PHYSICS 200B : CLASSICAL MECHANICS
SOLUTION SET #3

(1) Create your own pixelated image to iterate under the cat map. You can also find many interesting images over the web. (Nothing pornographic, please![1]) Iterate the pixel coordinates under the cat map. Show how your image gets scrambled after a few iterations of the map, but is nevertheless recurrent. You'll need to write a computer code to do this problem.

Solution: Here is the matlab code I wrote:

```
function cat_map(filename,num_iter)

orig_fig=importdata(filename);
 %import the figure, which ideally should be square (n by n)

len=length(orig_fig);

[p,q]=meshgrid(1:len); %generate the meshgrid

q_prime=q+p;
p_prime=q+2*p;
q_prime=mod(q_prime-1,len)+1;p_prime=mod(p_prime-1,len)+1; %the cat map

%q_prime is the new x coordinate; p_prime is the new y coordinate

linear_indx=sub2ind([len,len],p_prime,q_prime);
linear_indx=linear_indx(:);
old_fig=orig_fig;
old_fig=reshape(old_fig,[len*len,3]);
for i_iter=1:num_iter
    new_fig=old_fig(linear_indx,:);
    old_fig=new_fig;
end
%figure;
new_fig=reshape(new_fig,[len,len,3]);
imagesc(new_fig);
axis equal
axis off
xlim([1,len]);
ylim([1,len]);
```

[1]Well, I suppose animal sex is OK, if you must.

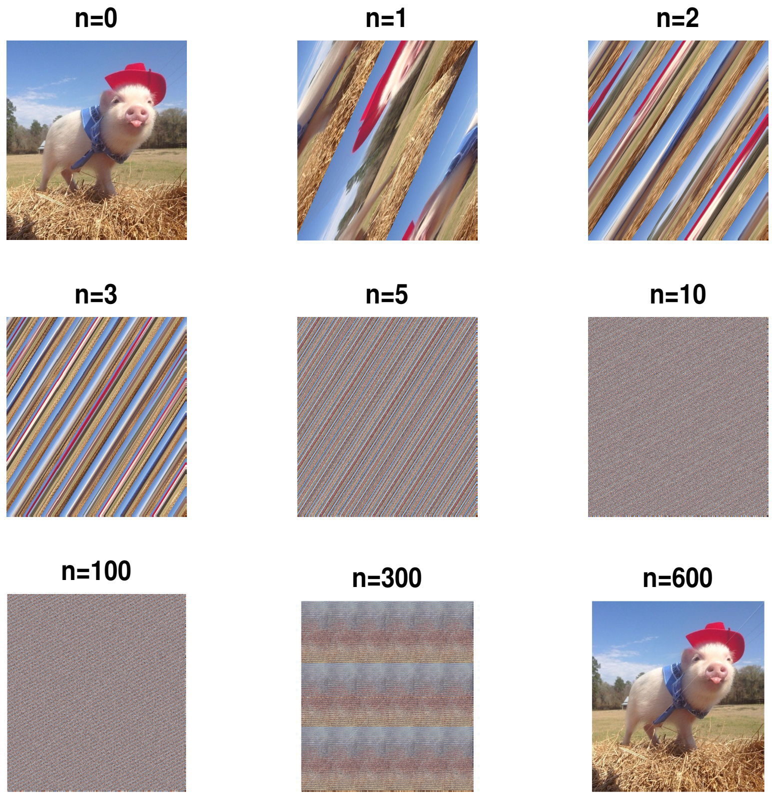

Figure 1: The cat map on a piglet.

The argument 'num_iter' is the number of the cat map you would like to perform on your cool figure. Here is mine.

(2) Numerically integrate the system

$$\dot{r} = r(1 - r^2) + \lambda r \cos\theta$$
$$\dot{\theta} = 1$$

with $0 < \lambda < 1$, and show that any initial condition lying between the concentric circles of radii $\sqrt{1 \pm \lambda}$ approaches a closed limit cycle in the long time limit. Choose whatever value of λ suits your taste.

Solution: One can use the function `ode45` to integrate the differential equation above with certain initial condition. Without loss of generality, $\theta(0)$ is set to be zero. The time evolution of r with various initial condition is shown in Fig. 2

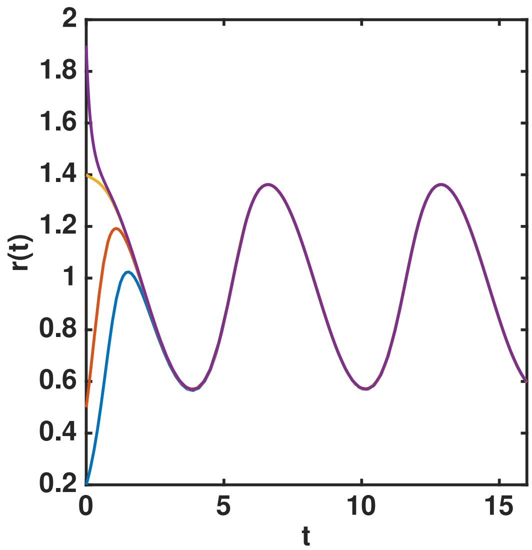

Figure 2:

(3) Consider the equation

$$\ddot{x} + x = \epsilon x^5$$

with $\epsilon \ll 1$.

(a) Develop a two term straightforward expansion for the solution and discuss its uniformity.

(b) Using the Poincaré-Lindstedt method, find a uniformly valid expansion to first order.

(c) Using the multiple time scale method, find a uniformly valid expansion to first order.

Solution: We can formally expand the solution as:

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \ldots \tag{1}$$

Plugging the expression into differentiate equation above, we obtain the equation for each order by matching the power of ϵ. The zeroth order simply describes harmonic oscillators:

$$\ddot{x}_0 + x_0 = 0 \tag{2}$$

The solution is $x_0(t) = A\cos(t + \phi)$. The second order equation is:

$$\ddot{x}_1 + x_1 = x_0^5 \tag{3}$$

The time dependence of right hand side (R.H.S) is known from the zeroth order solution:

$$x_0^5 = \frac{A^5}{16}(10\cos(t+\phi) + 5\cos(3t+3\phi) + \cos(5t+5\phi)) \tag{4}$$

The first term causes resonance in this order, and invalids the simple expansion.

Now we apply the Poincaré-Lindstedt method. We define $s = \Omega t$, then the Hamiltonian becomes:

$$\Omega^2 \frac{\partial^2 x}{\partial s^2} + x = \epsilon x^5 \tag{5}$$

Besides the expansion of x, we also need expand Ω^2 as $1+\epsilon a_1+\epsilon^2 a_2+\ldots$. Again we separate the equations by matching different orders of ϵ. The zeroth order is unchanged. The first order becomes:

$$\frac{\partial^2 x_1}{\partial s^2} + x_1 = x_0^5 - a_1 \frac{\partial^2 x_0}{\partial s^2} \tag{6}$$

Plugging in the zeroth order solution, the R.H.S reads:

$$x_0^5 = (\tfrac{5}{8}A^5 + a_1 A)\cos(s+\phi) + \tfrac{5}{16}A^5\cos(3s+3\phi) + \tfrac{1}{16}A^5\cos(5s+5\phi) \tag{7}$$

The resonant term can be removed by setting $a_1 = -\frac{5}{8}A^4$. Then the first order solution is:

$$x_1(s) = -\frac{A^5}{384}(15\cos(3s+3\phi) + \cos(5s+5\phi)) \tag{8}$$

Therefore, up to first order, we have

$$x(s) = A\cos(s+\phi) - \epsilon\frac{A^5}{384}(15\cos(3s+3\phi) + \cos(5s+5\phi)) \tag{9}$$

where $s = \sqrt{1 - \frac{5}{8}A^4\epsilon}\,t$.

Next, we apply the multiple time scale method to this problem. We define $\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1}\ldots$, along with the usual expansion of $x(t)$. The zeroth order equation is:

$$\frac{\partial^2 x_0}{\partial T_0^2} + x_0 = 0 \tag{10}$$

The solution is $x_0 = A\cos(T_0+\phi)$. For simplicity, we set θ to be $T_0+\phi$. Then we plug it into the first order equation:

$$\frac{\partial^2 x_1}{\partial \theta^2} + x_1 = A^5\cos^5\theta + 2\frac{\partial}{\partial T_1}A\sin\theta \tag{11}$$

The R.H.S reads:

$$\tfrac{5}{8}A^5\cos\theta + \tfrac{5}{16}A^5\cos 3\theta + \tfrac{1}{16}A^5\cos 5\theta + 2\frac{\partial}{\partial T_1}A\sin\theta \tag{12}$$

In order to remove the resonant term, we demand:

$$\begin{aligned} \frac{\partial}{\partial T_1}A &= 0 \\ \frac{\partial}{\partial T_1}\theta &= -\frac{5}{16}A^4 \end{aligned} \tag{13}$$

Therefore $\theta = T_0 - \frac{5}{16}A^4 T_1 + \theta_0 = (1 - \frac{5}{16}\epsilon A^4)t + \theta_0$. The solution of Eq. 17 is :

$$x_1(\theta) = -\frac{A^5}{384}(15\cos 3\theta + \cos 5\theta) \tag{14}$$

and the full solution is:

$$x(\theta) = A\cos\theta - \epsilon\frac{A^5}{384}(15\cos 3\theta + \cos 5\theta) \tag{15}$$

with $\theta = (1 - \frac{5}{16}\epsilon A^4)t + \theta_0$. This is consistent with what we got using he Poincaré-Lindstedt method.

(4) Consider the equation

$$\ddot{x} + \epsilon\,\dot{x}^3 + x = 0$$

with $\epsilon \ll 1$. Using the multiple time scale method, find a uniformly valid expansion to first order.

We apply the multiple time scale method to this problem. We define $\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1}...$, along with the usual expansion of $x(t)$. The zeroth order equation is:

$$\frac{\partial^2 x_0}{\partial T_0^2} + x_0 = 0 \tag{16}$$

The solution is $x_0 = A\cos(T_0 + \phi)$. For simplicity, we set θ to be $T_0 + \phi$. Then we plug it into the first order equation:

$$\frac{\partial^2 x_1}{\partial \theta^2} + x_1 = A^3\sin^3\theta + 2\frac{\partial}{\partial T_1}A\sin\theta \tag{17}$$

The R.H.S reads:

$$\frac{3}{4}A^3\sin\theta - \frac{1}{4}A^3\sin 3\theta + 2\frac{\partial}{\partial T_1}A\sin\theta \tag{18}$$

In order to remove the resonant term, we demand:

$$\begin{aligned}\frac{\partial}{\partial T_1}A &= -\frac{3}{8}A^3\\ \frac{\partial}{\partial T_1}\theta &= 0\end{aligned} \tag{19}$$

Therefore $\frac{1}{A^2} = \frac{3}{4}T_1 + A_0 = \frac{3}{4}\epsilon t + A_0$, where A_0 is an integrating constant, and $\theta = T_0 + \theta_0 = t + \theta_0$. The solution of the first order equation is :

$$x_1(\theta) = \frac{1}{32}A^3\sin 3\theta \tag{20}$$

and the full solution is:

$$x(\theta) = A\cos\theta - \epsilon\frac{A^3}{32}\sin 3\theta \tag{21}$$

with $A = (\frac{3}{4}\epsilon t + A_0)^{-1/2}$ and $\theta = t + \theta_0$.

(5) Analyze the forced oscillator

$$\ddot{x} + x = \epsilon\left(\dot{x} - \frac{1}{3}\dot{x}^3\right) + \epsilon\, f_0\,\cos(t + \epsilon\nu t)$$

using the discussion in §3.3.1 and §3.3.2 of the notes as a template.

Solution: We still apply the multiple time scale method. The zeroth order solution again is $x_0 = A\cos(T_0 + \phi)$. The first order equation is:

$$\frac{\partial^2 x_1}{\partial T_0^2} + x_1 = (\frac{A^3}{4} - A)\sin(T_0+\phi) - \frac{A^3}{12}\sin(3T_0+3\phi) + f_0\cos(T_0+\nu T_1) + 2\frac{\partial}{\partial T_1}A\sin(T_0+\phi) \quad (22)$$

In order to eliminate all the terms with frequency 1, we demand:

$$\begin{aligned}(\frac{A^3}{4} - A)\cos\phi - f_0\sin(\nu T_1) + 2\frac{\partial A}{\partial T_1}\cos\phi - 2A\frac{\partial\phi}{\partial T_1}\sin\phi = 0\\ (\frac{A^3}{4} - A)\sin\phi + f_0\cos(\nu T_1) + 2\frac{\partial A}{\partial T_1}\sin\phi + 2A\frac{\partial\phi}{\partial T_1}\cos\phi = 0\end{aligned} \quad (23)$$

Now we seek the fixed point solution $(A(T_1) = A, \phi(T_1) = \nu T_1 + \phi_0)$ (Here I've used a more symmetric convention compared with the one in the lecture notes). The above equation can be organized as:

$$(\frac{A^3}{4} - A + 2iA\nu)e^{i\phi_0} + if_0 = 0 \quad (24)$$

Therefore, A is given by the real solutions of the following equation:

$$(\frac{A^3}{4} - A)^2 + 4A^2\nu^2 = f_0^2 \quad (25)$$

Once A is obtained, one can find ϕ_0 easily.

The root structure is determined by the following polynomial:

$$y^3 - 8y^2 + 16(1+4\nu^2)y - 16f_0^2 = 0 \quad (26)$$

The two extrema are $y = \frac{4}{3}(2 \mp \sqrt{1-12\nu^2})$ with the corresponding values:

$$g^{\pm} = -16f_0^2 + \frac{128}{27}(1 \pm \sqrt{1-12\nu^2} + 12\nu^2(3 \mp \sqrt{1-12\nu^2})) \quad (27)$$

If $g^+ > 0$ and $g^- < 0$, then there are three real solutions. The number of real solutions of A depends on f_0 and ν and is illustrated in Fig. 3(a).

Next, we analyze the stability of these solutions. Adding the second equation to the first equation in Eq. 23 with an factor of i, we obtain:

$$(\frac{4}{3}A^3 - A)e^{i\phi_0} + if_0 + 2\frac{\partial A}{\partial T_1}e^{i\phi_0} + i2A\nu e^{i\phi_0} + iA\frac{\partial\phi_0}{\partial T_1}e^{i\phi_0} = 0 \quad (28)$$

After linearization around the fixed points, the equation can be simplified to

$$\frac{\partial}{\partial T_1}\begin{pmatrix}\delta A\\ \delta\phi_0\end{pmatrix} = \begin{pmatrix}\frac{1}{2} - \frac{3}{8}A^2 & A\nu\\ -\frac{\nu}{A}, & \frac{1}{2} - \frac{A^2}{8}\end{pmatrix}\begin{pmatrix}\delta A\\ \delta\phi_0\end{pmatrix} \quad (29)$$

The matrix has two eigenvalues:

$$E = \frac{1}{8}(4 - 2A^2 \pm \sqrt{A^4 - 64\nu^2}) \quad (30)$$

from which the property of the attractors can be determined straightforward: the curve $A^4 - 64\nu^2 = 0$ separates spiral region and node region. Within the spiral region, the curve $4 - 2A^2$ separates stable spiral region and unstable spiral region. Within the node region, if both of the eigenvalues are larger than zero, the attractor is a unstable node; if both of the eigenvalues are less than zero, the attractor is a stable node; otherwise the attractor is a saddle point. The phase diagram is plotted in Fig. 3(b).

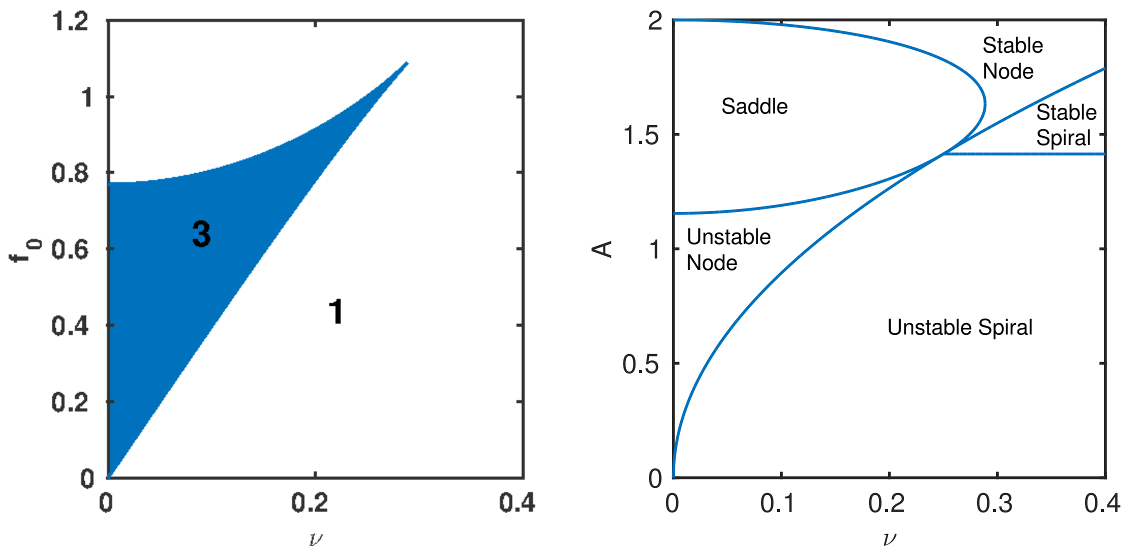

Figure 3: (a) The number of solutions that A can take. (b) The phase diagram of the attractors.