

**PHYSICS 200B : CLASSICAL MECHANICS**  
**SOLUTION SET #4**

(1) Consider two coupled nonlinear oscillators, with

$$\begin{aligned}\frac{d\varphi_1}{dt} &= V_1(\varphi_1) + \epsilon F_1(\varphi_1, \varphi_2) \\ \frac{d\varphi_2}{dt} &= V_2(\varphi_2) + \epsilon F_2(\varphi_1, \varphi_2) .\end{aligned}$$

Assume that for  $\epsilon = 0$  each of the oscillators exhibits at least one stable limit cycle. Assume further that the natural frequencies  $\omega_{1,2}$  of their respective limit cycles are close to resonance. That is, assume that the detuning

$$\nu = m\omega_1 - n\omega_2$$

is small for some integer values of  $m$  and  $n$ .

(a) Using the phase representation and isochrones for each oscillator, show that to lowest order one may write

$$\begin{aligned}\frac{d\phi_1}{dt} &= \omega_1 + \epsilon Q_1(\phi_1, \phi_2) \\ \frac{d\phi_2}{dt} &= \omega_2 + \epsilon Q_2(\phi_1, \phi_2) .\end{aligned}$$

Solution: We have :

$$\begin{aligned}\frac{d\phi_1}{dt} &= \frac{\partial\phi_1}{\partial\varphi_1} \frac{\partial\psi_1}{\partial t} + \frac{\partial\phi_1}{\partial\varphi_2} \frac{\partial\varphi_2}{\partial t} \\ &= \left( \frac{\partial\phi_1}{\partial\varphi_1} V_1(\varphi_1) + \frac{\partial\phi_1}{\partial\varphi_2} V_2(\varphi_2) \right) + \epsilon \left( \frac{\partial\phi_1}{\partial\varphi_1} F_1(\varphi_1, \varphi_2) + \frac{\partial\phi_1}{\partial\varphi_2} F_2(\varphi_1, \varphi_2) \right) \\ &= \omega_1 + \epsilon Q_1(\phi_1, \phi_2)\end{aligned}$$

In the last step, we assume  $\psi$ s are closing to the limit circle. Similarly, we have:

$$\frac{d\phi_2}{dt} = \omega_2 + \epsilon Q_2(\phi_1, \phi_2)$$

where:

$$Q_{1,2}(\phi_1, \phi_2) = \left. \frac{\partial\phi_{1,2}}{\partial\varphi_1} \right|_{\gamma_1(\phi_1)} F_1(\gamma_1(\phi_1), \gamma_2(\phi_2)) + \left. \frac{\partial\phi_{1,2}}{\partial\varphi_2} \right|_{\gamma_2(\phi_2)} F_2(\gamma_1(\phi_1), \gamma_2(\phi_2))$$

(b) Expand the functions  $Q_{1,2}(\phi_1, \phi_2)$  in a double Fourier series in their arguments. What terms satisfy the (near) resonance condition?

Solution: We can Fourier transform  $Q_{1,2}$ :

$$Q_{1,2}(\phi_1, \phi_2) = \sum_{m,n} Q_{1,2,m,n} e^{i(m\phi_1 - n\phi_2)}$$

The resonance condition occurs when  $m$  and  $n$  satisfy the following condition:

$$m\omega_1 - n\omega_2 = 0$$

This is only possible if  $\omega_1/\omega_2$  is a rational number  $p/q$ , where  $p$  and  $q$  are relatively prime integers. In this case, the resonant terms can be written as:

$$Q_{1,2}(\phi_1, \phi_2) = \sum_j Q_{1,2,jq,jp} e^{i(jq\phi_1 - jp\phi_2)}$$

- (c) Keeping only the terms which are nearly resonant, define  $\psi = m\phi_1 - n\phi_2$  and derive an ODE describing the behavior of  $\psi$ . Analyze this ODE and classify its fixed points. What is the condition for synchronization of the two nonlinear oscillators?

Solution: We define  $\psi = q\phi_1 - p\phi_2$ . Then the ODE of  $\psi$  reads:

$$\begin{aligned} \frac{d\psi}{dt} &= q\omega_1 - p\omega_2 + \epsilon \sum_j (Q_{1,jq,jp} + Q_{2,jq,jp}) e^{ij\psi} \\ &= \nu + \epsilon g(\psi) \end{aligned}$$

Therefore if the detuning  $\nu$  lies within the range  $[g_{min}, g_{max}]$ , the two nonlinear oscillators synchronize with each other.

- (2) Consider the function  $F(x)$  defined by

$$F(x) = \begin{cases} -x & \text{if } 0 \leq x \leq 1 \\ 3x - 4 & \text{if } 1 \leq x \leq 2 \\ -5x + 12 & \text{if } 2 \leq x \leq 3 \\ 7x - 24 & \text{if } x \geq 3 \end{cases}$$

with  $F(-x) \equiv -F(x)$ .

- (a) Sketch  $F(x)$  over the interval  $x \in [-4, 4]$ .

Solution: See Fig. 1.

- (b) Consider the nonlinear oscillator  $\ddot{x} + \mu F'(x) \dot{x} + x = 0$ . Find all the stable limit cycles and their periods for  $\mu \gg 1$ .

Solution: It is straightforward to see there are two limit circles, as illustrated in Fig. 1 (b), Now we compute their period, respectively. For the first one, we have  $a = 3$  and  $b = 27/7$ , then the period is :

$$T = -2\mu \int_b^a \frac{F'(x)}{x} = 14 \log\left(\frac{9}{7}\right)\mu$$

Similarly, for the second one, we have  $a = 1$  and  $b = 5/3$ , then the period is :

$$T = -2\mu \int_b^a \frac{F'(x)}{x} = 6 \log\left(\frac{5}{3}\right)\mu$$

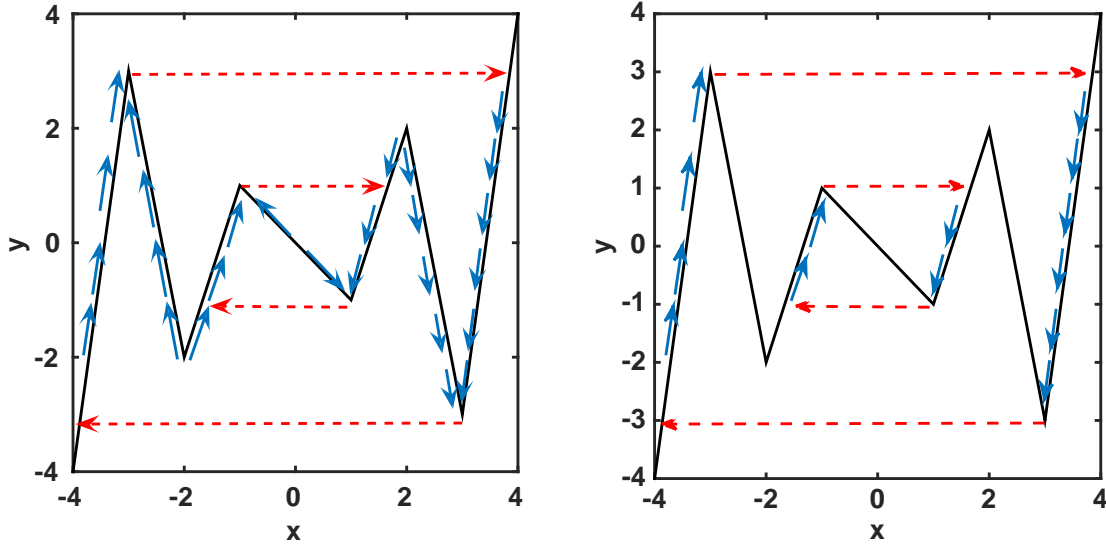


Figure 1: (a) The blue arrows indicate slow mode and the red arrows indicate fast mode. (b) The two limiting cycles are shown

(3) Use the method of characteristics to solve the quasilinear PDE

$$\phi_t + \gamma x \phi_x = -Bx^2 \phi \quad ,$$

subject to the initial condition  $\phi(x, 0) = f(x)$ .

Solution: using the method of characteristics, we have:

$$\begin{aligned} \frac{dt}{ds} &= 1 \\ \frac{dx}{ds} &= \gamma x \end{aligned}$$

which yields:  $t = s$  and  $x = \xi e^{\gamma s}$ . Then the ODE of  $\phi$  reads:

$$\frac{d\phi}{ds} = -B\xi^2 e^{2\gamma s} \phi$$

The solution that satisfies the initial condition is:

$$\begin{aligned} \phi &= f(\xi) e^{-\frac{B\xi^2}{2\gamma}(e^{2\gamma s} - 1)} \\ &= f\left(\frac{x}{e^{\gamma t}}\right) e^{-\frac{Bx^2}{2\gamma}(1 - e^{-2\gamma t})} \end{aligned}$$

(4) Consider shock formation in the equation\*  $c_t + c c_x = 0$  with initial conditions  $c(\zeta) = c(x = \zeta, t = 0)$  an odd function of its argument, *i.e.*  $c(-\zeta) = -c(\zeta)$ . Suppose further that  $c(\zeta)$  is monotonically decreasing.

(a) Find an expression for the time the wave first breaks,  $t_B$ .

Solution:  $t_B = \min\left(-\frac{1}{c'(\zeta)}\right)$

- (b) Show that  $\zeta_- = -\zeta_+$  and find an equation\* relating  $\zeta_+$  and  $t$  for  $t > t_B$ .

Solution:

$$\frac{1}{2}(\zeta_+ - \zeta_-)(c(\zeta_+) + c(\zeta_-)) = \int_{\zeta_-}^{\zeta_+} c(\zeta) d\zeta$$

This gives the relation  $\zeta_+ = \zeta_-$ , which obviously satisfy the equation\*. Then we have:

$$\zeta_+ + c(\zeta_+)t = \zeta_- + c(\zeta_-)t$$

Using the property of odd functions, we get that  $t = -\zeta_+/c(\zeta_+)$  and  $x_s(t) = \zeta_+ + c(\zeta_+)t = 0$ .

- (c) Show that the position of the shock remains at  $x_s(t) = 0$  for all  $t > t_B$ .

Solution: See above.

- (d) Find  $t_B$ ,  $\zeta_+(t)$ , and the shock discontinuity  $\Delta c(t)$  for the case

$$c(\zeta) = -\frac{c_0 \zeta}{\sqrt{a^2 + \zeta^2}} \quad .$$

Solution: After absorbing  $c_0$  into  $c$ , and  $a$  into  $\zeta$ . the initial condition reads:

$$c(\zeta) = -\frac{\zeta}{\sqrt{1+\zeta^2}}$$

The maximum of the derivative occurs at  $\zeta = 0$ , where  $c'(\zeta) = -1$ . Therefore  $t_B = 1$ . Furthermore, from above, we know that  $-\zeta_+/c(\zeta_+) = \sqrt{1 + \zeta^2}$ . Thus we obtain that  $\zeta_+ = \sqrt{t^2 - 1}$ . The shock discontinuity reads  $\Delta c(t) = -2c(\zeta_+) = \frac{\sqrt{t^2-1}}{t}$ .

- (e) Sketch the time evolution of  $c(x, t)$  with and without the shock fitting. (Without shock fitting, the function will eventually become multivalued.)

Solution: See Fig. 2

- (5) Consider shock formation in the equation\*  $c_t + c c_x = 0$  with initial conditions

$$c(x, t = 0) = \frac{c_0}{1 + x^2} \quad .$$

- (a) Find the position and time  $(x_B, t_B)$  where the wave first breaks.

Solution: Again we consider the dimensionless initial condition:

$$c(\zeta) = \frac{1}{1+\zeta^2}$$

The second derivative reads

$$c''(\zeta) = \frac{8\zeta^2}{(\zeta^2+1)^3} - \frac{2}{(\zeta^2+1)^2}$$

There are two solutions if we demand  $c''(\zeta) = 0$ . The solution  $x = 1/\sqrt{3}$  corresponds to the minimum of the first derivative  $c'(\zeta)$ , which equals  $-\frac{3\sqrt{3}}{8}$ . Therefore  $t_B = \frac{8\sqrt{3}}{9}$ .

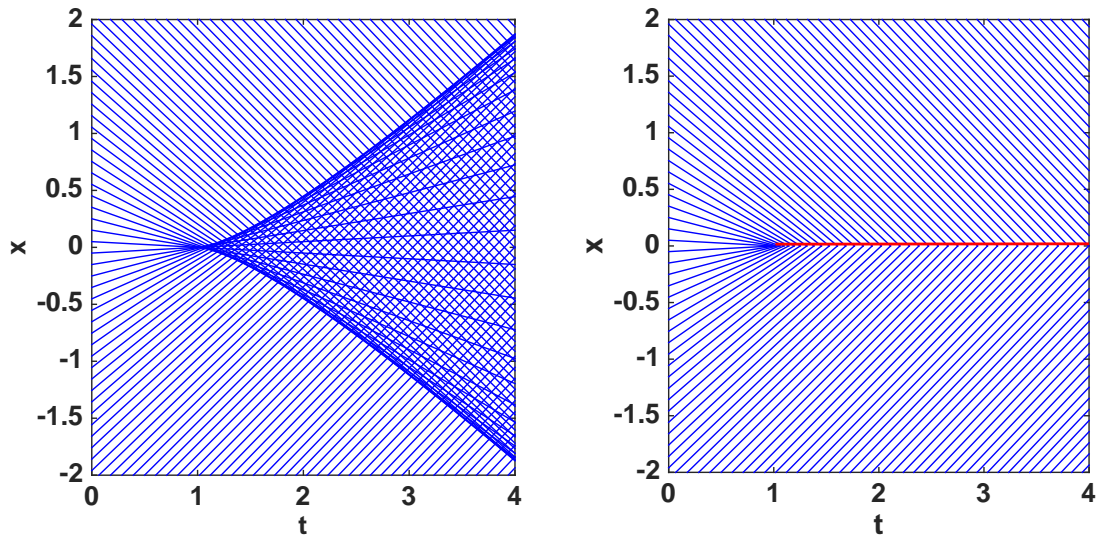


Figure 2:

- (b) Find the two equation\*s relating  $\zeta_+$ ,  $\zeta_-$ , and  $t$ .

Solution: Invoking the the shock fitting equation\*, we have:

$$\frac{1}{2}(\zeta_+ - \zeta_-)(c(\zeta_+) + c(\zeta_-)) = \int_{\zeta_-}^{\zeta_+} c(\zeta) d\zeta = \arctan(\zeta_+) - \arctan(\zeta_-)$$

Once we solve  $\zeta_-$  for  $\zeta_+$ ,  $t(\zeta_+)$  can be obtained from the following equation\*:

$$\zeta_+ + c(\zeta_+)t = \zeta_- + c(\zeta_-)t$$

- (c) For general  $t$ , your equation\*s from part (b) cannot be solved analytically. However, as  $t \rightarrow \infty$ , one can make progress. Show in the late time limit that  $\zeta_+ \propto \zeta_-^2$  and use the first shock fitting equation\* to obtain a relation between  $\zeta_+$  and  $\zeta_-$  valid for  $c_0 t \gg 1$ .

Solution: Starting from the shock fitting equation\*:

$$\frac{1}{2}\left(\frac{1}{1+\zeta_+^2} + \frac{1}{1+\zeta_-^2}\right)(\zeta_+ - \zeta_-) = \arctan \zeta_+ - \arctan \zeta_-$$

In the long time limit,  $\zeta_+ \rightarrow \infty$  and  $\zeta_- \rightarrow -\infty$ , and the equation\* above can be significantly simplified into:

$$\frac{\zeta_+}{\zeta_-^2} - \frac{\zeta_-}{\zeta_+^2} = 2\pi$$

We further require the shock happens at  $t > 0$ , which indicates that  $c(\zeta_+) < c(\zeta_-)$ , leading to  $|\zeta_+| > |\zeta_-|$ . Then the above equation\* gives:

$$\zeta_+ = 2\pi\zeta_-^2$$

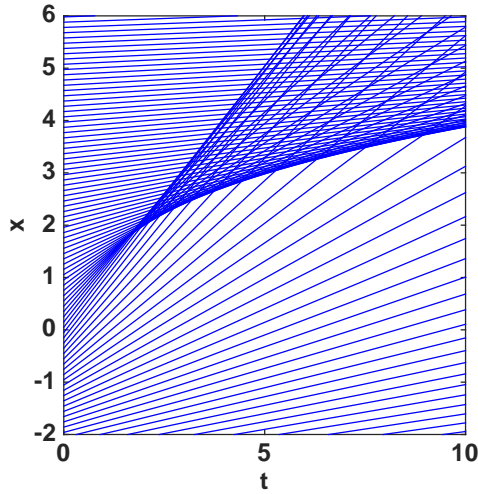


Figure 3:

- (d) Invoking the second shock equation\*, obtain expressions for  $\zeta_{\pm}(t)$  and  $c(\zeta_{\pm}(t))$ .

Solution: In the long time limit, we have

$$t = -\frac{\zeta_+ - \zeta_-}{c(\zeta_+) - c(\zeta_-)} = \lim_{\zeta_- \rightarrow \infty} -\frac{(\zeta_-^2 + 1)(4\pi^2 \zeta_-^4 + 1)}{\zeta_- (2\pi \zeta_- + 1)} = 2\pi \zeta_-^4$$

Therefore,

$$\begin{aligned}\zeta_- &= -\left(\frac{t}{2\pi}\right)^{1/4} \\ \zeta_+ &= 2\pi\left(\frac{t}{2\pi}\right)^{1/2}\end{aligned}$$

- (e) Find the motion of the shock  $x_s(t)$  at late times  $c_0 t \gg 1$ . Show that it agrees with the late time results derived in §4.5 of the Lecture Notes.

Solution:

$$x_s(t) = \lim_{t \rightarrow \infty} \zeta_-(t) + c(\zeta_-(t))t = \sqrt{2\pi t}$$

Compared with the lecture notes, there is no linear  $t$  dependence here. But this is OK, since for this particular initial profile,  $c_0$  (which is the coefficient of linear term) vanishes.