HW problem week 4

Your turned in assignment should be clearly written and easy to follow! Learning how to explain your work in a way that is as easy as possible to follow is an important part of your training as a physicist. An incoherent mess of equations with a correct final answer could receive less points than a solution which is clearly explained at every step but has an algebra mistake somewhere. Once you’ve solved the problem, you can rewrite it on a new piece of paper for clarity if you need to.

In class, we derived the Bohr model of the atom following the assumption that the angular momentum $\vec{L}$ is quantized in units of $\hbar$. In this problem, you will repeat the derivation using the relativistic expressions for energy and momentum. In this problem (and always in life), you should try to combine physical constants into $\alpha$, the fine structure constant, wherever possible.

1. Assume that the electron travels in a circular orbit with constant angular speed $\omega$. Using Newton’s law $\vec{F} = d\vec{p}/dt$ with the relativistic expression for the momentum, obtain an expression relating the radius $r$ and the velocity $v$. Hint: Since the speed is constant, the relativistic answer for $d\vec{p}/dt$ is related in a simple way to the nonrelativistic answer.

2. By quantizing the relativistic angular momentum (still $\vec{r} \times \vec{p}$) to integer multiples of $\hbar$, and using the result of part 1, show that the speed of the $n$'th Bohr orbit is the same as in the nonrelativistic case:

$$v_n = \frac{Z \alpha c}{n}.$$

3. Using the result of parts 1 and 2, calculate the radius of the $n$'th Bohr orbit.

4. The formula $E = \sqrt{(mc^2)^2 + p^2 c^2} = \gamma mc^2$ is for a free particle; in the presence of a potential we add the potential energy $U$. Using the result of the previous parts, calculate the relativistic answer for the total energy.

5. By expanding in powers of $\alpha$ using the Taylor expansion

$$\sqrt{1+\epsilon} \approx 1 + \epsilon/2 - \epsilon^2/8 + O(\epsilon^3)$$

for small $\epsilon$, show that the energy levels are of the form

$$E_n = (\text{rest energy}) + (\text{bohr result}) - \frac{mc^2}{8} \left(\frac{\alpha Z}{n}\right)^4 + O(\alpha^6).$$
6. The relativistic Bohr model of the atom actually makes a prediction for size of the largest stable element. By looking at the results derived in this problem and imposing some physical assumptions on the radius, velocity, or energy, find a condition on the atomic number \( Z \) of a hydrogen like atom.