(b) From Equation 7-5, \( E_{n_1 n_2} = \frac{\hbar^2 \pi^2}{2mL^2} (n_1^2 + n_2^2) \)

(c) The lowest energy degenerate states have quantum numbers \( n_1 = 1, \ n_2 = 2, \) and \( n_1 = 2, \ n_2 = 1. \)

7-9. (a) For \( n = 3, \ \ell = 0, 1, 2 \)

(b) For \( \ell = 0, m = 0. \) For \( \ell = 1, m = -1, 0, +1. \) For \( \ell = 2, m = -2, -1, 0, +1, +2. \)

(c) There are nine different \( m \)-states, each with two spin states, for a total of 18 states for \( n = 3. \)

7-10. (a) For \( \ell = 4 \)

\[ L = \sqrt{\ell (\ell + 1)} \hbar = \sqrt{4(5)} \hbar = \sqrt{20} \hbar \]

\[ m_\ell = 4\hbar \]

\[ \theta_{\min} = \cos^{-1} \frac{4}{\sqrt{20}} \rightarrow \theta_{\min} = 26.6^\circ \]

(b) For \( \ell = 2 \)

\[ L = \sqrt{6} \hbar \hspace{1cm} m_\ell = 2\hbar \]

\[ \theta_{\min} = \cos^{-1} \frac{2}{\sqrt{6}} \rightarrow \theta_{\min} = 35.3^\circ \]

7-12. (a)

\[ \ell = 1 \]

\[ |L| = \sqrt{2} \hbar \]
(d) $|L| = \sqrt{\ell(\ell+1)} \ h$  (See diagrams above.)

7-13. $L^2 = L_x^2 + L_y^2 + L_z^2 \rightarrow L_x^2 + L_y^2 = L^2 - L_z^2 = \ell(\ell+1) h^2 - (\ell h)^2 = (6 - m^2) h^2$
(a) \((L_x^2 + L_y^2)_{\text{min}} = (6 - 2^2)h^2 = 2h^2\)

(b) \((L_x^2 + L_y^2)_{\text{max}} = (6 - 0^2)h^2 = 6h^2\)

(c) \(L_x^2 + L_y^2 = (6 - 1)h^2 = 5h^2\) \(L_x\) and \(L_y\) cannot be determined separately.

(d) \(n = 3\)

7-15. \[ L = r \times p \quad \frac{dL}{dt} = \frac{dr}{dt} \times p + r \times \frac{dp}{dt} \]

\[ \frac{dr}{dt} \times p = v \times m v = m v \times v = 0 \quad \text{and} \quad r \times \frac{dp}{dt} = r \times F. \]  
Since for \(V = V(r)\), i.e., central forces, \(F\) is parallel to \(r\), then \(r \times F = 0\) and \(\frac{dL}{dt} = 0\)

7-16. (a) For \(\ell = 3, n = 4, 5, 6, \ldots\) and \(m = -3, -2, -1, 0, 1, 2, 3\)

(b) For \(\ell = 4, n = 5, 6, 7, \ldots\) and \(m = -4, -3, -2, -1, 0, 1, 2, 3, 4\)

(c) For \(\ell = 0, n = 1\) and \(m = 0\)

(d) The energy depends only on \(n\). The minimum in each case is:

\[ E_4 = -13.6eV / n^2 = -13.6eV / 4^2 = -0.85eV \]

\[ E_5 = -13.6eV / 5^2 = -0.54eV \]

\[ E_1 = -13.6eV \]

7-17. (a) \(6f\) state: \(n = 6, \ell = 3\)

(b) \(E_6 = -13.6eV / n^2 = -13.6eV / 6^2 = -0.38eV\)

(c) \(L = \sqrt{\ell(\ell + 1)}h = \sqrt{3(3+1)}h = \sqrt{12}h = 3.65 \times 10^{-34} \text{J}\cdot\text{s}\)

(d) \(L_z = m\hbar \quad L_z = -3\hbar, -2\hbar, -1\hbar, 0\hbar, 1\hbar, 2\hbar, 3\hbar\)
7-21. (a) For the ground state, \( P(r) \Delta r = \psi^3 \left( 4\pi r^3 \right) \Delta r = \frac{4r^2}{a_0^3} e^{-2r/a_0} \Delta r \)

For \( \Delta r = 0.03a_0 \), at \( r = a_0 \) we have \( P(r) \Delta r = \frac{4a_0^2}{a_0^3} e^{-2(0.03a_0)} = 0.0162 \)

(b) For \( \Delta r = 0.03a_0 \), at \( r = 2a_0 \) we have \( P(r) \Delta r = \frac{4(2a_0)^3}{a_0^3} e^{-2(0.03a_0)} = 0.0088 \)

7-22. \( P(r) = Cr^2 e^{-2r/a_0} \) For \( P(r) \) to be a maximum,

\[
\frac{dP}{dr} = C \left[ r^2 \left( \frac{-2Z}{a_0} \right) e^{-2r/a_0} + 2re^{-2r/a_0} \right] = 0 \rightarrow C \frac{2Zr}{a_0} \left( \frac{a_0}{Z} - r \right) e^{-2r/a_0} = 0
\]

This condition is satisfied with \( r = 0 \) or \( r = a_0/Z \). For \( r = 0 \), \( P(r) = 0 \) so the maximum \( P(r) \) occurs for \( r = a_0/Z \).

7-27. For the most likely value of \( r \), \( P(r) \) is a maximum, which requires that (see Problem 7-25)

\[
\frac{dP}{dr} = A \cos^2 \theta \left[ r^4 \left( \frac{-Z}{a_0} \right) e^{-2r/a_0} + 4r^3 e^{-2r/a_0} \right] = 0
\]

For hydrogen \( Z = 1 \) and \( A \cos^2 \theta (r^3/a_0) (4a_0 - r) e^{-r/a_0} = 0 \). This is satisfied for \( r = 0 \) and \( r = 4a_0 \). For \( r = 0 \), \( P(r) = 0 \) so the maximum \( P(r) \) occurs for \( r = 4a_0 \).

7-65. \( \psi_{100} = \frac{1}{\sqrt{\pi}} \left( \frac{Z}{a_0} \right)^{3/2} e^{-2r/a_0} \) (Equations 7-30 and 7-31)

\[ P(r) = 4\pi r^3 \psi_{100}^* \psi_{100} \] (Equation 7-32)

\[ = 4\pi r^3 \left( \frac{Z^3}{\pi a_0^3} e^{-2r/a_0} \right) = \frac{4Z^3}{a_0^3} r^3 e^{-2Zr/a_0} \]

\[
\langle r \rangle = \int_0^\infty rP(r) dr = \int_0^\infty \frac{4Z^3}{a_0^3} r^3 e^{-2Zr/a_0} dr
\]

\[ = \frac{a_0}{4Z} \left( \frac{2Zr}{a_0} \right)^3 e^{-2Zr/a_0} d \left( 2Zr/a_0 \right) = \frac{a_0^2}{4Z} \times 3! = \frac{3a_0}{2Z} \]
7-68. \( \theta_{\min} = \cos^{-1}\left[ \frac{m_i h}{\sqrt{\ell (\ell + 1)} h} \right] \) with \( m_i = \ell \).

\[ \cos \theta_{\min} = \ell \sqrt{\ell (\ell + 1)}. \] Thus, \( \cos^2 \theta_{\min} = \ell^2 \left[ \frac{\ell}{\ell (\ell + 1)} \right] = 1 - \sin^2 \theta_{\min} \)

or, \( \sin^2 \theta_{\min} = 1 - \frac{\ell^2}{\ell (\ell + 1)} = \frac{\ell (\ell + 1) - \ell^2}{\ell (\ell + 1)} = \frac{\ell^3 + \ell - \ell^2}{\ell (\ell + 1)} \)

And, \( \sin \theta_{\min} = \left( \frac{1}{\ell + 1} \right)^{1/2} \) For large \( \ell \), \( \theta_{\min} \) is small.

Then \( \sin \theta_{\min} \approx \theta_{\min} = \left( \frac{1}{\ell + 1} \right)^{1/2} \approx \frac{1}{(\ell)^{1/2}} \)

7-70. \( P(r) = \frac{4Z^3}{a_0^3} r^2 e^{-2r/a_0} \) (See Problem 7-65)

For hydrogen, \( Z = 1 \) and at the edge of the proton \( r = R_0 = 10^{-15} m \). At that point, the exponential factor in \( P(r) \) has decreased to:

\[ e^{-2R_0/a_0} = e^{-2(10^{-15})/(0.529 \times 10^{-10} m)} = e^{-3.78 \times 10^{-5}} \approx 1 - 3.78 \times 10^{-5} \approx 1 \]

Thus, the probability of the electron in the hydrogen ground state being inside the nucleus, to better than four figures, is:

\[ P(r) = \frac{4r^2}{a_0^3} \int_0^{R_0} P(r) dr = \frac{4r^2}{a_0^3} \frac{4r^2}{a_0^3} \frac{r^3}{3} \bigg|_0^{R_0} = \frac{4}{a_0^3} \left( \frac{R_0^3}{3} \right) = \frac{4(10^{-15} m)^3}{3(0.529 \times 10^{-10} m)^3} = 9.0 \times 10^{-15} \]

7-72. (a) Substituting \( \psi(r, \theta) \) into Equation 7-9 and carrying out the indicated operations yields (eventually)

\[ -\frac{\hbar^2}{2\mu} \psi(r, \theta) \left[ 2l^2 r^2 - \frac{1}{4a_0^2} \right] - \frac{\hbar^2}{2\mu} \psi(r, \theta) (-2l^2) + V \psi(r, \theta) = E \psi(r, \theta) \]
Canceling $\psi(r, \theta)$ and recalling that $r^2 = 4a_0^2$ (because $\psi$ given is for $n = 2$) we have
\[
\frac{\hbar^2}{2\mu} \left( -\frac{1}{4a_0^2} \right) + V = E
\]

The circumference of the $n = 2$ orbit is: $C = 2\pi(4a_0) = 2\lambda \rightarrow a_0 = \lambda / 4\pi = 1 / 2k$.

Thus, $-\frac{\hbar^2}{2\mu} \left( -\frac{1}{4/4k^2} \right) + V = E \rightarrow \frac{\hbar^2k^3}{2\mu} + V = E$

(b) or $\frac{p^2}{2m} + V = E$ and Equation 7-9 is satisfied.

\[
\int_0^\infty \psi^2 dx = \int A^2 \left( \frac{r}{a_0} \right)^2 e^{-r/a_0} \cos^2 \theta r^3 \sin \theta dr d\theta d\phi = 1
\]

\[
A^2 \int_0^\infty \left( \frac{r}{a_0} \right)^2 e^{r/a_0} r^3 dr \left[ \cos^2 \theta \sin \theta d\theta \right] \int_0^{2\pi} d\phi = 1
\]

Integrating (see Problem 7-23),
\[
A^2 \left( 6a_0^3 \right) (2/3) (2\pi) = 1
\]

\[
A^2 = 1 / 8a_0^3 \pi \rightarrow A = \sqrt{1 / 8a_0^3 \pi}
\]