t) = f\big(\zeta(x,t)\big) \qquad \Longrightarrow \qquad \begin{cases} \rho_t = f'(\zeta)\, \zeta_t \\ \\ \rho_x = f'(\zeta)\, \zeta_x \end{cases} \tag{4.16}$$

To find ζ_t and ζ_x, we invoke $x = \zeta + g(\zeta)\, t$, hence

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\big[\zeta + g(\zeta)\, t - x\big] = \zeta_t + \zeta_t\, g'(\zeta)\, t + g(\zeta) \\ 0 &= \frac{\partial}{\partial x}\big[\zeta + g(\zeta)\, t - x\big] = \zeta_x + \zeta_x\, g'(\zeta)\, t - 1 \ , \end{aligned} \tag{4.17}$$

from which we conclude

$$\rho_t = -\frac{f'(\zeta)\, g(\zeta)}{1 + g'(\zeta)\, t} \qquad , \qquad \rho_x = \frac{f'(\zeta)}{1 + g'(\zeta)\, t} \ . \tag{4.18}$$

Thus, $\rho_t + c(\rho)\, \rho_x = 0$, since $c(\rho) = g(\zeta)$.

As any wave disturbance propagates, different values of ρ propagate with their own velocities. Thus, the solution $\rho(x,t)$ can be constructed by splitting the curve $\rho(x, t=0)$ into level sets of constant ρ, and then shifting each such set by a distance $c(\rho)\, t$. For $c(\rho) = c_0$, the entire curve is shifted uniformly. When $c(\rho)$ varies, different level sets are shifted by different amounts.

We see that ρ_x diverges when $1 + g'(\zeta)\, t = 0$. At this time, the wave is said to *break*. The break time $t_{\rm B}$ is defined to be the smallest value of t for which $\rho_x = \infty$ anywhere. Thus,

$$t_{\rm B} = \min_{\substack{\zeta \\ g'(\zeta)<0}} \bigg(-\frac{1}{g'(\zeta)} \bigg) \equiv -\frac{1}{g'(\zeta_{\rm B})} \ . \tag{4.19}$$

Breaking can only occur when $g'(\zeta) < 0$, and differentiating $g(\zeta) = c\big(f(\zeta)\big)$, we have that $g'(\zeta) = c'(f)\, f'(\zeta)$. We then conclude

$$\begin{aligned} c' < 0 \quad &\Longrightarrow \quad \text{need } f' > 0 \text{ to break} \\ c' > 0 \quad &\Longrightarrow \quad \text{need } f' < 0 \text{ to break} \ . \end{aligned} \tag{4.20}$$

Thus, if $\rho(x = \zeta, t = 0) = f(\zeta)$ has a hump profile, then the wave breaks forward (*i.e.* in the direction of its motion) if $c' > 0$ and backward (*i.e.* opposite to the direction of its motion) if $c' < 0$. In Fig. 4.1 we sketch the breaking of a wave with $\rho(x, t = 0) = 1/(1 + x^2)$ for the cases $c = 1 + \rho$ and $c = 2 - \rho$. Note that it is possible for different regions of a wave to break at different times, if, say, it has multiple humps.

Wave breaking occurs when neighboring characteristic cross. We can see this by comparing two neighboring characteristics,

$$x_\zeta(t) = \zeta + g(\zeta)\, t \tag{4.21}$$

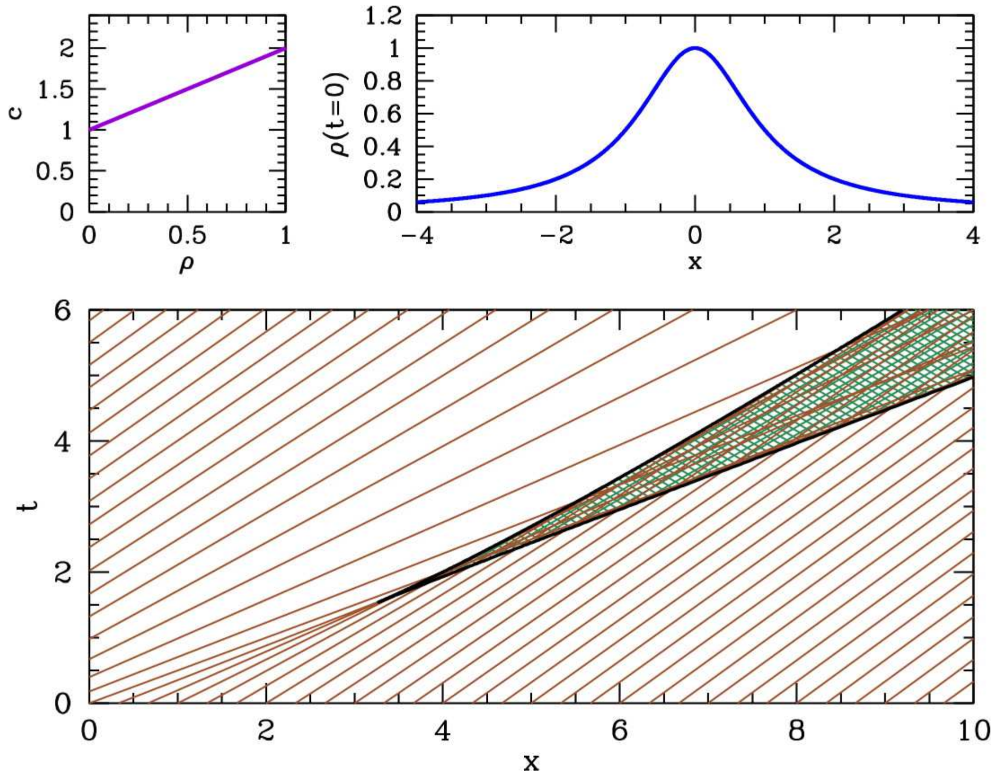

Figure 4.2: Crossing of characteristics of the nonlinear chiral wave equation $\rho_t + c(\rho)\,\rho_x = 0$, with $c(\rho) = 1 + \rho$ and $\rho(x, t = 0) = 1/(1 + x^2)$. Within the green hatched region of the (x, t) plane, the characteristics cross, and the function $\rho(x, t)$ is apparently multivalued.

and

$$\begin{aligned} x_{\zeta+\delta\zeta}(t) &= \zeta + \delta\zeta + g(\zeta + \delta\zeta)\, t \\ &= \zeta + g(\zeta)\, t + \big(1 + g'(\zeta)\, t\big)\, \delta\zeta + \ldots \ . \end{aligned} \tag{4.22}$$

For these characteristics to cross, we demand

$$x_\zeta(t) = x_{\zeta+\delta\zeta}(t) \quad \Longrightarrow \quad t = -\frac{1}{g'(\zeta)} \ . \tag{4.23}$$

Usually, in most physical settings, the function $\rho(x, t)$ is single-valued. In such cases, when the wave breaks, the multivalued solution ceases to be applicable[2]. Generally speaking, this means that some important physics has been left out. For example, if we neglect viscosity η and thermal conductivity κ, then the equations of gas dynamics have breaking wave solutions similar to those just discussed. When the gradients are steep – just before breaking – the effects of η and κ are no longer negligible, even if these parameters are small. This is because these parameters enter into the coefficients of higher derivative terms in the governing PDEs, and even if they are small their effect is magnified in the presence of steep gradients. In mathematical parlance, they constitute *singular perturbations*. The shock wave is then a thin

[2]This is even true for water waves, where one might think that a multivalued height function $h(x, t)$ is physically possible.

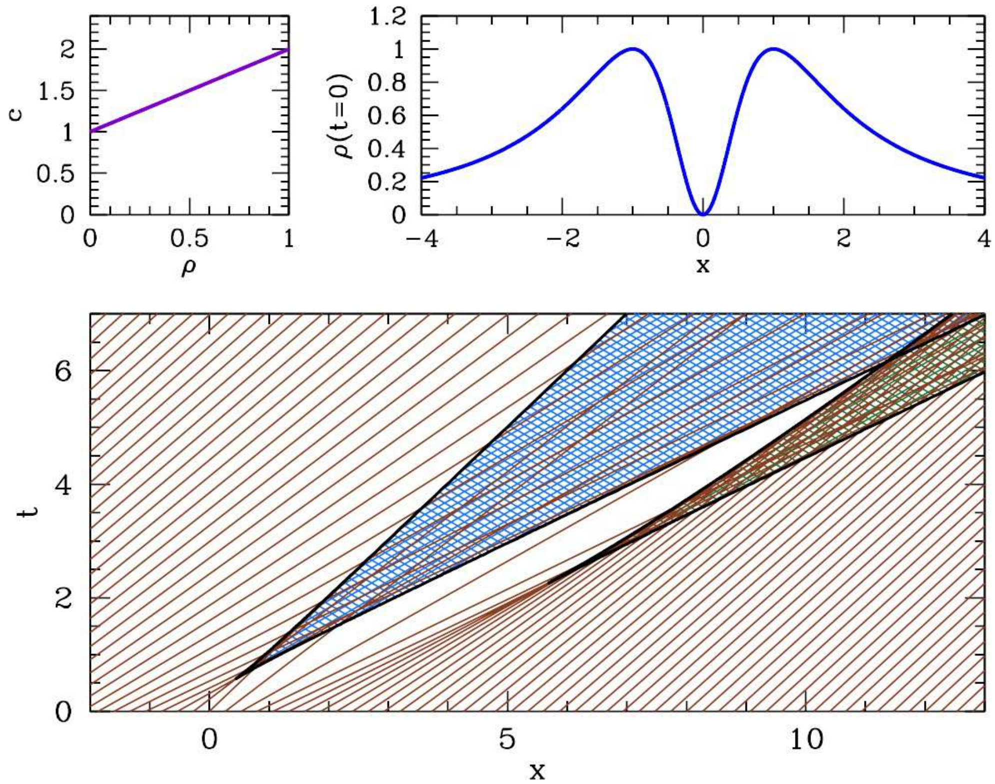

Figure 4.3: Crossing of characteristics of the nonlinear chiral wave equation $\rho_t + c(\rho)\,\rho_x = 0$, with $c(\rho) = 1 + \rho$ and $\rho(x, t = 0) = \big[x/(1 + x^2)\big]^2$. The wave now breaks in two places and is multivalued in both hatched regions. The left hump is the first to break.

region in which η and κ are crucially important, and the flow changes rapidly throughout this region. If one is not interested in this small scale physics, the shock region can be approximated as being infinitely thin, *i.e.* as a discontinuity in the inviscid limit of the theory. What remains is a set of *shock conditions* which govern the discontinuities of various quantities across the shocks.

4.2 Shocks

We now show that a solution to Eqn. 4.3 exists which is single valued for almost all (x, t), *i.e.* everywhere with the exception of a set of zero measure, but which has a discontinuity along a curve $x = x_{\rm s}(t)$. This discontinuity is the shock wave.

The velocity of the shock is determined by mass conservation, and is most easily obtained in the frame of the shock. The situation is as depicted in Fig. 4.4. If the density and current are $(\rho_- \,,\, j_-)$ to the left of the shock and $(\rho_+ \,,\, j_+)$ to the right of the shock, and if the shock moves with velocity $v_{\rm s}$, then making a Galilean transformation to the frame of the shock, the densities do not change but the currents transform as $j \to j' = j - \rho v$. Thus, in the frame where the shock is stationary, the current on the left and right are

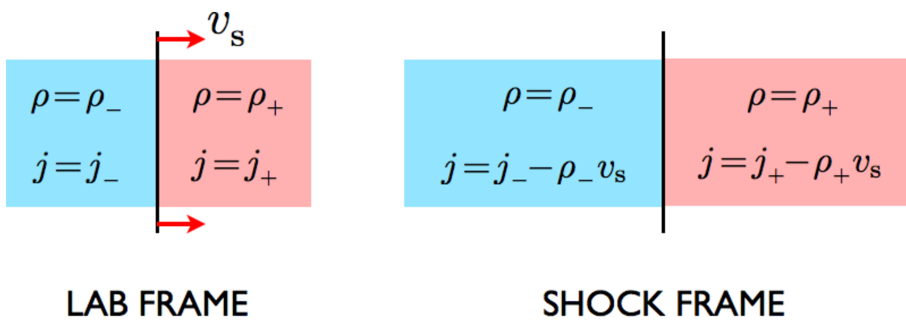

Figure 4.4: Current conservation in the shock frame yields the shock velocity, $v_s = \Delta j/\Delta\rho$.

$j_{\pm} = j_{\pm} - \rho_{\pm} v_s$. Current conservation then requires

$$v_s = \frac{j_+ - j_-}{\rho_+ - \rho_-} = \frac{\Delta j}{\Delta \rho} \ . \tag{4.24}$$

The special case of quadratic $J(\rho)$ bears mention. Suppose

$$J(\rho) = \alpha\rho^2 + \beta\rho + \gamma \ . \tag{4.25}$$

Then $c = 2\alpha\rho + \beta$ and

$$\begin{aligned} v_s &= \alpha(\rho_+ + \rho_-) + \beta \\ &= \tfrac{1}{2}(c_+ + c_-) \ . \end{aligned} \tag{4.26}$$

So for quadratic $J(\rho)$, the shock velocity is simply the average of the flow velocity on either side of the shock.

Consider, for example, a model with $J(\rho) = 2\rho(1-\rho)$, for which $c(\rho) = 2 - 4\rho$. Consider an initial condition $\rho(x = \zeta, t = 0) = f(\zeta) = \frac{3}{16} + \frac{1}{8}\,\Theta(\zeta)$, so initially $\rho = \rho_1 = \frac{3}{16}$ for $x < 0$ and $\rho = \rho_2 = \frac{5}{16}$ for $x > 0$. The lower density part moves faster, so in order to avoid multiple-valuedness, a shock must propagate. We find $c_- = \frac{5}{4}$ and $c_+ = \frac{3}{4}$. The shock velocity is then $v_s = 1$. Ths situation is depicted in Fig. 4.5.

4.3 Internal Shock Structure

At this point, our model of a shock is a discontinuity which propagates with a finite velocity. This may be less problematic than a multivalued solution, but it is nevertheless unphysical. We should at least understand how the discontinuity is resolved in a more complete model. To this end, consider a model where

$$j = J(\rho, \rho_x) = J(\rho) - \nu\rho_x \ . \tag{4.27}$$

The $J(\rho)$ term contains a nonlinearity which leads to steepening and broadening of regions where $\frac{dc}{dx} > 0$ and $\frac{dc}{dx} < 0$, respectively. The second term, $-\nu\rho_x$, is due to diffusion, and recapitulates *Fick's law*, which

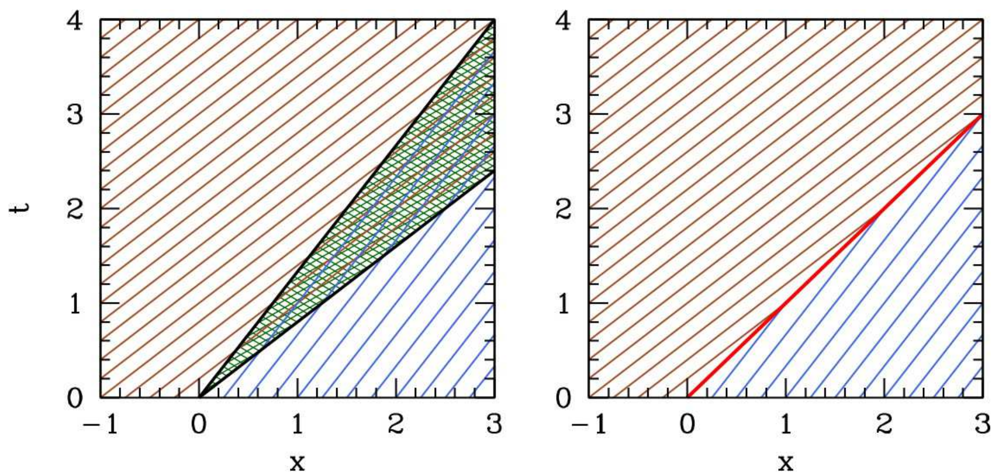

Figure 4.5: A resulting shock wave arising from $c_- = \frac{5}{4}$ and $c_+ = \frac{3}{4}$. With no shock fitting, there is a region of (x, t) where the characteristics cross, shown as the hatched region on the left. With the shock, the solution remains single valued. A quadratic behavior of $J(\rho)$ is assumed, leading to $v_s = \frac{1}{2}(c_+ + c_-) = 1$.

says that a diffusion current flows in such a way as to reduce gradients. The continuity equation then reads

$$\rho_t + c(\rho)\,\rho_x = \nu \rho_{xx} \,, \tag{4.28}$$

with $c(\rho) = J'(\rho)$. Even if ν is small, its importance is enhanced in regions where $|\rho_x|$ is large, and indeed $-\nu\rho_x$ dominates over $J(\rho)$ in such regions. Elsewhere, if ν is small, it may be neglected, or treated perturbatively.

General solutions to Eqn. 4.28 are of the form $\rho = \rho(x, t)$. In §4.9 below, we will show how to obtain such general solutions for the case where $c(\rho)$ is a linear function of the density, *i.e.* $c''(\rho) = 0$, using a nifty nonlinear transformation. Here, we will content ourselves with obtaining a special class of solution to Eqn. 4.28 by assuming a form corresponding to *front propagation*,

$$\rho(x, t) = \rho(x - v_s t) \equiv \rho(\xi) \qquad ; \qquad \xi = x - v_s t \,, \tag{4.29}$$

where the propagation speed v_s is a constant to be determined. The above form of solution corresponds to a fixed shape $\rho(\xi)$ which moves with constant propagation speed. Thus, $\rho_t = -v_s\,\rho_x$ and $\rho_x = \rho_\xi$, leading to

$$-v_s\,\rho_\xi + c(\rho)\,\rho_\xi = \nu \rho_{\xi\xi} \,. \tag{4.30}$$

By assuming the propagating front form, we have achieved a remarkable simplification, transforming a first order PDE in two variables to a second order ODE. Moreover, the ODE is simple to integrate. Integrating once, we have

$$J(\rho) - v_s\,\rho + A = \nu\,\rho_\xi \,, \tag{4.31}$$

where A is a constant. Integrating a second time, we have

$$\xi - \xi_0 = \nu \int_{\rho_0}^{\rho} \frac{d\rho'}{J(\rho') - v_s\,\rho' + A} \,. \tag{4.32}$$

Suppose ρ interpolates between the values ρ_1 and ρ_2 over the interval $\xi \in (-\infty, +\infty)$. Then we must have

$$\begin{aligned} J(\rho_1) - v_s\,\rho_1 + A = 0 \\ J(\rho_2) - v_s\,\rho_2 + A = 0 \,, \end{aligned} \tag{4.33}$$

which in turn requires

$$v_s = \frac{J_2 - J_1}{\rho_2 - \rho_1} \,, \tag{4.34}$$

where $J_{1,2} = J(\rho_{1,2})$, exactly as before! We also conclude that the constant A must be

$$A = \frac{\rho_1 J_2 - \rho_2 J_1}{\rho_2 - \rho_1} \,. \tag{4.35}$$

4.3.1 Quadratic $J(\rho)$

For the special case where $J(\rho)$ is quadratic, with $J(\rho) = \alpha\rho^2 + \beta\rho + \gamma$, we may write

$$J(\rho) - v_s\,\rho + A = \alpha(\rho - \rho_2)(\rho - \rho_1) \,. \tag{4.36}$$

We then have $v_s = \alpha(\rho_1 + \rho_2) + \beta$, as well as $A = \alpha\rho_1\rho_2 - \gamma$. The moving front solution then obeys

$$d\xi = \frac{\nu\, d\rho}{\alpha(\rho - \rho_2)(\rho - \rho_1)} = \frac{\nu}{\alpha(\rho_2 - \rho_1)}\, d\ln\!\left(\frac{\rho_2 - \rho}{\rho - \rho_1}\right) , \tag{4.37}$$

which is integrated to yield

$$\rho(x,t) = \frac{\rho_2 + \rho_1 \exp\Big[\alpha(\rho_2 - \rho_1)\big(x - v_s t\big)/\nu\Big]}{1 + \exp\Big[\alpha(\rho_2 - \rho_1)\big(x - v_s t\big)/\nu\Big]} \,. \tag{4.38}$$

We consider the case $\alpha > 0$ and $\rho_1 < \rho_2$. Then $\rho(\pm\infty, t) = \rho_{1,2}$. Note that

$$\rho(x,t) = \begin{cases} \rho_1 & \text{if } x - v_s t \gg \delta \\ \rho_2 & \text{if } x - v_s t \ll -\delta \,, \end{cases} \tag{4.39}$$

where

$$\delta = \frac{\nu}{\alpha\,(\rho_2 - \rho_1)} \tag{4.40}$$

is the thickness of the shock region. In the limit $\nu \to 0$, the shock is discontinuous. All that remains is the *shock condition*,

$$v_s = \alpha(\rho_1 + \rho_2) + \beta = \tfrac{1}{2}\big(c_1 + c_2\big) \,. \tag{4.41}$$

We stress that we have limited our attention here to the case where $J(\rho)$ is quadratic. It is worth remarking that for *weak shocks* where $\Delta\rho = \rho_+ - \rho_-$ is small, we can expand $J(\rho)$ about the average $\frac{1}{2}(\rho_+ + \rho_-)$, in which case we find $v_s = \frac{1}{2}(c_+ + c_-) + \mathcal{O}\big((\Delta\rho)^2\big)$.

4.4 Shock Fitting

When we neglect diffusion currents, we have $j = J$. We now consider how to fit discontinuous shocks satisfying

$$v_{\rm s} = \frac{J_+ - J_-}{\rho_+ - \rho_-} \tag{4.42}$$

into the continuous solution of Eqn. 4.3, which are described by

$$x = \zeta + g(\zeta)\,t \qquad , \qquad \rho = f(\zeta) \ , \tag{4.43}$$

with $g(\zeta) = c\big(f(\zeta)\big)$, such that the multivalued parts of the continuous solution are eliminated and replaced with the shock discontinuity. The guiding principle here is number conservation:

$$\frac{d}{dt}\int\limits_{-\infty}^{\infty}\!dx\,\rho(x,t) = 0 \ . \tag{4.44}$$

We'll first learn how do fit shocks when $J(\rho)$ is quadratic, with $J(\rho) = \alpha\rho^2 + \beta\rho + \gamma$. We'll assume $\alpha > 0$ for the sake of definiteness.

4.4.1 An important caveat

Let's multiply the continuity equation $\rho_t + c(\rho)\,\rho_x = 0$ by $c'(\rho)$. Thus results in

$$c_t + c\,c_x = 0 \ . \tag{4.45}$$

If we define $q = \frac{1}{2}c^2$, then this takes the form of a continuity equation:

$$c_t + q_x = 0 \ . \tag{4.46}$$

Now consider a shock wave. Invoking Eqn. 4.24, we would find, *mutatis mutandis*, a shock velocity

$$u_{\rm s} = \frac{q_+ - q_-}{c_+ - c_-} = \tfrac{1}{2}(c_+ + c_-) \ . \tag{4.47}$$

This agrees with the velocity $v_{\rm s} = \Delta j/\Delta\rho$ only when $J(\rho)$ is quadratic. Something is wrong – there cannot be two velocities for the same shock.

The problem is that Eqn. 4.45 is not valid across the shock and cannot be used to determine the shock velocity. There is no conservation law for c as there is for ρ. One way we can appreciate the difference is to add diffusion into the mix. Multiplying Eqn. 4.28 by $c'(\rho)$, and invoking $c_{xx} = c'(\rho)\,\rho_{xx} + c''(\rho)\,\rho_x^2$, we obtain

$$c_t + c\,c_x = \nu c_{xx} - \nu c''(\rho)\,\rho_x^2 \ . \tag{4.48}$$

We now see explicitly how nonzero $c''(\rho)$ leads to a different term on the RHS. When $c''(\rho) = 0$, the above equation is universal, independent of the coefficients in the quadratic $J(\rho)$, and is known as *Burgers' equation*,

$$c_t + c\,c_x = \nu c_{xx} \ . \tag{4.49}$$

Later on we shall see how this nonlinear PDE may be linearized, and how we can explicitly solve for shock behavior, including the merging of shocks.

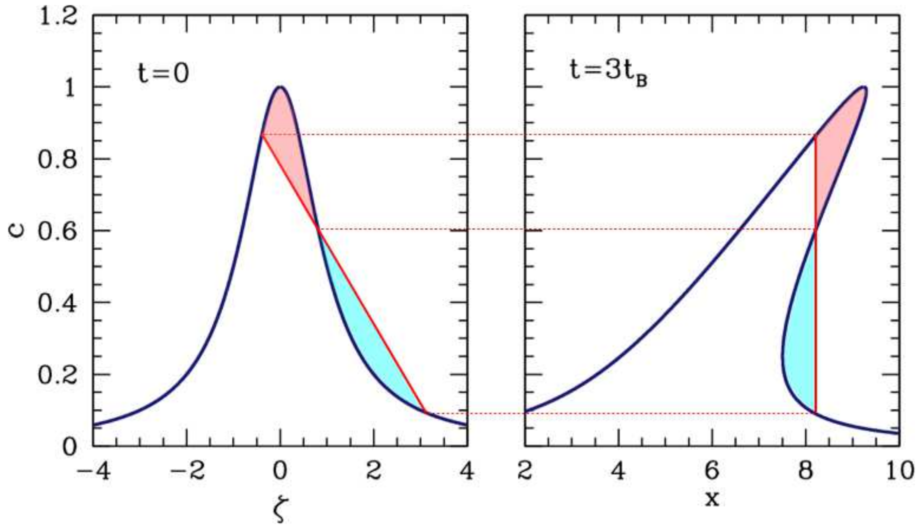

Figure 4.6: Shock fitting for quadratic $J(\rho)$.

4.4.2 Recipe for shock fitting when $J'''(\rho) = 0$

Number conservation means that when we replace the multivalued solution by the discontinuous one, the area under the curve must remain the same. If $J(\rho)$ is quadratic, then we can base our analysis on the equation $c_t + c\,c_x = 0$, since it gives the correct shock velocity $v_s = \frac{1}{2}(c_+ + c_-)$. We then may then follow the following rules:

(i) Sketch $c(\zeta) = c(x = \zeta, t = 0)$.

(ii) Draw a straight line connecting two points on this curve at ζ_- and ζ_+ which obeys the equal area law, *i.e.*

$$\tfrac{1}{2}(\zeta_+ - \zeta_-)\Big(c(\zeta_+) + c(\zeta_-)\Big) = \int\limits_{\zeta_-}^{\zeta_+} d\zeta\, c(\zeta)\ . \tag{4.50}$$

(iii) This line evolves into the shock front after a time t such that

$$x_s(t) = \zeta_- + c(\zeta_-)\,t = \zeta_+ + c(\zeta_+)\,t\ . \tag{4.51}$$

Thus,

$$t = \frac{\zeta_+ - \zeta_-}{c(\zeta_-) - c(\zeta_+)}\ . \tag{4.52}$$

Alternatively, we can fix t and solve for $\zeta_\pm$. See Fig. 4.6 for a graphical description.

(iv) The position of the shock at this time is $x = x_s(t)$. The strength of the shock is $\Delta c = c(\zeta_-) - c(\zeta_+)$. Since $J(\rho) = \alpha\rho^2 + \beta\rho + \gamma$, we have $c(\rho) = 2\alpha\rho + \beta$ and hence the density discontinuity at the shock is $\Delta\rho = \Delta c/2\alpha$.

(v) The break time, when the shock first forms, is given by finding the steepest chord satisfying the equal area law. Such a chord is tangent to $c(\zeta)$ and hence corresponds to zero net area. The break time is

$$t_{\rm B} = \min_{\substack{\zeta \\ c'(\zeta)>0}} \bigg(-\frac{1}{c'(\zeta)} \bigg) \equiv -\frac{1}{c'(\zeta_{\rm B})} \ . \tag{4.53}$$

(vi) If $c(\infty) = c(-\infty)$, the shock strength vanishes as $t \to \infty$. If $c(-\infty) > c(+\infty)$ then asymptotically the shock strength approaches $\Delta c = c(-\infty) - c(+\infty)$.

(vii) To plot $c(x,t)$ *versus* x at fixed t: If $t \le t_{\rm B}$, plot $c(\zeta)$ *versus* $x = \zeta + c(\zeta)\,t$ for $\zeta \in (-\infty, +\infty)$. If $t > t_{\rm B}$, plot $c(\zeta)$ *versus* $x = \zeta + c(\zeta)\,t$ separately for $\zeta \in (-\infty, \zeta_-]$ and $\zeta \in [\zeta_+, +\infty)$.

4.4.3 Example : shock fitting an inverted parabola

Consider the shock fitting problem for the initial condition

$$c(x, t=0) = c_0 \bigg(1 - \frac{x^2}{a^2} \bigg)\, \Theta(a^2 - x^2) \ , \tag{4.54}$$

which is to say a truncated inverted parabola. We assume $J'''(\rho) = 0$. Clearly $-c_x(x,0)$ is maximized at $x = a$, where $-c_x(a,0) = 2c_0/a$, hence breaking first occurs at

$$\big(x_{\rm B} \, , \, t_{\rm B} \big) = \bigg(a \, , \, \frac{a}{2c_0} \bigg) \ . \tag{4.55}$$

Clearly $\zeta_+ > a$, hence $c_+ = 0$. Shock fitting then requires

$$\tfrac{1}{2}(\zeta_+ - \zeta_-)(c_+ + c_-) = \int\limits_{\zeta_-}^{\zeta_+} \! d\zeta \; c(\zeta) = \frac{c_0}{3a^2}\,(2a + \zeta_-)(a - \zeta_-)^2 \ . \tag{4.56}$$

Since $c_+ + c_- = \frac{c_0}{a^2}\,(a^2 - \zeta_-^2)$, we have

$$\zeta_+ - \zeta_- = \tfrac{2}{3}\,(2a + \zeta_-) \bigg(\frac{a - \zeta_-}{a + \zeta_-} \bigg) \ . \tag{4.57}$$

The second shock-fitting equation is $\zeta_+ - \zeta_- = (c_- - c_+)\,t$. Eliminating ζ_+ from the two shock-fitting equations, and invoking $c_+ = 0$, we have

$$t = \frac{2a^2}{3c_0} \cdot \frac{2a + \zeta_-}{(a + \zeta_-)^2} \ . \tag{4.58}$$

Inverting to find $\zeta_-(t)$, we obtain

$$\frac{\zeta_-(t)}{a} = \frac{a}{3c_0 t} - 1 + \frac{a}{3c_0 t} \sqrt{1 + \frac{6c_0 t}{a}} \ . \tag{4.59}$$

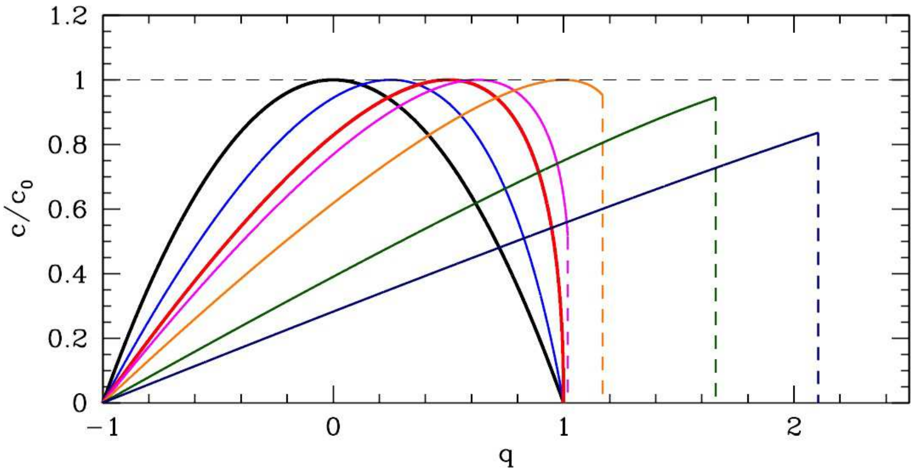

Figure 4.7: Evolution of the inverted parabola profile for $\tau = 0$ (black), $\tau = 0.5$ (blue), $\tau = 1$ (red), $\tau = 1.25$ (magenta), $\tau = 2.0$ (green), and $\tau = 6.0$ (navy blue). The wave first breaks at time $\tau = \tau_B = 1$.

The shock position is then $x_s(t) = \zeta_-(t) + c_-\big(\zeta_-(t)\big)\, t = \zeta_+(t)$.

It is convenient to rescale lengths by a and times by $t_B = a/2c_0$, defining q and τ from $x \equiv aq$ and $t \equiv a\tau/2c_0$. Then we have, for $q_\pm(\tau) = \zeta_\pm(t)/a$,

$$q_-(\tau) = \frac{2}{3\tau} - 1 + \frac{2}{3\tau}\sqrt{1+3\tau}$$
$$q_s(\tau) = q_+(\tau) = \frac{2}{9\tau} - 1 + \frac{2}{9\tau}\,(1+3\tau)^{3/2} \quad . \tag{4.60}$$

The dimensionless break time is $\tau_B = 1$. Note that $q_-(1) = q_+(1) = 1$, *i.e.* the wave starts breaking at $q_B = 1$. The dimensionless shock discontinuity is given by

$$\frac{\Delta c(\tau)}{c_0} = 1 - q_-^2 = -\frac{8}{9\tau^2}\Big[1 + \sqrt{1+3\tau}\Big] + \frac{4}{3\tau}\sqrt{1+3\tau} \quad . \tag{4.61}$$

Note also that $\Delta c(\tau_B) = 0$. The dimensionless shock velocity is

$$\dot q_s = -\frac{2}{9\tau^2}\Big[1 + (1+3\tau)^{3/2}\Big] + \frac{1}{\tau}\,(1+3\tau)^{1/2}$$
$$= \tfrac{3}{4}\,(\tau - 1) + \tfrac{81}{64}\,(\tau-1)^2 + \ldots \ , \tag{4.62}$$

with $v_s = 2c_0\,\dot q_s = \frac{1}{2}c_-$, if we restore units. Note that $\dot q_s(\tau = 1) = 0$, so the shock curve initially rises vertically in the (x,t) plane. In Fig. 4.7 we plot the evolution of $c(x,t)$. Another example of shock fitting is worked out in the appendix §4.11 below. Interestingly, $v_s \propto (\tau - 1)$ here, while for the example in §4.11, where $c(x,0)$ has a similar profile, we find $v_s \propto (\tau-1)^{1/2}$ (see Eqn. 4.170).

4.5 Long-time Behavior of Shocks

Starting with an initial profile $\rho(x,t)$, almost all the original details are lost in the $t\to\infty$ limit. What remains is a set of propagating triangular waves, where only certain gross features of the original shape, such as its area, are preserved.

4.5.1 Fate of a hump

Once again, we consider the case $c''(\rho)=0$ and analyze the equation $c_t + c\,c_x = 0$. The late time profile of $c(x,t)$ in Fig. 4.14 is that of a triangular wave. This is a general result. Following Whitham, we consider the late time evolution of a hump profile $c(\zeta)=c(x=\zeta,t=0)$. We assume $c(\zeta)=c_0$ for $|\zeta|>L$. Shock fitting requires

$$\tfrac{1}{2}\Big[c(\zeta_+)+c(\zeta_-)-2c_0\Big](\zeta_+-\zeta_-)=\int\limits_{\zeta_-}^{\zeta_+}\!d\zeta\,\big(c(\zeta)-c_0\big)\ . \tag{4.63}$$

Eventually the point ζ_+ must pass $x=L$, in which case $c(\zeta_+)=c_0$. Then

$$\tfrac{1}{2}\Big[c(\zeta_+)-c_0\Big](\zeta_+-\zeta_-)=\int\limits_{\zeta_-}^{L}\!d\zeta\,\big(c(\zeta)-c_0\big) \tag{4.64}$$

and therefore

$$t=\frac{\zeta_+-\zeta_-}{c(\zeta_-)-c_0}\ . \tag{4.65}$$

Using this equation to eliminate ζ_+, we have

$$\tfrac{1}{2}\big(c(\zeta_-)-c_0\big)^2\,t=\int\limits_{\zeta_-}^{L}\!d\zeta\,\big(c(\zeta)-c_0\big)\ . \tag{4.66}$$

As $t\to\infty$ we must have $\zeta_-\to -L$, hence

$$\tfrac{1}{2}\big(c(\zeta_-)-c_0\big)^2\,t\approx\int\limits_{-L}^{L}\!d\zeta\,\big(c(\zeta)-c_0\big)\equiv A\ , \tag{4.67}$$

where A is the area under the hump to the line $c=c_0$. Thus,

$$c(\zeta_-)-c_0\approx\sqrt{\frac{2A}{t}}\ , \tag{4.68}$$

and the late time motion of the shock is given by

$$\begin{aligned}x_\mathrm{s}(t)&=-L+c_0t+\sqrt{2At}\\ v_\mathrm{s}(t)&=c_0+\sqrt{\frac{A}{2t}}\ .\end{aligned} \tag{4.69}$$

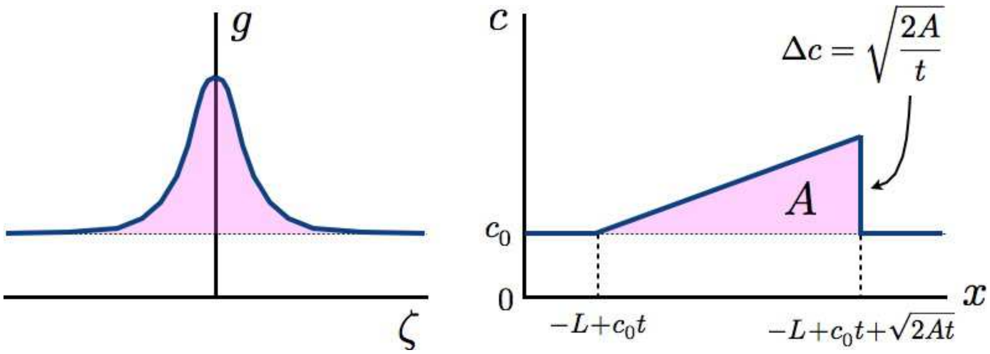

Figure 4.8: Initial and late time configurations for a hump profile. For late times, the profile is triangular, and all the details of the initial shape are lost, save for the area A.

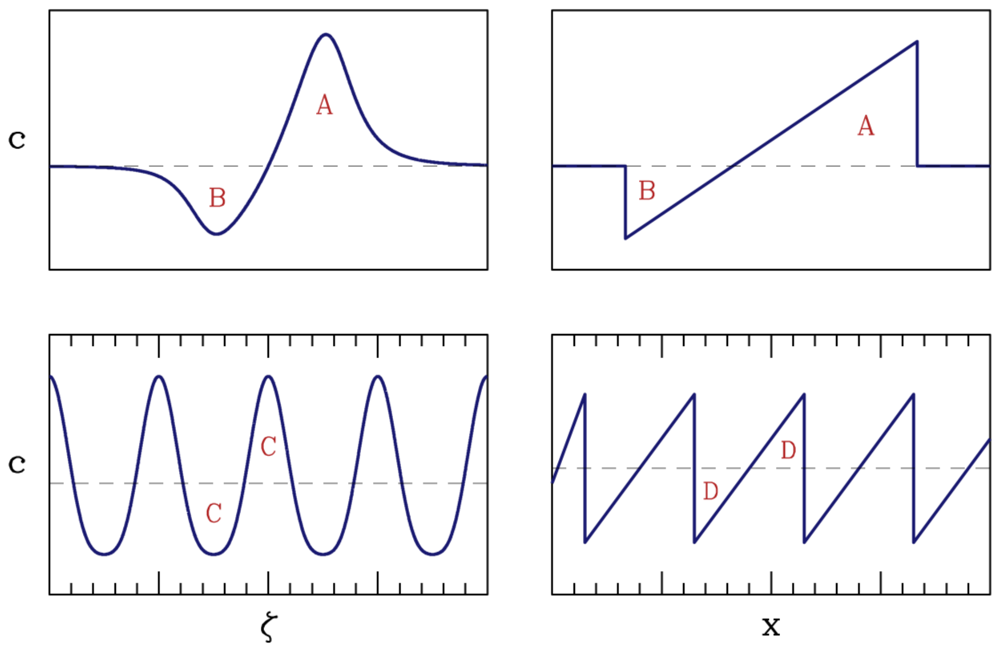

Figure 4.9: Top panels : An N-wave, showing initial (left) and late time (right) profiles. As the N-wave propagates, the areas A and B are preserved. Bottom panels : A P-wave. The area D eventually decreases to zero as the shock amplitude dissipates.

The shock strength is $\Delta c = c(\zeta_-) - c_0 = \sqrt{2A/t}$. Behind the shock, we have $c = c(\zeta)$ and $x = \zeta + c(\zeta)\, t$, hence

$$c = \frac{x+L}{t} \qquad \text{for} \quad -L + c_0 t < x < -L + c_0 t + \sqrt{2At}\ . \tag{4.70}$$

As $t \to \infty$, the details of the original profile $c(x,0)$ are lost, and all that remains conserved is the area A. Both shock velocity and the shock strength decrease as $t^{-1/2}$ at long times, with $v_{\rm s}(t) \to c_0$ and $\Delta c(t) \to 0$.

4.5.2 N-wave and P-wave

Consider the initial profile in the top left panel of Fig. 4.9. Now there are two propagating shocks, since there are two compression regions where $g'(\zeta) < 0$. As $t \to \infty$, we have $(\zeta_- \, , \, \zeta_+)_{\rm A} \to (0 \, , \, \infty)$ for the A shock, and $(\zeta_- \, , \, \zeta_+)_{\rm B} \to (-\infty \, , \, 0)$ for the B shock. Asymptotically, the shock strength

$$\Delta c(t) \equiv c\big(x_{\rm s}^-(t), t\big) - c\big(x_{\rm s}^+(t), t\big) \tag{4.71}$$

for the two shocks is given by

$$\begin{aligned} x_{\rm s}^{\rm A}(t) &\approx c_0 t + \sqrt{2At} \quad , \quad \Delta c_{\rm A} \approx +\sqrt{\frac{2A}{t}} \\ x_{\rm s}^{\rm B}(t) &\approx c_0 t - \sqrt{2Bt} \quad , \quad \Delta c_{\rm B} \approx -\sqrt{\frac{2B}{t}} \ , \end{aligned} \tag{4.72}$$

where A and B are the areas associated with the two features This feature is called an N-wave, for its N (or inverted N) shape.

The initial and late stages of a periodic (P) wave, where $g(\zeta + \lambda) = g(\zeta)$, are shown in the bottom panels of Fig. 4.9. In the $t \to \infty$ limit, we evidently have $\zeta_+ - \zeta_- = \lambda$, the wavelength. Asymptotically the shock strength is given by

$$\Delta c(t) \equiv g(\zeta_-) - g(\zeta_+) = \frac{\zeta_+ - \zeta_-}{t} = \frac{\lambda}{t} \ , \tag{4.73}$$

where we have invoked Eqn. 4.52. In this limit, the shock train travels with constant velocity c_0, which is the spatial average of $c(x, 0)$ over one wavelength:

$$c_0 = \frac{1}{\lambda} \int_0^\lambda d\zeta \, g(\zeta) \ . \tag{4.74}$$

4.6 Shock Merging

It is possible for several shock waves to develop, and in general these shocks form at different times, have different strengths, and propagate with different velocities. Under such circumstances, it is quite possible that one shock overtakes another. These two shocks then merge and propagate on as a single shock. The situation is depicted in Fig. 4.10. We label the shocks by A and B when they are distinct, and the late time single shock by C. We must have

$$\begin{aligned} v_{\rm s}^{\rm A} &= \tfrac{1}{2}\, g\big(\zeta_+^{\rm A}\big) + \tfrac{1}{2}\, g\big(\zeta_-^{\rm A}\big) \\ v_{\rm s}^{\rm B} &= \tfrac{1}{2}\, g\big(\zeta_+^{\rm B}\big) + \tfrac{1}{2}\, g\big(\zeta_-^{\rm B}\big) \ . \end{aligned} \tag{4.75}$$

The merging condition requires

$$\zeta_+^{\rm A} = \zeta_-^{\rm B} \equiv \xi \tag{4.76}$$

as well as

$$\zeta_+^{\rm C} = \zeta_+^{\rm B} \qquad , \qquad \zeta_-^{\rm C} = \zeta_-^{\rm A} \, . \tag{4.77}$$

The merge occurs at time t, where

$$t = \frac{\zeta_+ - \xi}{g(\xi) - g(\zeta_+)} = \frac{\xi - \zeta_-}{g(\zeta_-) - g(\xi)} \, . \tag{4.78}$$

Thus, the slopes of the A and B shock construction lines are equal when they merge.

4.7 Shock Fitting for General $J(\rho)$

When $J(\rho)$ is quadratic, we may analyze the equation $c_t + c\,c_x$, as it is valid across any shocks in that it yields the correct shock velocity. If $J'''(\rho) \neq 0$, this is no longer the case, and we must base our analysis on the original equation $\rho_t + c(\rho)\,\rho_x = 0$.

The coordinate transformation

$$(x, c) \longrightarrow (x + ct \, , \, c) \tag{4.79}$$

preserves areas in the (x, c) plane and also maps lines to lines. However, while

$$(x, \rho) \longrightarrow (x + c(\rho)\,t \, , \, \rho) \tag{4.80}$$

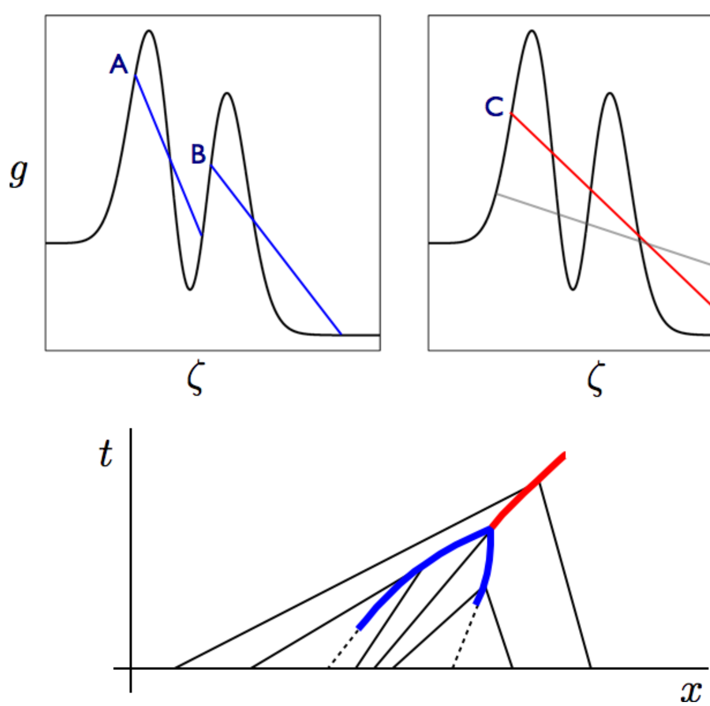

Figure 4.10: Merging of two shocks. The shocks initially propagate independently (upper left), and then merge and propagate as a single shock (upper right). Bottom : characteristics for the merging shocks.

does preserve areas in the (x, ρ) plane, it does not map lines to lines. Thus, the 'pre-image' of the shock front in the (x, ρ) plane is not a simple straight line, and our equal area construction fails. Still, we can make progress. We once again follow Whitham, §2.9.

Let $x(\rho, t)$ be the inverse of $\rho(x, t)$, with $\zeta(\rho) \equiv x(\rho, t = 0)$. Then

$$x(\rho, t) = \zeta(\rho) + c(\rho)\, t \ . \tag{4.81}$$

Note that $\rho(x, t)$ is in general multi-valued. We still have that the shock solution covers the same area as the multivalued solution $\rho(x, t)$. Let $\rho_\pm$ denote the value of ρ just to the right $(+)$ or left $(-)$ of the shock. For purposes of illustration, we assume $c'(\rho) > 0$, which means that $\rho_x < 0$ is required for breaking, although the method works equally well for $c'(\rho) < 0$. Assuming a hump-like profile, we then have $\rho_- > \rho_+$, with the shock breaking to the right. Area conservation requires

$$\int\limits_{\rho_+}^{\rho_-} d\rho\, x(\rho, t) = \int\limits_{\rho_+}^{\rho_-} d\rho\, \big[\zeta(\rho) + c(\rho)\, t\,\big] = (\rho_- - \rho_+)\, x_{\rm s}(t) \ . \tag{4.82}$$

Since $c(\rho) = J'(\rho)$, the above equation may be written as

$$(J_+ - J_-)\, t - (\rho_+ - \rho_-)\, x_{\rm s}(t) = \int\limits_{\rho_+}^{\rho_-} d\rho\, \zeta(\rho) = \rho_-\, \zeta_- - \rho_+\, \zeta_+ - \int\limits_{\zeta_+}^{\zeta_-} d\zeta\, \rho(\zeta) \ , \tag{4.83}$$

where we have integrated by parts in obtaining the RHS, writing $\zeta\, d\rho = d(\rho\, \zeta) - \rho\, d\zeta$. Next, the shock position $x_{\rm s}(t)$ is given by

$$x_{\rm s}(t) = \zeta_- + c_-\, t = \zeta_+ + c_+ t \ , \tag{4.84}$$

which yields

$$t = \frac{\zeta_- - \zeta_+}{c_+ - c_-} \quad , \tag{4.85}$$

and therefore

$$\Big[(J_+ - \rho_+ c_+) - (J_- - \rho_- c_-)\Big] = -\frac{c_+ - c_-}{\zeta_+ - \zeta_-} \int\limits_{\zeta_-}^{\zeta_+} d\zeta\, \rho(\zeta) \ . \tag{4.86}$$

This is a useful result because $J_\pm$, $\rho_\pm$, and $c_\pm$ are all functions of $\zeta_\pm$, hence what we have here is a relation between ζ_+ and ζ_-. When $J(\rho)$ is quadratic, this reduces to our earlier result in Eqn. 4.50. For a hump, we still have $x_{\rm s} \approx c_0 t + \sqrt{2At}$ and $c - c_0 \approx \sqrt{2A/t}$ as before, with

$$A = c'(\rho_0) \int\limits_{-\infty}^{\infty} d\zeta\, \big[\rho(\zeta) - \rho_0\big] \ . \tag{4.87}$$

4.8 Sources

Consider the continuity equation in the presence of a source term,

$$\rho_t + c\,\rho_x = \sigma\ , \tag{4.88}$$

where $c = c(x,t,\rho)$ and $\sigma = \sigma(x,t,\rho)$. Note that we are allowing for more than just $c = c(\rho)$ here. According to the discussion in the Appendix, the characteristic obey the coupled ODEs[3],

$$\frac{dx}{dt} = c(x,t,\rho) \qquad , \qquad \frac{d\rho}{dt} = \sigma(x,t,\rho) \quad . \tag{4.89}$$

In general, the characteristics no longer are straight lines.

4.8.1 Examples

Whitham analyzes the equation

$$c_t + c\,c_x = -\alpha\, c\ , \tag{4.90}$$

so that the characteristics obey

$$\frac{dc}{dt} = -\alpha\, c \qquad , \qquad \frac{dx}{dt} = c\ . \tag{4.91}$$

The solution is

$$\begin{aligned} c_\zeta(t) &= e^{-\alpha t}\, g(\zeta) \\ x_\zeta(t) &= \zeta + \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right) g(\zeta)\ , \end{aligned} \tag{4.92}$$

where $\zeta = x_\zeta(0)$ labels the characteristics. Clearly $x_\zeta(t)$ is not a straight line. Neighboring characteristics will cross at time t if

$$\frac{\partial x_\zeta(t)}{\partial \zeta} = 1 + \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right) g'(\zeta) = 0\ . \tag{4.93}$$

Thus, the break time is

$$t_{\rm B} = \min_{\substack{\zeta \\ t_{\rm B} > 0}} \left[-\frac{1}{\alpha}\, \ln\left(1 + \frac{\alpha}{g'(\zeta)}\right)\right] . \tag{4.94}$$

This requires $g'(\zeta) < -\alpha$ in order for wave breaking to occur.

For another example, consider

$$c_t + c\,c_x = -\alpha\, c^2\ , \tag{4.95}$$

so that the characteristics obey

$$\frac{dc}{dt} = -\alpha\, c^2 \qquad , \qquad \frac{dx}{dt} = c\ . \tag{4.96}$$

[3] We skip the step where we write $dt/ds = 1$ since this is immediately integrated to yield $s = t$.

The solution is now

$$\begin{aligned} c_\zeta(t) &= \frac{g(\zeta)}{1+\alpha\, g(\zeta)\, t} \\ x_\zeta(t) &= \zeta + \frac{1}{\alpha}\,\ln\big(1+\alpha\, g(\zeta)\, t\big)\ . \end{aligned} \tag{4.97}$$

4.8.2 Moving sources

Consider a source moving with velocity u. We then have

$$c_t + c\, c_x = \sigma(x - ut)\ , \tag{4.98}$$

where u is a constant. We seek a moving wave solution $c = c(\xi) = c(x - ut)$. This leads immediately to the ODE

$$(c-u)\, c_\xi = \sigma(\xi)\ . \tag{4.99}$$

This may be integrated to yield

$$\tfrac{1}{2}(u - c_\infty)^2 - \tfrac{1}{2}(u-c)^2 = \int\limits_\xi^\infty \! d\xi'\ \sigma(\xi')\ . \tag{4.100}$$

Consider the supersonic case where $u > c$. Then we have a smooth solution,

$$c(\xi) = u - \left[(u-c_\infty)^2 - 2\int\limits_\xi^\infty \! d\xi'\ \sigma(\xi')\right]^{1/2}\ , \tag{4.101}$$

provided that the term inside the large rectangular brackets is positive. This is always the case for $\sigma < 0$. For $\sigma > 0$ we must require

$$u - c_\infty > \sqrt{2\int\limits_\xi^\infty \! d\xi'\ \sigma(\xi')} \tag{4.102}$$

for all ξ. If $\sigma(\xi)$ is monotonic, the lower limit on the above integral may be extended to $-\infty$. Thus, if the source strength is sufficiently small, no shocks are generated. When the above equation is satisfied as an equality, a shock develops, and transients from the initial conditions overtake the wave. A complete solution of the problem then requires a detailed analysis of the transients. What is surprising here is that a supersonic source need not produce a shock wave, if the source itself is sufficiently weak.

4.9 Burgers' Equation

The simplest equation describing both nonlinear wave propagation and diffusion equation is the one-dimensional *Burgers' equation*,

$$c_t + c\, c_x = \nu\, c_{xx}\ . \tag{4.103}$$

As we've seen, this follows from the continuity equation $\rho_t + j_x$ when $j = J(\rho) - \nu\rho_x$, with $c = J'(\rho)$ and $c''(\rho) = 0$.

We have already obtained, in §4.3.1, a solution to Burgers' equation in the form of a propagating front. However, we can do much better than this; we can find *all* the solutions to the one-dimensional Burgers' equation. The trick is to employ a nonlinear transformation of the field $c(x,t)$, known as the *Cole-Hopf transformation*, which linearizes the PDE. Once again, we follow the exceptionally clear discussion in the book by Whitham (ch. 4).

The Cole-Hopf transformation is defined as follows:

$$c \equiv -2\nu\,\frac{\varphi_x}{\varphi} = \frac{\partial}{\partial x}\big(-2\nu\ln\varphi\big)\ . \tag{4.104}$$

Plugging into Burgers' equation, one finds that $\varphi(x,t)$ satisfies the *linear* diffusion equation,

$$\varphi_t = \nu\,\varphi_{xx}\ . \tag{4.105}$$

Isn't that just about the coolest thing you've ever heard?

Suppose the initial conditions on $\varphi(x,t)$ are

$$\varphi(x,0) = \Phi(x)\ . \tag{4.106}$$

We can then solve the diffusion equation 4.105 by Laplace transform. The result is

$$\varphi(x,t) = \frac{1}{\sqrt{4\pi\nu t}}\int\limits_{-\infty}^{\infty}\!dx'\;e^{-(x-x')^2/4\nu t}\,\Phi(x')\ . \tag{4.107}$$

Thus, if $c(x,t=0) = g(x)$, then the solution for subsequent times is

$$c(x,t) = \frac{\int_{-\infty}^{\infty} dx'\,(x-x')\,e^{-H(x,x',t)/2\nu}}{t\int_{-\infty}^{\infty} dx'\,e^{-H(x,x',t)/2\nu}}\ , \tag{4.108}$$

where

$$H(x,x',t) = \int\limits_0^{x'}\!dx''\,g(x'') + \frac{(x-x')^2}{2t}\ . \tag{4.109}$$

4.9.1 The limit $\nu \to 0$

In the limit $\nu \to 0$, the integrals in the numerator and denominator of Eqn. 4.108 may be computed via the method of steepest descents. This means that extremize $H(x,x',t)$ with respect to x', which entails solving

$$\frac{\partial H}{\partial x'} = g(x') - \frac{x-x'}{t}\ . \tag{4.110}$$

Let $\zeta = \zeta(x,t)$ be a solution to this equation for x', so that

$$x = \zeta + g(\zeta)\, t \ . \tag{4.111}$$

We now expand about $x' = \zeta$, writing $x' = \zeta + s$, in which case

$$H(x') = H(\zeta) + \tfrac{1}{2} H''(\zeta)\, s^2 + \mathcal{O}(s^3) \ , \tag{4.112}$$

where the x and t dependence is here implicit. If $F(x')$ is an arbitrary function which is slowly varying on distance scales on the order of $\nu^{1/2}$, then we have

$$\int\limits_{-\infty}^{\infty} \! dx'\, F(x')\, e^{-H(x')/2\nu} \approx \sqrt{\frac{4\pi\nu}{H''(\zeta)}}\, e^{-H(\zeta)/2\nu}\, F(\zeta) \ . \tag{4.113}$$

Applying this result to Eqn. 4.108, we find

$$c \approx \frac{x-\zeta}{t} \ , \tag{4.114}$$

which is to say

$$\begin{aligned} c &= g(\zeta) \\ x &= \zeta + g(\zeta)\, t \ . \end{aligned} \tag{4.115}$$

This is precisely what we found for the characteristics of $c_t + c\, c_x = 0$.

What about multivaluedness? This is obviated by the presence of an additional saddle point solution. *I.e.* beyond some critical time, we have a discontinuous change of saddles as a function of x:

$$x = \zeta_\pm + g(\zeta_\pm)\, t \quad \longrightarrow \quad \zeta_\pm = \zeta_\pm(x,t) \ . \tag{4.116}$$

Then

$$c \sim \frac{1}{t} \cdot \frac{\frac{x-\zeta_-}{\sqrt{H''(\zeta_-)}}\, e^{-H(\zeta_-)/2\nu} + \frac{x-\zeta_+}{\sqrt{H''(\zeta_+)}}\, e^{-H(\zeta_+)/2\nu}}{\frac{1}{\sqrt{H''(\zeta_-)}}\, e^{-H(\zeta_-)/2\nu} + \frac{1}{\sqrt{H''(\zeta_+)}}\, e^{-H(\zeta_+)/2\nu}} \ . \tag{4.117}$$

Thus,

$$\begin{aligned} H(\zeta_+) > H(\zeta_-) \qquad &\Rightarrow \qquad c \approx \frac{x-\zeta_-}{t} \\ H(\zeta_+) < H(\zeta_-) \qquad &\Rightarrow \qquad c \approx \frac{x-\zeta_+}{t} \ . \end{aligned} \tag{4.118}$$

At the shock, these solutions are degenerate:

$$H(\zeta_+) = H(\zeta_-) \qquad \Rightarrow \qquad \tfrac{1}{2}(\zeta_+ - \zeta_-)\big(g(\zeta_+) + g(\zeta_-)\big) = \int\limits_{\zeta_-}^{\zeta_+} \! d\zeta\, g(\zeta) \ , \tag{4.119}$$

which is again exactly as before. We stress that for ν small but finite the shock fronts are smoothed out on a distance scale proportional to ν.

What does it mean for ν to be small? The dimensions of ν are $[\nu] = L^2/T$, so we must find some other quantity in the problem with these dimensions. The desired quantity is the area,

$$A = \int\limits_{-\infty}^{\infty} dx \, \big[g(x) - c_0 \big] \ , \tag{4.120}$$

where $c_0 = c(x = \pm\infty)$. We can now define the dimensionless ratio,

$$\mathsf{R} \equiv \frac{A}{2\nu} \ , \tag{4.121}$$

which is analogous to the Reynolds number in viscous fluid flow. R is proportional to the ratio of the nonlinear term $(c - c_0)\, c_x$ to the diffusion term νc_{xx}.

4.9.2 Examples

Whitham discusses three examples: diffusion of an initial step, a hump, and an N-wave. Here we simply reproduce the functional forms of these solutions. For details, see chapter 4 of Whitham's book.

For an initial step configuration,

$$c(x, t = 0) = \begin{cases} c_1 & \text{if } x < 0 \\ c_2 & \text{if } x > 0 \ . \end{cases} \tag{4.122}$$

We are interested in the case $c_1 > c_2$. Using the Cole-Hopf transformation and applying the appropriate initial conditions to the resulting linear diffusion equation, one obtains the complete solution,

$$c(x, t) = c_2 + \frac{c_1 - c_2}{1 + h(x, t) \ \exp\big[(c_1 - c_2)(x - v_{\rm s} t)/2\nu\big]} \ , \tag{4.123}$$

where

$$v_{\rm s} = \tfrac{1}{2}(c_1 + c_2) \tag{4.124}$$

and

$$h(x, t) = \frac{\text{erfc}\Big(-\frac{x - c_2 t}{\sqrt{4\nu t}}\Big)}{\text{erfc}\Big(+\frac{x - c_1 t}{\sqrt{4\nu t}}\Big)} \ . \tag{4.125}$$

Recall that $\text{erfc}(z)$ is the complimentary error function:

$$\begin{aligned} \text{erf}(z) &= \frac{2}{\sqrt{\pi}} \int\limits_0^z du \, e^{-u^2} \\ \text{erfc}(z) &= \frac{2}{\sqrt{\pi}} \int\limits_z^\infty du \, e^{-u^2} = 1 - \text{erf}(z) \ . \end{aligned} \tag{4.126}$$

Note the limiting values $\mathrm{erfc}(-\infty) = 2$, $\mathrm{erfc}(0) = 1$ and $\mathrm{erfc}(\infty) = 0$. If $c_2 < x/t < c_1$, then $h(x,t) \to 1$ as $t \to \infty$, in which case the solution resembles a propagating front. It is convenient to adimensionalize $(x,t) \to (y,\tau)$ by writing

$$x = \frac{\nu\, y}{\sqrt{c_1 c_2}} \quad , \quad t = \frac{\nu\, \tau}{c_1 c_2} \quad , \quad r \equiv \sqrt{\frac{c_1}{c_2}} \, . \tag{4.127}$$

We then have

$$\frac{c(z,\tau)}{\sqrt{c_1 c_2}} = r^{-1} + \frac{2\alpha}{1 + h(z,\tau)\, \exp(\alpha z)} \, , \tag{4.128}$$

where

$$h(z,\tau) = \mathrm{erfc}\bigg(-\frac{z+\alpha\tau}{2\sqrt{\tau}}\bigg) \bigg/ \mathrm{erfc}\bigg(+\frac{z-\alpha\tau}{2\sqrt{\tau}}\bigg) \tag{4.129}$$

and

$$\alpha \equiv \tfrac{1}{2}\big(r - r^{-1}\big) \quad , \quad z \equiv y - \tfrac{1}{2}\big(r + r^{-1}\big)\, \tau \, . \tag{4.130}$$

The second example involves the evolution of an infinitely thin hump, where

$$c(x, t=0) = c_0 + A\, \delta(x) \, . \tag{4.131}$$

The solution for subsequent times is

$$c(x,t) = c_0 + \sqrt{\frac{\nu}{\pi t}} \cdot \frac{(e^{\mathsf{R}} - 1)\, \exp\Big(-\frac{x-c_0 t}{4\nu t}\Big)}{1 + \frac{1}{2}(e^{\mathsf{R}} - 1)\, \mathrm{erfc}\Big(\frac{x - c_0 t}{\sqrt{4\nu t}}\Big)} \, , \tag{4.132}$$

where $\mathsf{R} = A/2\nu$. Defining

$$z \equiv \frac{x - c_0 t}{\sqrt{2At}} \, , \tag{4.133}$$

we have the solution

$$c = c_0 + \bigg(\frac{2A}{t}\bigg)^{1/2} \frac{1}{\sqrt{4\pi\mathsf{R}}} \cdot \frac{(e^{\mathsf{R}} - 1)\, e^{-\mathsf{R} z^2}}{1 + \frac{1}{2}(e^{\mathsf{R}} - 1)\, \mathrm{erfc}(\sqrt{\mathsf{R}}\, z)} \, . \tag{4.134}$$

Asymptotically, for $t \to \infty$ with x/t fixed, we have

$$c(x,t) = \begin{cases} x/t & \text{if } 0 < x < \sqrt{2At} \\ 0 & \text{otherwise} \, . \end{cases} \tag{4.135}$$

This recapitulates the triangular wave solution with the two counterpropagating shock fronts and dissipating shock strengths.

Finally, there is the N-wave. If we take the following solution to the linear diffusion equation,

$$\varphi(x,t) = 1 + \sqrt{\frac{a}{t}}\, e^{-x^2/4\nu t} \, , \tag{4.136}$$

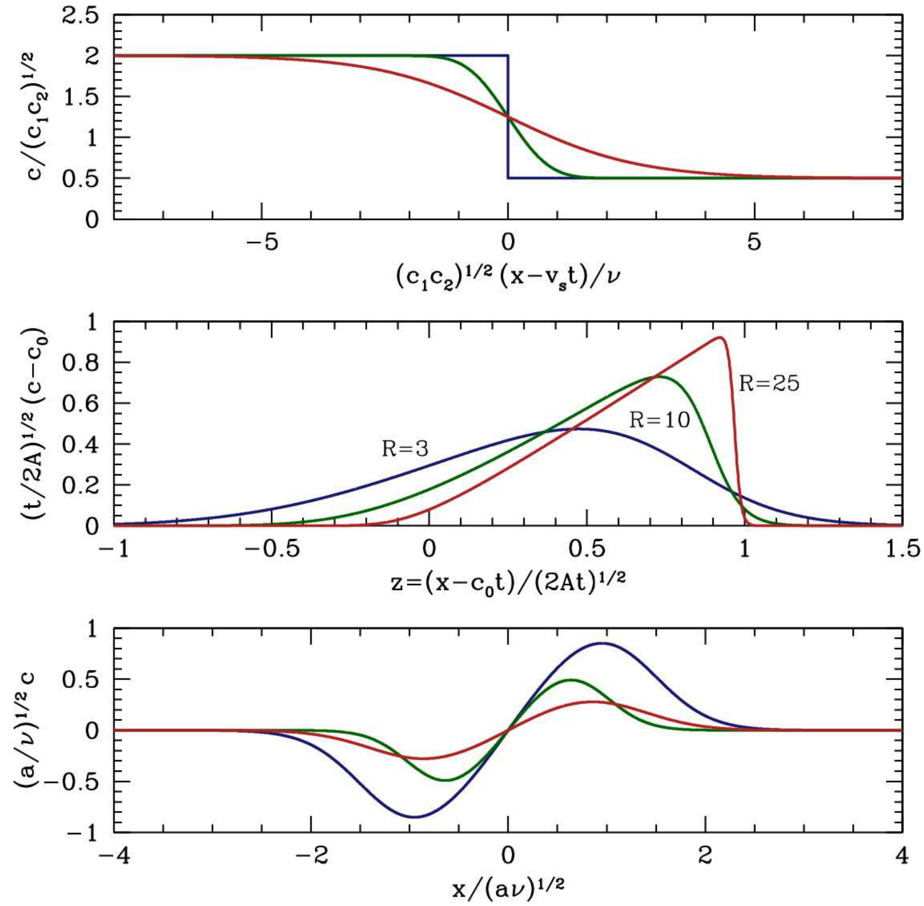

Figure 4.11: Evolution of profiles for Burgers' equation. Top : a step discontinuity evolving into a front at times $\tau = 0$ (blue), $\tau = \frac{1}{5}$ (green), and $\tau = 5$ (red).. Middle : a narrow hump $c_0 + A\delta(x)$ evolves into a triangular wave. Bottom : dissipation of an N-wave at times $\tau = \frac{1}{4}$ (blue), $\tau = \frac{1}{2}$ (green), and $\tau = 1$ (red).

then we obtain

$$c(x,t) = \frac{x}{t} \cdot \frac{e^{-x^2/4\nu t}}{\sqrt{\frac{t}{a}} + e^{-x^2/4\nu t}} \ . \tag{4.137}$$

In terms of dimensionless variables (y, τ), where

$$x = \sqrt{a\nu}\; y \qquad , \qquad t = a\tau \ , \tag{4.138}$$

we have

$$c = \sqrt{\frac{\nu}{a}}\, \frac{y}{\tau} \cdot \frac{e^{-y^2/4\tau}}{\sqrt{\tau} + e^{-y^2/4\tau}} \ . \tag{4.139}$$

The evolving profiles for these three cases are plotted in Fig. 4.11.

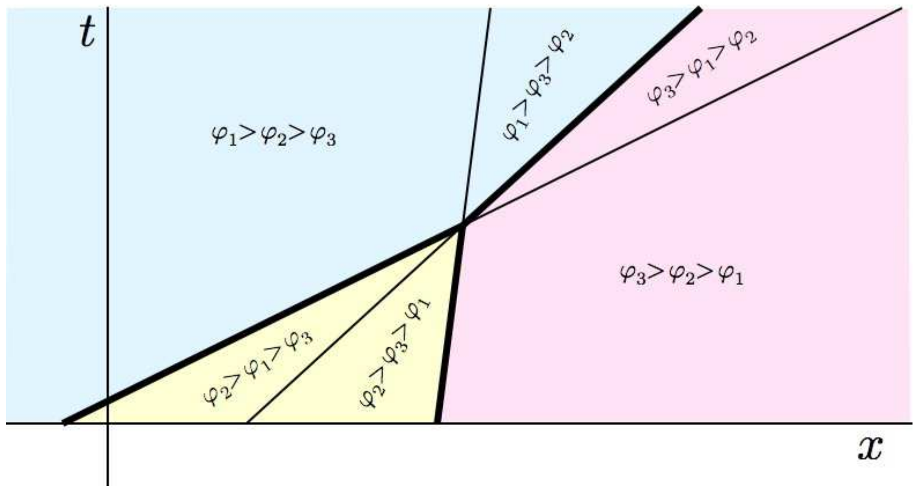

Figure 4.12: Merging of two shocks for piecewise constant initial data. The (x,t) plane is broken up into regions labeled by the local value of $c(x,t)$. For the shocks to form, we require $c_1 > c_2 > c_3$. When the function $\varphi_j(x,t)$ dominates over the others, then $c(x,t) \approx c_j$.

4.9.3 Confluence of shocks

The fact that the diffusion equation 4.105 is linear means that we can superpose solutions:

$$\varphi(x,t) = \varphi_1(x,t) + \varphi_2(x,t) + \ldots + \varphi_N(x,t) \;, \tag{4.140}$$

where

$$\varphi_j(x,t) = e^{-c_j(x-b_j)/2\nu}\, e^{+c_j^2 t/4\nu} \;. \tag{4.141}$$

We then have

$$c(x,t) = -\frac{2\nu\varphi_x}{\varphi} = \frac{\sum_i c_i\, \varphi_i(x,t)}{\sum_i \varphi_i(x,t)} \;. \tag{4.142}$$

Consider the case $N = 2$, which describes a single shock. If $c_1 > c_2$, then at a fixed time t we have that φ_1 dominates as $x \to -\infty$ and φ_2 as $x \to +\infty$. Therefore $c(-\infty,t) = c_1$ and $c(+\infty) = c_2$. The shock center is defined by $\varphi_1 = \varphi_2$, where $x = \frac{1}{2}(c_1 + c_2)\, t$.

Next consider $N = 3$, where there are two shocks. We assume $c_1 > c_2 > c_3$. We identify regions in the (x,t) plane where φ_1, φ_2, and φ_3 are dominant. One finds

$$\begin{aligned}
\varphi_1 > \varphi_2 \quad &: \quad x < \tfrac{1}{2}(c_1 + c_2)\, t + \frac{b_1 c_1 - b_2 c_2}{c_1 - c_2} \\
\varphi_1 > \varphi_3 \quad &: \quad x < \tfrac{1}{2}(c_1 + c_3)\, t + \frac{b_1 c_1 - b_3 c_3}{c_1 - c_3} \\
\varphi_2 > \varphi_3 \quad &: \quad x < \tfrac{1}{2}(c_2 + c_3)\, t + \frac{b_2 c_2 - b_3 c_3}{c_2 - c_3} \;.
\end{aligned} \tag{4.143}$$

These curves all meet in a single point at $(x_{\mathrm{m}}, t_{\mathrm{m}})$, as shown in Fig. 4.12. The shocks are the locus of points along which two of the φ_j are equally dominant. We assume that the intercepts of these lines with the x-axis are ordered as in the figure, with $x^*_{12} < x^*_{13} < x^*_{23}$, where

$$x^*_{ij} \equiv \frac{b_i c_i - b_j c_j}{c_i - c_j} \ . \tag{4.144}$$

When a given $\varphi_i(x,t)$ dominates over the others, we have from Eqn. 4.142 that $c \approx c_i$. We see that for $t < t^*$ one has that φ_1 is dominant for $x < x^*_{12}$, and φ_3 is dominant for $x > x^*_{23}$, while φ_2 dominates in the intermediate regime $x^*_{12} < x < x^*_{23}$. The boundaries between these different regions are the two propagating shocks. After the merge, for $t > t_{\mathrm{m}}$, however, φ_2 *never dominates*, and hence there is only one shock.

4.10 Appendix I : The Method of Characteristics

Consider the quasilinear PDE

$$a_1(\boldsymbol{x},\phi)\,\frac{\partial\phi}{\partial x_1} + a_2(\boldsymbol{x},\phi)\,\frac{\partial\phi}{\partial x_2} + \ldots + a_N(\boldsymbol{x},\phi)\,\frac{\partial\phi}{\partial x_N} = b(\boldsymbol{x},\phi) \ . \tag{4.145}$$

This PDE is called 'quasilinear' because it is linear in the derivatives $\partial\phi/\partial x_j$. The N independent variables are the elements of the vector $\boldsymbol{x} = (x_1, \ldots, x_N)$. A solution is a function $\phi(\boldsymbol{x})$ which satisfies the PDE.

Now consider a curve $\boldsymbol{x}(s)$ parameterized by a single real variable s satisfying

$$\frac{dx_j}{ds} = a_j\big(\boldsymbol{x}, \phi(\boldsymbol{x})\big) \ , \tag{4.146}$$

where $\phi(\boldsymbol{x})$ is a solution of Eqn. 4.145. Along such a curve, which is called a *characteristic*, the variation of ϕ is

$$\frac{d\phi}{ds} = \sum_{j=1}^N \frac{\partial\phi}{\partial x_j}\,\frac{dx_j}{ds} = b\big(\boldsymbol{x}(s),\phi\big) \ . \tag{4.147}$$

Thus, we have converted our PDE into a set of $(N+1)$ ODEs. To integrate, we must supply some initial conditions of the form

$$g\big(\boldsymbol{x},\phi\big)\Big|_{s=0} = 0 \ . \tag{4.148}$$

This defines an $(N-1)$-dimensional hypersurface, parameterized by $\{\zeta_1, \ldots, \zeta_{N-1}\}$:

$$\begin{aligned} x_j(s=0) &= h_j(\zeta_1, \ldots, \zeta_{N-1}) \qquad , \qquad j = 1, \ldots, N \\ \phi(s=0) &= f(\zeta_1, \ldots, \zeta_{N-1}) \ . \end{aligned} \tag{4.149}$$

If we can solve for all the characteristic curves, then the solution of the PDE follows. For every $\boldsymbol{x}$, we identify the characteristic curve upon which $\boldsymbol{x}$ lies. The characteristics are identified by their parameters $(\zeta_1, \ldots, \zeta_{N-1})$. The value of $\phi(\boldsymbol{x})$ is then $\phi(\boldsymbol{x}) = f(\zeta_1, \ldots, \zeta_{N-1})$. If two or more characteristics cross, the solution is multi-valued, or a shock has occurred.

4.10.1 Example

Consider the PDE

$$\phi_t + t^2\,\phi_x = -x\,\phi\ . \tag{4.150}$$

We identify $a_1(t,x,\phi) = 1$ and $a_2(t,x,\phi) = t^2$, as well as $b(t,x,\phi) = -x\,\phi$. The characteristics are curves $\big(t(s), x(s)\big)$ satisfing

$$\frac{dt}{ds} = 1 \qquad , \qquad \frac{dx}{ds} = t^2\ . \tag{4.151}$$

The variation of ϕ along each of the characteristics is given by

$$\frac{d\phi}{ds} = -x\,\phi\ . \tag{4.152}$$

The initial data are expressed parametrically as

$$t(s=0) = 0 \qquad , \qquad x(s=0) = \zeta \qquad , \qquad \phi(s=0) = f(\zeta)\ . \tag{4.153}$$

We now solve for the characteristics. We have

$$\frac{dt}{ds} = 1 \quad \Rightarrow \quad t(s,\zeta) = s\ . \tag{4.154}$$

It then follows that

$$\frac{dx}{ds} = t^2 = s^2 \quad \Rightarrow \quad x(s,\zeta) = \zeta + \tfrac{1}{3}s^3\ . \tag{4.155}$$

Finally, we have

$$\frac{d\phi}{ds} = -x\,\phi = -\big(\zeta + \tfrac{1}{3}s^3\big)\,\phi \quad \Rightarrow \quad \phi(s,\zeta) = f(\zeta)\,\exp\big(-\tfrac{1}{12}s^4 - s\zeta\big)\ . \tag{4.156}$$

We may now eliminate (ζ, s) in favor of (x,t), writing $s = t$ and $\zeta = x - \frac{1}{3}t^3$, yielding the solution

$$\phi(x,t) = f\big(x - \tfrac{1}{3}t^3\big)\,\exp\big(\tfrac{1}{4}t^4 - xt\big)\ . \tag{4.157}$$

4.11 Appendix II : Worked Problem in Shock Fitting

Consider the equation $c_t + c\,c_x = 0$. Suppose the $c(\rho)$ and $\rho(x, t=0)$ are such that $c(x, t=0)$ is given by

$$c(x,0) = c_0 \cos\left(\frac{\pi x}{2\ell}\right)\Theta\big(\ell - |x|\big)\ , \tag{4.158}$$

where $\Theta(s)$ is the step function, which vanishes identically for negative values of its argument. Thus, $c(x,0) = 0$ for $|x| \geq \ell$.

(a) Find the time $t_{\rm B}$ at which the wave breaks and a shock front develops. Find the position of the shock $x_{\rm s}(t_{\rm B})$ at the moment it forms.

Solution : Breaking first occurs at time

$$t_B = \min_x \bigg(-\frac{1}{c'(x,0)} \bigg) \ . \tag{4.159}$$

Thus, we look for the maximum negative slope in $g(x) \equiv c(x,0)$, which occurs at $x = \ell$, where $c'(\ell,0) = -\pi c_0/2\ell$. Therefore,

$$t_B = \frac{2\ell}{\pi c_0} \quad , \quad x_B = \ell \ . \tag{4.160}$$

(b) Use the shock-fitting equations to derive $\zeta_\pm(t)$.

Solution : The shock fitting equations are

$$\tfrac{1}{2}\,(\zeta_+ - \zeta_-)\,\Big(g(\zeta_+) + g(\zeta_-)\Big) = \int\limits_{\zeta_-}^{\zeta_+}\! d\zeta\; g(\zeta) \tag{4.161}$$

and

$$t = \frac{\zeta_+ - \zeta_-}{g(\zeta_-) - g(\zeta_+)} \ . \tag{4.162}$$

Clearly $\zeta_+ > \ell$, hence $g(\zeta_+) = 0$ and

$$\int\limits_{\zeta_-}^{\zeta_+}\! d\zeta\; g(\zeta) = c_0 \cdot \frac{2\ell}{\pi} \int\limits_{\pi\zeta_-/2\ell}^{\pi/2}\! dz\; \cos z = \frac{2\ell\, c_0}{\pi} \left\{ 1 - \sin\!\bigg(\frac{\pi\zeta_-}{2\ell}\bigg) \right\} . \tag{4.163}$$

Thus, the first shock fitting equation yields

$$\tfrac{1}{2}\,(\zeta_+ - \zeta_-)\, c_0 \cos\!\bigg(\frac{\pi\zeta_-}{2\ell}\bigg) = \frac{2\ell\, c_0}{\pi} \left\{ 1 - \sin\!\bigg(\frac{\pi\zeta_-}{2\ell}\bigg) \right\} . \tag{4.164}$$

The second shock fitting equation gives

$$t = \frac{\zeta_+ - \zeta_-}{c_0 \cos\big(\frac{\pi\zeta_-}{2\ell}\big)} \ . \tag{4.165}$$

Eliminating $\zeta_+ - \zeta_-$, we obtain the relation

$$\sin\!\bigg(\frac{\pi\zeta_-}{2\ell}\bigg) = \frac{4\ell}{\pi c_0 t} - 1 \ . \tag{4.166}$$

Thus,

$$\zeta_-(t) = \frac{2\ell}{\pi}\, \sin^{-1}\!\bigg(\frac{4\ell}{\pi c_0 t} - 1\bigg) \tag{4.167}$$

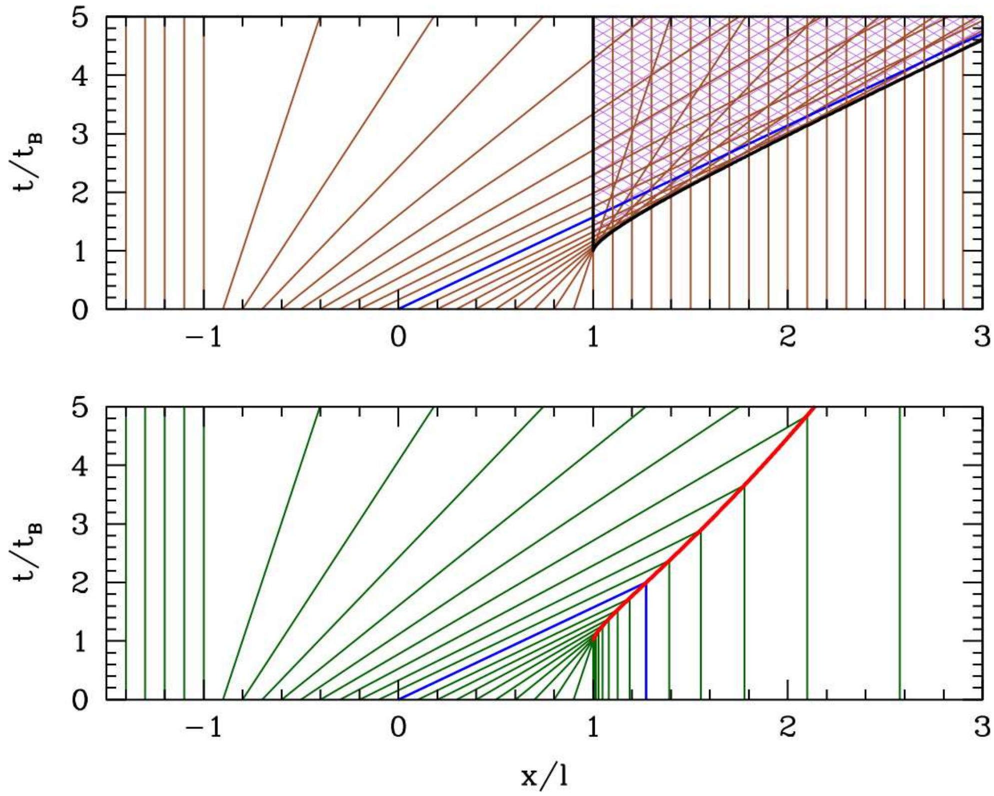

Figure 4.13: Top : crossing characteristics (purple hatched region) in the absence of shock fitting. Bottom : characteristics in the presence of the shock.

and

$$\begin{aligned}\zeta_+(t) &= \zeta_- + \frac{4\ell}{\pi} \cdot \frac{1-\sin(\pi\zeta_-/2\ell)}{\cos(\pi\zeta_-/2\ell)} \\ &= \frac{2\ell}{\pi}\left\{ \sin^{-1}\!\left(\frac{4\ell}{\pi c_0 t} - 1\right) + 2\sqrt{\frac{\pi c_0 t}{2\ell} - 1}\,\right\},\end{aligned} \tag{4.168}$$

where $t \geq t_{\mathrm{B}} = 2\ell/\pi c_0$.

(c) Find the shock motion $x_s(t)$.

Solution : The shock position is

$$\begin{aligned}x_{\mathrm{s}}(t) &= \zeta_- + g(\zeta_-)\, t \\ &= \frac{2\ell}{\pi} \sin^{-1}\!\left(\frac{2}{\tau} - 1\right) + \frac{4\ell}{\pi}\sqrt{\tau - 1}\,,\end{aligned} \tag{4.169}$$

where $\tau = t/t_{\mathrm{B}} = \pi c_0 t/2\ell$, and $\tau \geq 1$.

(d) Sketch the characteristics for the multivalued solution with no shock fitting, identifying the region in (x, t) where characteristics cross. Then sketch the characteristics for the discontinuous shock solution.

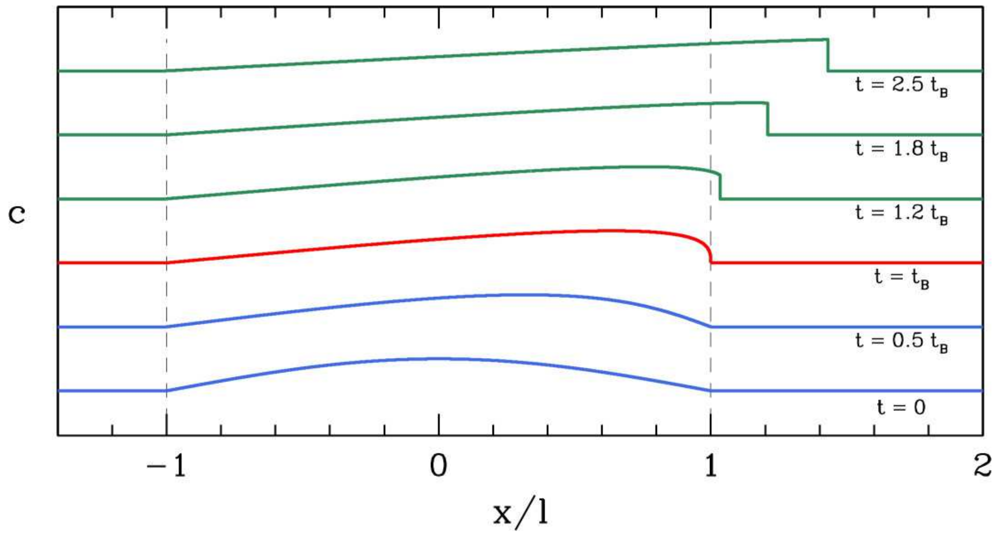

Figure 4.14: Evolution of $c(x,t)$ for a series of time values.

Solution : See Fig. 4.13.

(e) Find the shock discontinuity $\Delta c(t)$.

Solution : The shock discontinuity is

$$\begin{aligned}\Delta c(t) &= g(\zeta_-) - g(\zeta_+) = c_0 \cos\left(\frac{\pi\zeta_-}{2\ell}\right) \\ &= \sqrt{\frac{8\ell\, c_0}{\pi t}\left(1 - \frac{2\ell}{\pi c_0 t}\right)} = 2c_0\,\frac{\sqrt{\tau - 1}}{\tau}\ .\end{aligned} \tag{4.170}$$

(f) Find the shock velocity $v_s(t)$.

Solution : The shock wave velocity is

$$\begin{aligned}c_s(t) &= \tfrac{1}{2}\Big[g(\zeta_-) + g(\zeta_+)\Big] = \tfrac{1}{2}\,\Delta c(t) \\ &= \sqrt{\frac{2\ell\, c_0}{\pi t}\left(1 - \frac{2\ell}{\pi c_0 t}\right)}\ .\end{aligned} \tag{4.171}$$

(g) Sketch the evolution of the wave, showing the breaking of the wave at $t = t_B$ and the subsequent evolution of the shock front.

Solution : A sketch is provided in Fig. 4.14.

4.12 Appendix III : The Traffic Light

Problem : Consider vehicular traffic flow where the flow velocity is given by

$$u(\rho) = u_0 \left(1 - \frac{\rho^2}{\rho_{\rm BB}^2}\right) . \tag{4.172}$$

Solve for the density $\rho(x,t)$ with the initial conditions appropriate to a green light starting at $t = 0$, *i.e.* $\rho(x,0) = \rho_{\rm BB}\,\Theta(-x)$. Solve for the motion of a vehicle in the flow, assuming initial position $x(0) = x_0 < 0$. How long does it take the car to pass the light?

Solution : Consider the one-parameter family of velocity functions,

$$u(\rho) = u_0 \left(1 - \frac{\rho^\alpha}{\rho_{\rm BB}^\alpha}\right) , \tag{4.173}$$

where $\alpha > 0$. The speed of wave propagation is

$$\begin{aligned} c(\rho) &= u(\rho) + \rho\, u'(\rho) \\ &= u_0 \left(1 - (1+\alpha)\,\frac{\rho^\alpha}{\rho_{\rm BB}^\alpha}\right) . \end{aligned} \tag{4.174}$$

The characteristics are shown in figure 4.15. At $t = 0$, the characteristics emanating from the x-axis have slope $-1/(\alpha u_0)$ for $x < 0$ and slope $+1/u_0$ for $x > 0$. This is because the slope is $1/c$ (we're plotting t on the y-axis), and $c(0) = u_0$ while $c(\rho_{\rm BB}) = -\alpha u_0$. Interpolating between these two sets is the fan region, shown in blue in the figure. All characteristics in the fan emanate from $x = 0$, where the density is discontinuous at $t = 0$. Hence, the entire interval $[0, \rho_{\rm BB}]$ is represented at $t = 0$.

For the fan region, the characteristics satisfy

$$c(\rho) = \frac{x}{t} \quad \Rightarrow \quad \frac{\rho^\alpha}{\rho_{\rm BB}^\alpha} = \frac{1}{1+\alpha}\left(1 - \frac{x}{u_0\, t}\right) . \tag{4.175}$$

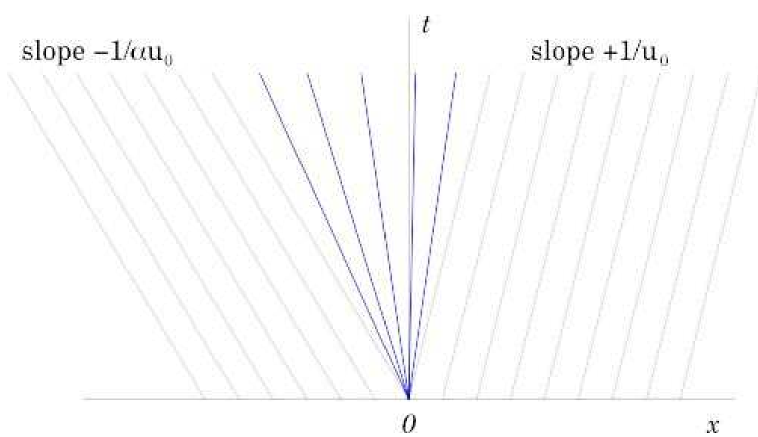

Figure 4.15: Characteristics for the green light problem. For $x < 0$, the density at $t = 0$ is $\rho = \rho_{\rm BB}$, and $c(x < 0, t = 0) = -\alpha\, u_0$. For $x > 0$, the density at $t = 0$ is $\rho = 0$ and $c(x > 0, t = 0) = u_0$.

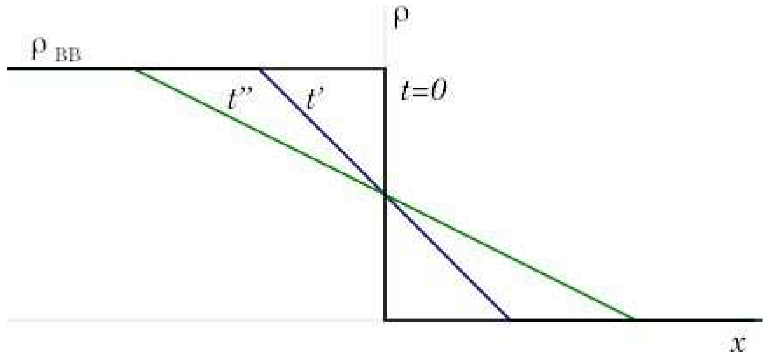

Figure 4.16: Density *versus* position for the green light problem at three successive times. The initial discontinuity at $x = 0$ is smoothed into a straight line interpolating between $\rho = \rho_{\rm BB}$ and $\rho = 0$. Because $dc/dx > 0$ for this problem, the initial profile smooths out. When $dc/dx < 0$, the wave steepens.

This is valid throughout the region $-\alpha u_0 < x/t < u_0$. The motion of a vehicle in this flow is determined by the equation

$$\frac{dx}{dt} = u(\rho) = \frac{\alpha u_0}{1+\alpha} + \frac{1}{1+\alpha}\,\frac{x}{t} \,, \tag{4.176}$$

which is obtained by eliminating ρ in terms of the ratio x/t. Thus, we obtain the linear, non-autonomous, inhomogeneous equation

$$t\,\frac{dx}{dt} - \frac{x}{1+\alpha} = \frac{\alpha u_0 t}{1+\alpha} \,, \tag{4.177}$$

whose solution is

$$x(t) = \mathcal{C}\, t^{1/(1+\alpha)} + u_0 t \,, \tag{4.178}$$

where $\mathcal{C}$ is a constant, fixed by the initial conditions.

What are the initial conditions? Consider a vehicle starting at $x(t = 0) = -|x_0|$. It remains stationary until a time $t^* = |x_0|/\alpha u_0$, at which point $x/t^* = -\alpha u_0$, and it enters the fan region. At this point (see figure 4.16), the trailing edge of the smoothing front region, which was absolutely sharp and discontinuous at $t = 0$, passes by the vehicle, the local density decreases below $\rho_{\rm BB}$, and it starts to move. We solve for $\mathcal{C}$ by setting

$$x(t^*) = -|x_0| \quad \Rightarrow \quad \mathcal{C} = -(1+\alpha)\,\alpha^{-\alpha/(1+\alpha)}\,|x_0|^{\alpha/(1+\alpha)}\,u_0^{1/(1+\alpha)} \,. \tag{4.179}$$

Thus,

$$\begin{aligned} x(t) &= -\,|x_0| \qquad (t < t^* = |x_0|/\alpha u_0) \\ &= u_0 t - (1+\alpha)\,\alpha^{-\alpha/(1+\alpha)}\,|x_0|^{\alpha/(1+\alpha)}\,(u_0\,t)^{1/(1+\alpha)} \qquad (t > t^*) \,. \end{aligned} \tag{4.180}$$

Finally, we set $x(t_{\rm c}) = 0$ to obtain

$$x(t_{\rm c}) = 0 \quad \Rightarrow \quad t_{\rm c} = \frac{1}{\alpha}\,(1+\alpha)^{1+\alpha^{-1}} \cdot \frac{|x_0|}{u_0} \,. \tag{4.181}$$

For $\alpha = 2$, find $t_{\rm c} = \frac{3\sqrt{3}}{2}\,|x_0|/u_0$.

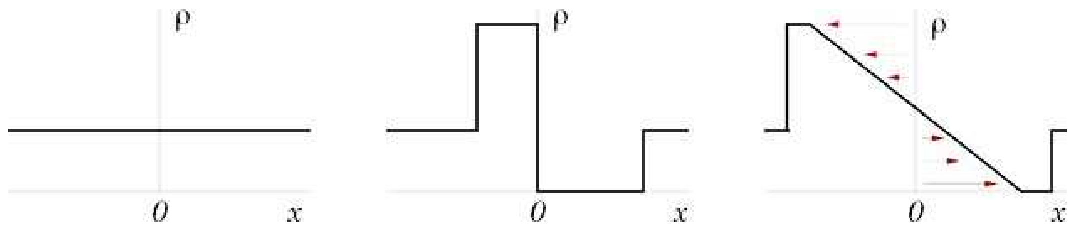

Figure 4.17: Evolution of traffic density before, during, and after a red light.

4.13 Appendix IV : Characteristics for a Periodic Signal

Problem : Consider traffic flow in the presence of a signal on a single-lane roadway. Assume initial conditions $\rho(x,0)=\rho_1$. At time $t=0$, the signal turns red for an interval $T_{\rm red}=r\,T$, after which the signal is green for $T_{\rm green}=g\,T$, with $r+g=1$. The cycle length is T. (Traffic engineers call g the 'green-to-cycle ratio'.) It is useful to work with dimensionless quantities,

$$\bar\rho\equiv\rho/\rho_{\rm BB}\qquad \bar c\equiv c/u_0\qquad \bar x\equiv x/u_0 T \tag{4.182}$$

$$\bar j\equiv j/u_0\,\rho_{\rm BB}\qquad \bar u\equiv u/u_0\qquad \tau\equiv t/T\ .$$

The continuity equation is then

$$\frac{\partial\bar\rho}{\partial\tau}+\bar c(\bar\rho)\,\frac{\partial\bar\rho}{\partial\bar x}=0\ . \tag{4.183}$$

Note : Assume $\bar j(\bar\rho)=\bar\rho(1-\bar\rho)$, so

$$\bar u(\bar\rho)=1-\bar\rho\qquad,\qquad \bar c(\bar\rho)=1-2\bar\rho\ . \tag{4.184}$$

(a) During the red phase, two shocks propagate – one behind the light (*i.e.* $\bar x_{\rm s}^-<0$) and one ahead of the light (*i.e.* $\bar x_{\rm s}^+>0$). Find the velocities and discontinuities at these shocks.

Solution : During the red phase, $0\le\tau\le r$, a shock is formed behind the light at the boundary of the $\bar\rho=\bar\rho_1$ and $\bar\rho=1$ regions. The velocity of this shock is

$$v_{\rm s}=\frac{\bar j(1)-\bar j(\bar\rho_1)}{1-\bar\rho_1}=-\bar\rho_1\ . \tag{4.185}$$

This shock forms immediately upon the appearance of the red light, hence $\bar x_{\rm s}^-(\tau)=-\bar\rho_1\,\tau$. Another shock forms ahead of the light, at the boundary between $\bar\rho=0$ and $\bar\rho=\bar\rho_1$ regions. There,

$$v_{\rm s}=\frac{\bar j(\bar\rho_1)-\bar j(0)}{\bar\rho_1-0}=1-\bar\rho_1\ . \tag{4.186}$$

Thus, $\bar x_{\rm s}^+(\tau)=(1-\bar\rho_1)\,\tau$.

(b) When the light changes to green, the characteristics develop a fan, and the discontinuity at $x=0$ starts to spread. Let $\bar x_>(\tau)$ denote the minimum value of $\bar x$ for which $\bar\rho(\tau)=0$, and $\bar x_<(\tau)$ the maximum value of $\bar{x}$ for which $\bar{\rho}(\tau) = 1$. Show that $\bar{x}_>$ overtakes $\bar{x}_\text{s}^+$ after a time τ_+ and that $\bar{x}_<$ overtakes $\bar{x}_\text{s}^-$ after a time τ_-. Compute $\tau_\pm$.

Solution : The situation is depicted in figure 4.17. At $\tau = r$, the light turns green, and the queue behind the light is released. The discontinuity at $\bar{x} = 0$ spreads as shown in the figure. (Recall that regions with $dc/dx > 0$ spread while those with $dc/dx < 0$ steepen.) The boundaries of the spreading region are $\bar{x}_>(\tau) = \tau - r$ and $\bar{x}_<(\tau) = r - \tau$, and travel at speeds $\bar{c}_> = +1$ and $\bar{c}_< = -1$. Setting $\bar{x}_\text{s}^+(\tau_+) = \bar{x}_>(\tau_+)$ gives $\tau_+ = r/\bar{\rho}_1$. Similarly, setting $\bar{x}_\text{s}^-(\tau_-) = \bar{x}_<(\tau_-)$ gives $\tau_- = r/(1-\bar{\rho}_1)$. For light traffic ($\bar{\rho}_1 < \frac{1}{2}$), we have $\tau_- < \tau_+$, while for heavy traffic ($\bar{\rho}_1 > \frac{1}{2}$), $\tau_- > \tau_+$.

(c) Impose the proper shock conditions at $\bar{x}_\text{s}^\pm$ for $\tau > \tau_\pm$, respectively. Show that each shock obeys the equation

$$\frac{d\bar{x}_\text{s}}{d\tau} = \tfrac{1}{2} - \bar{\rho}_1 + \frac{\bar{x}_\text{s}}{2(\tau - r)} \ .$$

Apply the proper boundary conditions for the ahead and behind shocks and obtain explicit expressions for $\bar{x}_\text{s}^\pm(\tau)$ for all $\tau \geq 0$. (You will have to break up your solution into different cases, depending on τ.)

Solution : For a shock discontinuity between $\bar{\rho} = \bar{\rho}_-$ and $\bar{\rho} = \bar{\rho}_+$, we have in general

$$v_s = \frac{\bar{j}_+ - \bar{j}_-}{\bar{\rho}_+ - \bar{\rho}_-} = 1 - \bar{\rho}_+ - \bar{\rho}_- \ . \tag{4.187}$$

Let's first apply this at the shock $\bar{x}_\text{s}^+$ for $\tau > \tau_+$. For $x < \bar{x}_\text{s}^+$, we are in a fan region of the characteristics, with

$$\bar{x}(\tau) = \bar{c}(\bar{\rho})(\tau - r) = (1 - 2\bar{\rho})\,(\tau - r) \ . \tag{4.188}$$

Thus,

$$\bar{\rho} = \tfrac{1}{2}\bigg(1 - \frac{\bar{x}}{\tau - r}\bigg) \ . \tag{4.189}$$

Now invoke the shock condition at $\bar{x} = \bar{x}_\text{s}^+$:

$$\begin{aligned}\frac{d\bar{x}_\text{s}^+}{d\tau} &= 1 - \bar{\rho}_1 - \tfrac{1}{2}\bigg(1 - \frac{\bar{x}_\text{s}^+}{\tau - r}\bigg) \\ &= \tfrac{1}{2} - \bar{\rho}_1 - \frac{\bar{x}_\text{s}^+}{2(\tau - r)} \ ,\end{aligned} \tag{4.190}$$

the solution of which is

$$\bar{x}_\text{s}^+(\tau) = A_+\,(\tau - r)^{1/2} + (1 - 2\bar{\rho}_1)\,(\tau - r) \ . \tag{4.191}$$

The constant A_+ is determined by setting

$$\bar{x}_\text{s}^+(\tau_+) = \bar{x}_>(\tau_+) = \frac{\bar{\rho}_1\, r}{1 - \bar{\rho}_1} \quad \Rightarrow \quad A_+ = 2\sqrt{r\bar{\rho}_1(1 - \bar{\rho}_1)} \ . \tag{4.192}$$

Putting this all together, we have

$$\bar{x}_\text{s}^+(\tau) = \begin{cases} (1 - \bar{\rho}_1)\,\tau & \text{for } \tau < \frac{r}{\bar{\rho}_1} \\ \\ \sqrt{4r\bar{\rho}_1(1 - \bar{\rho}_1)}\,(\tau - r)^{1/2} + (1 - 2\bar{\rho}_1)\,(\tau - r) & \text{for } \tau > \frac{r}{\bar{\rho}_1} \ . \end{cases} \tag{4.193}$$

The identical analysis can be applied to the shock at $\bar{x}_s^-$ for $\tau > \tau_-$. The solution is once again of the form

$$\bar{x}_s^-(\tau) = A_-\,(\tau - r)^{1/2} + (1 - 2\bar{\rho}_1)\,(\tau - r)\ , \tag{4.194}$$

now with A_- chosen to satisfy

$$\bar{x}_s^-(\tau_-) = \bar{x}_<(\tau_-) = -\frac{\bar{\rho}_1\, r}{1-\bar{\rho}_1} \quad \Rightarrow \quad A_- = -2\sqrt{r\bar{\rho}_1(1-\bar{\rho}_1)} = -A_+\ . \tag{4.195}$$

Thus,

$$\bar{x}_s^-(\tau) = \begin{cases} -\bar{\rho}_1\,\tau & \text{for } \tau < \frac{r}{1-\bar{\rho}_1} \\ \\ -\sqrt{4r\bar{\rho}_1(1-\bar{\rho}_1)}\,(\tau - r)^{1/2} + (1 - 2\bar{\rho}_1)\,(\tau - r) & \text{for } \tau > \frac{r}{1-\bar{\rho}_1}\ . \end{cases} \tag{4.196}$$

(d) Compute the discontinuity $\Delta\bar{\rho}^{\pm}(\tau)$ at the shocks.

Solution : For the shock at $\bar{x}_s^+$, we have

$$\begin{aligned} \Delta\bar{\rho}^+(\tau) &= \bar{\rho}_1 - \tfrac{1}{2}\left(1 - \frac{\bar{x}_s^+}{\tau - r}\right) \\ &= \sqrt{r\bar{\rho}_1(1-\bar{\rho}_1)}\,(\tau - r)^{-1/2}\ , \end{aligned} \tag{4.197}$$

which is valid for $\tau > \tau_+ = r/\bar{\rho}_1$. For the shock at $\bar{x}_s^-$,

$$\begin{aligned} \Delta\bar{\rho}^-(\tau) &= \tfrac{1}{2}\left(1 - \frac{\bar{x}_s^-}{\tau - r}\right) - \bar{\rho}_1 \\ &= \sqrt{r\bar{\rho}_1(1-\bar{\rho}_1)}\,(\tau - r)^{-1/2}\ , \end{aligned} \tag{4.198}$$

which is valid for $\tau > \tau_- = r/(1-\bar{\rho}_1)$. Note $\Delta\bar{\rho}^+(\tau) = \Delta\bar{\rho}^-(\tau)$ for $\tau > \max(\tau_+, \tau_-)$.

(e) If $\bar{\rho}_1 < \frac{1}{2}$, show that the shock which starts out behind the light passes the light after a finite time τ^*. What is the condition that the shock passes the light before the start of the next red phase? Show that this condition is equivalent to demanding that the number of cars passing the light during the green phase must be greater than the incoming flux at $\bar{x} = -\infty$, integrated over the cycle length, under the assumption that the shock just barely manages to pass through before the next red.

Solution : For light traffic with $\bar{\rho}_1 < \frac{1}{2}$, the shock behind the light passes the signal at time τ^*, where $\bar{x}_s^-(\tau^*) = 0$, *i.e.*

$$\bar{x}_s^-(\tau^*) = (1 - 2\bar{\rho}_1)\,(\tau^* - r) - \sqrt{4r\bar{\rho}_1(1-\bar{\rho}_1)}\,(\tau^* - r)^{1/2} = 0\ . \tag{4.199}$$

Solving for τ^*, we have

$$\tau^* = r + \frac{4r\bar{\rho}_1(1-\bar{\rho}_1)}{(1-2\bar{\rho}_1)^2} = \frac{r}{(1-2\bar{\rho}_1)^2}\ . \tag{4.200}$$

Similarly, if $\bar{\rho}_1 > \frac{1}{2}$, the shock ahead of the light reverses direction and passes behind the signal at time $\tau^* = r/(2\bar{\rho}_1 - 1)^2$.

For the case of light traffic, we see that if $r > (1 - 2\bar{\rho}_1)^2$, the trailing shock remains behind the signal when the next green phase starts – the shock never clears the signal. In traffic engineering parlance, the capacity of the intersection is insufficient, and the traffic will back up. We can derive the result in a very simple way. Until the trailing shock at $\bar{x}_{\rm s}^-$ passes the signal, the density of vehicles at the signal is fixed at $\bar{\rho} = \frac{1}{2}$. Hence the flux of vehicles at the signal is maximum, and equal to $\bar{j}(\bar{\rho} = \frac{1}{2}) = \frac{1}{4}$. The maximum number of vehicles that the signal can clear is therefore $\frac{1}{4}\, g = \frac{1}{4}\,(1-r)$, which pertains when the trailing shock just barely passes the signal before the start of the next red phase. We now require that the incident flux of cars from $\bar{x} = -\infty$, integrated over the *entire* cycle, is less than this number:

$$\bar{j}(\bar{\rho}_1) \times 1 = \bar{\rho}_1(1-\bar{\rho}_1) < \tfrac{1}{4}\,(1-r) \qquad \Rightarrow \qquad r < (1-2\bar{\rho}_1)^2 \ . \tag{4.201}$$

Merging shocks : So far, so good. We now ask: does the shock at $\bar{x}_{\rm s}^-(\tau)$ ever catch up with the shock at $\bar{x}_{\rm s}^+(\tau-1)$ from the next cycle? Let's assume that the equation $\bar{x}_{\rm s}^-(\tau) = \bar{x}_{\rm s}^+(\tau-1)$ has a solution, which means we set

$$-a\,\sqrt{\tau-r} + b\,(\tau - r) = \begin{cases} a\,\sqrt{\tau-1-r} + b\,(\tau-1-r) & \text{if } \tau > 1 + \frac{r}{\bar{\rho}_1} \\ \\ (1-\bar{\rho}_1)\,(\tau-1) & \text{if } 1 < \tau < 1 + \frac{r}{\bar{\rho}_1} \end{cases} \tag{4.202}$$

with

$$a \equiv \sqrt{4r\bar{\rho}_1(1-\bar{\rho}_1)} \quad , \quad b \equiv 1 - 2\bar{\rho}_1 \ . \tag{4.203}$$

We first look for a solution satisfying $\tau > 1 + \tau_+ = 1 + \frac{r}{\bar{\rho}_1}$. Let us call this a 'case I merge'. It is convenient to define $s = \tau - \frac{1}{2} - r$, in which case

$$a\,\sqrt{s-\tfrac{1}{2}} + a\,\sqrt{s+\tfrac{1}{2}} = b\,, \tag{4.204}$$

which is solved by

$$s = \tfrac{1}{4}\bigg(\frac{a^2}{b^2} + \frac{b^2}{a^2}\bigg) \ . \tag{4.205}$$

Note $s > \frac{1}{2}$, so $\tau > 1 + r$. Solving for τ, we obtain

$$\tau = \tau_{\rm I} \equiv 1 + r + \tfrac{1}{4}\bigg(\frac{a}{b} - \frac{b}{a}\bigg)^2 \ . \tag{4.206}$$

We must now check that $\tau > 1 + \frac{r}{\bar{\rho}_1}$. We have

$$\begin{aligned} 0 > f(r) &\equiv 1 + \frac{r}{\bar{\rho}_1} - \tau_{\rm I} \\ &= \frac{(1-\bar{\rho}_1)^2(1-3\bar{\rho}_1)}{(1-2\bar{\rho}_1)^2\bar{\rho}_1}\, r + \frac{1}{2} - \frac{(1-2\bar{\rho}_1)^2}{16r\bar{\rho}_1(1-\bar{\rho}_1)} \\ &\equiv \frac{(1-\bar{\rho}_1)^2(1-3\bar{\rho}_1)}{(1-2\bar{\rho}_1)^2\bar{\rho}_1\, r}\,(r-r_1)(r-r_2) \end{aligned} \tag{4.207}$$

with

$$r_1 = \bigg(\frac{1-2\bar{\rho}_1}{2-2\bar{\rho}_1}\bigg)^2 \quad , \quad r_2 = -\frac{(1-2\bar{\rho}_1)^2}{4(1-\bar{\rho}_1)(1-3\bar{\rho}_1)} \ . \tag{4.208}$$

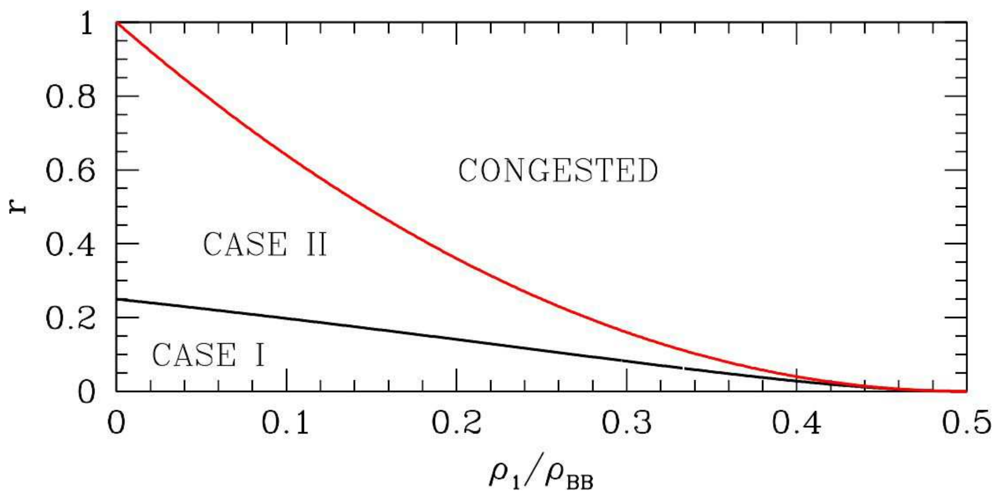

Figure 4.18: Phase diagram for the shock merge analysis. For $r > (1 - 2\bar{\rho}_1)^2$ (red line), the signal is congested, and the trailing shock does not clear the signal before the start of the following red phase. In case II, the shocks merge when the leading shock is still in its constant velocity phase. In case I, the shocks merge after the leading shock has already begun to decelerate.

When $\bar{\rho}_1 < \frac{1}{3}$, we have $r_2 > 0 > r_1$, and we require $0 < r < r_2$. When $\bar{\rho}_1 > \frac{1}{3}$, $r_1 > r_2$ so again we require $0 < r < r_2$.

Now let's solve for the merging shocks. We write the condition that two characteristics meet. Thus,

$$\alpha\,(\tau - r) = \beta\,(\tau - 1 - r) \tag{4.209}$$

with $\bar{x} = \alpha\,(\tau - r)$. Let $\sigma = \tau - r$. Then $\bar{x} = \alpha\,\sigma = \beta\,(\sigma - 1)$, which yields

$$\alpha = \frac{\bar{x}}{\sigma} \quad , \quad \beta = \frac{\bar{x}}{\sigma - 1} \ . \tag{4.210}$$

The combined shock moves at velocity v_s, where

$$v_s = \dot{\bar{x}} = \tfrac{1}{2}(\alpha + \beta) = \frac{\bar{x}}{2\sigma} + \frac{\bar{x}}{2(\sigma - 1)} \ . \tag{4.211}$$

This equation is easily integrated to yield

$$\bar{x}^{\rm I}(\sigma) = \mathcal{C}\,\sqrt{\sigma(\sigma - 1)} \ . \tag{4.212}$$

The constant $\mathcal{C}$ is fixed by the initial conditions at time $\tau_{\rm I}$, when the shocks merge. We have

$$\begin{aligned} \tau_{\rm I} - r &= \tfrac{1}{4}\left(\frac{a}{b} + \frac{b}{a}\right)^2 \\ \bar{x}(\tau_{\rm I}) &= -a\,\sqrt{\tau_{\rm I} - r} + b\,(\tau_{\rm I} - r) \\ &= \tfrac{1}{4}\left(\frac{b^2}{a^2} - \frac{a^2}{b^2}\right) b \ . \end{aligned} \tag{4.213}$$

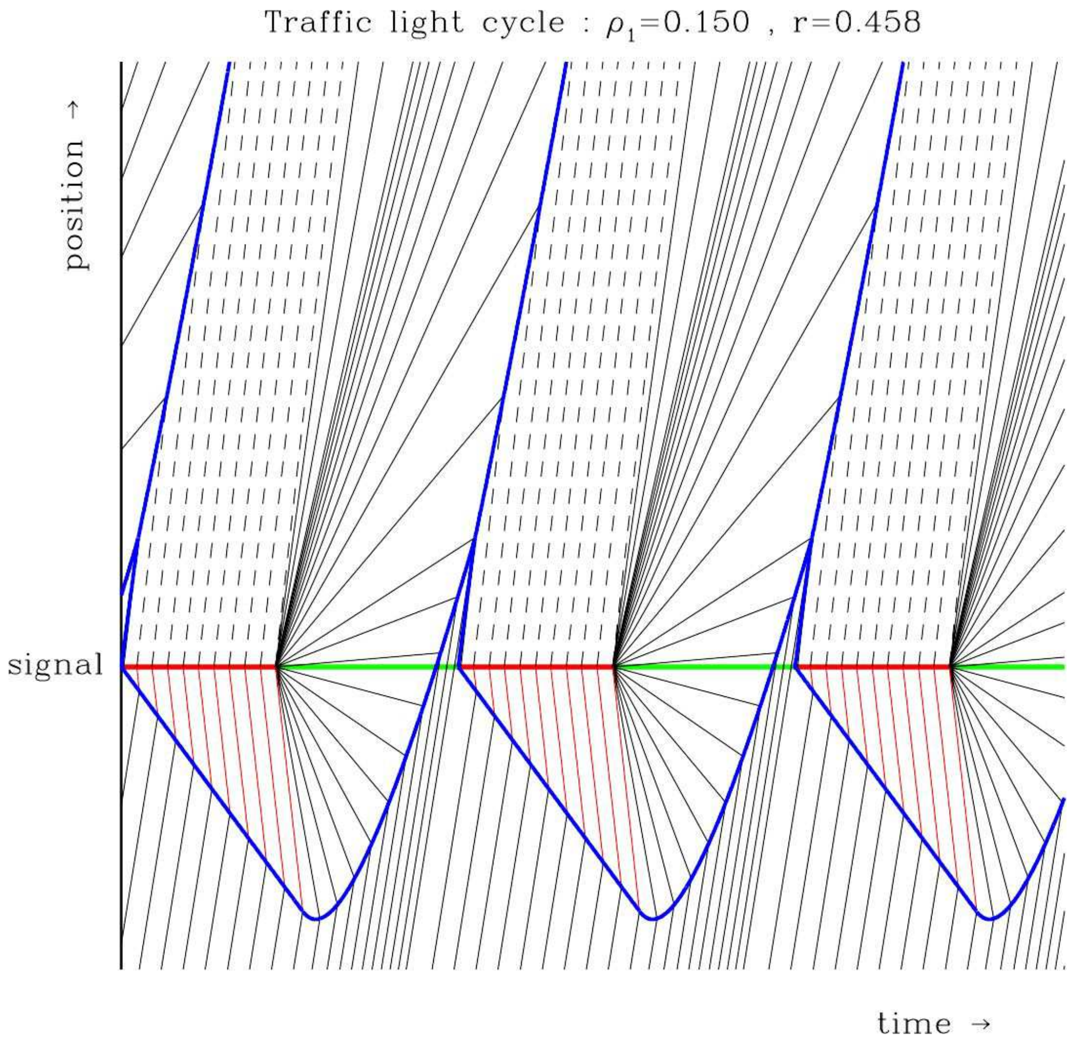

Figure 4.19: Characteristics for the light cycle problem. Red interval signifies red light; green interval signifies green light. Red characteristics correspond bumper-to-bumper traffic, *i.e.* $\bar{\rho} = 1$. Dotted line characteristics correspond to $\bar{\rho} = 0$. Note the merging of shocks. Congested regions are those with negatively-sloped characteristics, since $\bar{c} < 0$ for $\bar{\rho} > \frac{1}{2}$.

We then have

$$\sigma\,(\sigma - 1) = \tfrac{1}{16}\bigg(\frac{a^2}{b^2} - \frac{b^2}{a^2}\bigg)^2 ,$$

and we set

$$\tfrac{1}{4}\left(\frac{a^2}{b^2} - \frac{b^2}{a^2}\right)\mathcal{C} = \tfrac{1}{4}\bigg(\frac{b^2}{a^2} - \frac{a^2}{b^2}\bigg)\, b\,,$$

which simply yields $\mathcal{C} = b = 1 - 2\bar{\rho}_1$.

Finally, consider the 'case II merge' where $\bar{x}_{\rm s}^-(\tau) = \bar{x}_{\rm s}^+(\tau - 1)$ requires $\tau < 1 + \frac{r}{\bar{\rho}_1}$. We now must solve

$$-a\,\sqrt{\tau - r} + b\,(\tau - r) = (1 - \bar{\rho}_1)\,(\tau - 1)\,, \tag{4.214}$$

the solution of which is

$$\tau_{\text{II}} = r + \big(1-\sqrt{r}\big)^2\,\frac{1-\bar{\rho}_1}{\bar{\rho}_1}\ . \tag{4.215}$$

One can check that $\tau_{\text{II}} < 1 + \frac{r}{\bar{\rho}_1}$ is equivalent to $r > r_2$. In this case, the characteristics to the right of the shock all have slope 1, corresponding to $\bar{\rho} = 0$, and are not part of the fan. Thus, the shock condition becomes

$$\dot{\bar{x}} = \tfrac{1}{2}\bigg(1 + \frac{\bar{x}}{\sigma}\bigg) \tag{4.216}$$

with $\sigma = \tau - r$ as before. The solution here is

$$\bar{x}^{\text{II}} = \mathcal{C}'\,\sqrt{\sigma} + \sigma\ . \tag{4.217}$$

Again, $\mathcal{C}'$ is determined by the merging shock condition, $\bar{x}^-_{\text{s}}(\tau) = \bar{x}^+_{\text{s}}(\tau-1)$. We therefore set

$$\mathcal{C}'\,\big(1-\sqrt{r}\big)\sqrt{\frac{1-\bar{\rho}_1}{\bar{\rho}_1}} + \big(1-\sqrt{r}\big)^2\,\frac{1-\bar{\rho}_1}{\bar{\rho}_1} = (1-\bar{\rho}_1)\bigg\{r + \big(1-\sqrt{r}\big)^2\,\frac{1-\bar{\rho}_1}{\bar{\rho}_1} - 1\bigg\}\ , \tag{4.218}$$

yielding

$$\mathcal{C}' = -\sqrt{4\bar{\rho}_1(1-\bar{\rho}_1)} = -\frac{a}{\sqrt{r}}\ . \tag{4.219}$$

This solution is valid until the characteristics on the right side of the shock extrapolate back to the $\bar{x}$ axis at time $\tau = 1 + r$, when the fan begins. Thus,

$$\bar{x} = \tau - 1 - r = \sigma - 1 = \mathcal{C}'\,\sqrt{\sigma} + \sigma\ , \tag{4.220}$$

which yields $\sigma = \sigma_{\text{m}} \equiv \frac{1}{4\bar{\rho}_1(1-\bar{\rho}_1)}$.

For $\sigma > \sigma_{\text{m}}$, we match with the profile $\mathcal{C}''\,\sqrt{\sigma(\sigma-1)}$, which we obtained earlier. Thus,

$$\mathcal{C}''\sqrt{\sigma_{\text{m}}(\sigma_{\text{m}}-1)} = \sigma_{\text{m}} - \sqrt{4\bar{\rho}_1(1-\bar{\rho}_1)}\,\sqrt{\sigma_{\text{m}}}\ , \tag{4.221}$$

which yields $\mathcal{C}'' = b = 1 - 2\bar{\rho}_1$, as before.

Putting this all together, we plot the results in figure 4.19. The regions with negatively-sloped characteristics are regions of local congestion, since $\bar{c} < 0$ implies $\bar{\rho} > \frac{1}{2}$. Note that the diagram is periodic in time, and it is presumed that several cycles have passed in order for the cycle to equilibrate. *I.e.* the conditions at $t = 0$ in the diagram do not satisfy $\bar{\rho}(x) = \bar{\rho}_1$ for $x > 0$.

4.14 Appendix V : Car-Following Models

So-called 'car-following' models are defined by equations of motion for individual vehicles. Let $x_n(t)$ be the motion of the n^{th} vehicle in a group, with $x_{n+1} < x_n$, so that the lead vehicle ($n = 0$) is the rightmost vehicle. Three classes of models have traditionally been considered in the literature:

- *Traditional car-following model (CFM)* – Cars accelerate (or decelerate) in an attempt to match velocity with the car ahead, subject to a delay time τ:

$$\ddot{x}_n(t+\tau) = \alpha\Big\{\dot{x}_{n-1}(t) - \dot{x}_n(t)\Big\} .$$

- *Optimal velocity model (OVM)* – Drivers instantaneously accelerate in order to maintain a velocity which is optimally matched to the distance to the next car. (The distance to the next vehicle is known as the *headway* in traffic engineering parlance.) Thus,

$$\ddot{x}_n = \alpha\Big\{V(x_{n-1} - x_n) - \dot{x}_n\Big\} , \tag{4.222}$$

where $V(\Delta x)$ is the optimum velocity function.

- *Optimal headway model (OHM)* – Drivers instantaneously accelerate in order to maintain an optimal headway, given their current velocity:

$$\ddot{x}_n = \alpha\Big\{H(\dot{x}_n) - (x_{n-1} - x_n)\Big\} . \tag{4.223}$$

The optimal headway function is just the inverse of the optimal velocity function of the OVM: $H = V^{-1}$.

(a) The CFM equation above is a linear equation. Solve it by Fourier transforming the function $v_n(t) = \dot{x}_n(t)$. Show that

$$\hat{v}_n(\omega) = r(\omega)\, e^{i\theta(\omega)}\, \hat{v}_{n-1}(\omega) . \tag{4.224}$$

Thus, given the velocity $v_0(t)$ of the lead vehicle, the velocity of every other vehicle is determined, since $\hat{v}_n(\omega) = r^n(\omega)\, e^{in\theta(\omega)}\, \hat{v}_0(\omega)$. Derive the stability criterion $|r| < 1$ in terms of ω, τ, and α. Show that, if the motion is stable, no matter how erratic the lead vehicle moves, for $n \to \infty$ the variations in $v_n(t)$ are damped out and the velocity approaches a constant.

Solution : We have the linear equation

$$\dot{v}_n(t+\tau) = \alpha\Big\{v_{n-1}(t) - v_n(t)\Big\} . \tag{4.225}$$

Solve by Fourier transform:

$$v_n(t) = \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \hat{v}_n(\omega)\, e^{-i\omega t} . \tag{4.226}$$

This yields

$$\hat{v}_n(\omega) = z(\omega)\, \hat{v}_{n-1}(\omega) \tag{4.227}$$

with

$$z(\omega) = \frac{\alpha}{\alpha - i\omega\, \exp(-i\omega\tau)} . \tag{4.228}$$

Thus, writing $z = r\, e^{i\theta}$, we have

$$\theta(\omega) = \tan^{-1}\left(\frac{\omega\cos\omega\tau}{\alpha - \omega\sin\omega\tau}\right) \tag{4.229}$$

and

$$r(\omega) = \frac{\alpha}{\sqrt{\alpha^2 + \omega^2 - 2\alpha\omega\sin(\omega\tau)}} \,. \tag{4.230}$$

The velocity of the n^{th} vehicle is now given in terms of that of the lead vehicle, according to the relation

$$\hat{v}_n(\omega) = r^n(\omega)\, e^{in\theta(\omega)}\, \hat{v}_0(\omega) \,. \tag{4.231}$$

For $\omega = 0$ we have $z(\omega) = 1$, which says that the time-averaged velocity of each vehicle is the same. For $\hat{v}_{n\to\infty}(\omega)$ to be bounded requires $r(\omega) \le 1$, which gives

$$r(\omega) \le 1 \quad \Longleftrightarrow \quad 2\alpha\tau \cdot \frac{\sin\omega\tau}{\omega\tau} \le 1 \,. \tag{4.232}$$

The maximum of the function $x^{-1}\sin x$ is unity, for $x = 0$. This means that the instability first sets in for infinitesimal ω. We can ensure that every frequency component is stable by requiring

$$\alpha\,\tau < \tfrac{1}{2} \,. \tag{4.233}$$

The interpretation of this result is straightforward. The motion is stable if the product of the response time τ and the sensitivity α is sufficiently small. Slow response times and hypersensitive drivers result in an unstable flow. This is somewhat counterintuitive, as we expect that increased driver awareness should improve traffic flow.

(b) Linearize the OVM about the steady state solution,

$$x_n^0(t) = -na + V(a)t \,, \tag{4.234}$$

where a is the distance between cars. Write $x_n(t) = x_n^0(t) + \delta x_n(t)$ and find the linearized equation for $\delta x_n(t)$. Then try a solution of the form

$$\delta x_n(t) = A\, e^{ikna}\, e^{-\beta t} \,, \tag{4.235}$$

and show this solves the linearized dynamics provided $\beta = \beta(k)$. What is the stability condition in terms of α, k, and a? Show that all small perturbations about the steady state solution are stable provided $V'(a) < \frac{1}{2}\,\alpha$. Interpret this result physically. If you get stuck, see M. Bando *et al.*, *Phys. Rev. E* **51**, 1035 (1995).

Solution : Writing $x_n = -na + V(a)t + \delta x_n$, we linearize the OVM and obtain

$$\delta\ddot{x}_n = \alpha\Big\{V'(a)\big(\delta x_{n-1} - \delta x_n\big) - \delta\dot{x}_n\Big\} + \ldots \,. \tag{4.236}$$

We now write

$$\delta x_n(t) = A\, e^{inka}\, e^{-\beta t} \,, \tag{4.237}$$

and obtain the equation

$$\beta^2 - \alpha\beta + \alpha V'(a)\,(1 - e^{-ika}) = 0 \,. \tag{4.238}$$

This determines the growth rate $\beta(k)$ for each wavelength k. Solving the quadratic equation,

$$\beta(k) = \tfrac{1}{2}\alpha \pm \tfrac{1}{2}\sqrt{\alpha^2 - 4\alpha V'(a)\,(1 - e^{-ika})} \,. \tag{4.239}$$

Let us separate β into its real and imaginary parts: $\beta \equiv \mu + i\nu$. Then

$$\mu - \tfrac{1}{2}\alpha + i\nu = \pm\tfrac{1}{2}\sqrt{\alpha^2 - 4\alpha V'(a)\,(1 - e^{-ika})} \tag{4.240}$$

which, when squared, gives two equations for the real and imaginary parts:

$$\begin{aligned} \nu^2 + \mu\alpha - \mu^2 &= \alpha V'(a)\,(1 - \cos ka) \\ \alpha\nu - 2\mu\nu &= \alpha V'(a) \sin ka \ . \end{aligned} \tag{4.241}$$

We set $\mu = \mathrm{Re}\,\beta = 0$ to obtain the stability boundary. We therefore obtain

$$V'(a) < \frac{\alpha}{2\cos^2 \frac{1}{2}ka} \quad \Leftrightarrow \quad \mathrm{Re}\,\beta(k) > 0 \ . \tag{4.242}$$

The uniform $k = 0$ mode is the first to go unstable. We therefore have that all k modes are linearly stable provided

$$V'(a) < \tfrac{1}{2}\alpha \ . \tag{4.243}$$

This says that the stability places a *lower* limit on the sensitivity α – exactly the opposite of what we found for the CFM. The OVM result is more intuitive. Physically, we expect that the OVM is more realistic as well, since drivers generally find it easier to gauge the distance to the next car than to gauge the difference in velocity.