Stochastic population genetics: homework 4
To be returned on May 24

May 17, 2017

1 First passage time for a Brownian particle in 3D

Consider an absorbing sphere of radius a. An isotropically diffusing particle (diffusivity D) is started at radius $r_0 > a$ and is free to diffuse in the region $a < r < \infty$.

a. Determine and solve the equation for the probability of absorption $p_{\rm abs}(r_0)$. Specify the boundary conditions.

b. What is the probability of absorption if the particle diffuses inside the sphere, i.e. $r_0 < a$?

c. Write the equation satisfied by the probability distribution $\mathcal{P}_{\rm abs}(r_0, t)$ for the time t of absorption conditional to the starting position r_0. Specify the boundary conditions.

d. Take the Laplace transform of the equation in c. and specify the boundary conditions.

e. For the case where $r_0 > a$, look for the solution in the form $\mathcal{P}_{\rm abs}(r_0, t) = \frac{a}{r_0}\Phi(r_0, t)$ and write down the equation for Φ. You should obtain that the resulting equation is the same as for a 1D diffusing particle. Use results derived in class to write down the solution, the inverse Laplace transform and the expression for $\mathcal{P}_{\rm abs}(r_0, t)$.

2 Probability of extinction in noisy Malthusian growth

Consider the evolution of a population governed by the following forward Kolmogorov (Fokker-Planck) equation

$$\partial_t P(N, t|N_0, 0) + \alpha\partial_N \left(NP(N, t|N_0, 0)\right) = \beta\partial_N^2 \left(N^2 P(N, t|N_0, 0)\right) , \qquad (1)$$

where α and β are two constants ($\beta > 0$). In class we have calculated the probability of extinction, i.e. reaching a level $N = a$, where a is order unity. Here, we shall be looking at the times of extinction.

a. Write down the equation satisfied by the probability distribution $\mathcal{P}_{\text{abs}}(N_0, t)$ for the time t of absorption conditional to the initial size N_0 of the population. Specify the boundary conditions.

b. Make the change of variable $z = \log\left(\frac{N_0}{a}\right)$. You should obtain that the resulting equation is analogous to the equation for a particle moving with constant velocity and diffusivity, which we have solved in class. Use those results to obtain the expression for $\mathcal{P}_{\text{abs}}(N_0, t)$. Identify and discuss the two possible regimes that should emerge from the solution.