Calculation of the Kolmogorov Entropy for Motion Along a Stochastic Magnetic Field

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An expression for the Kolmogorov entropy has been derived. Excellent agreement between a probability description and direct dynamical computations has been found.

A typical magnetic field of interest to magnetic confinement physics may be taken to be of the form

$$\vec{B} = B_0 \hat{z} + B_1(x) \hat{y} + \delta \vec{B},$$  \hspace{1cm} (1)$$

where $|\delta \vec{B}| \ll B_0, B_1$ might result from any short-wavelength plasma microinstabilities. When $\delta B$ exceeds a very small value such that neighboring magnetic islands overlap, the field structure becomes stochastic, and can be characterized by two geometrical properties.\(^1\) First, a given field line diffuses in $x$, and, second, two neighboring field lines diverge from each other—the mean distance between them increasing as $d \sim \exp(z/L_\perp)$. The diffusion of a field line has obvious implications for the confinement of particles orbiting along the field and has been calculated using quasilinear theory.\(^2\) However, the divergence of field lines is also important in calculating transport since it prevents the reversible wandering of a particle back and forth along a single field line as velocity is reversed by collisions.\(^1\)

The value $1/L_\perp$ is called the Kolmogorov entropy and is defined by the formula\(^3\)

$$h = \lim_{z \to \infty} \lim_{d_0 \to 0} \left[ z \ln \left( \frac{d(z)}{d_0} \right) \right].$$  \hspace{1cm} (2)$$

Since exponential divergence is a statistical property, one expects that averaging along trajectories ($z \to \infty$) can be replaced by phase-space averaging with a proper distribution function. It is our purpose in this note to introduce such a statistical description and to give a derivation of the dependence of $h$ on the properties of the fields given by Eq. (1). Our main result, Eq. (19), is similar to that already obtained by Krommes, Kleva, and Oberman,\(^4\) but we have tried to make the basic assumptions more transparent and the derivation more quantitative. We have also developed a simplified model of a stochastic field for which we have computed $h$ directly using Eq. (2). This model allows detailed numerical study which agrees very well with the theoretical predictions.
Consider a magnetic field given by Eq. (1). In order to model poloidal and toroidal periodicity we assume that the system is periodic in the $y$ direction with period $2\pi a$ and in the $z$ direction with period $2\pi \ell$. Then $\delta B$ can be written in the form

$$\delta B = B_0 \sum_{m,n} \tilde{B}_{mn}(x) \exp[i(my/a - nz/R)] + \text{c.c.} \quad (3)$$

The turbulent nature of $\tilde{B}_{mn}$ is introduced by assuming that they have random phases but constant, saturated, amplitudes. Equations for the coordinates of the field lines are

$$dx/dz = B_y/B_\phi; \quad dy/dz = B_z/B_\theta. \quad (4)$$

Assuming shear, $L_z^{-1} = B_0^{-1} dB_\phi/dx$, being constant, we can write (4) in the dimensionless form

$$dx'/dz' = b; \quad dy'/dz' = x'. \quad (5)$$

Here $b = \delta B_\phi/B_\phi$ is a small parameter in our problem and $x' = x/L_{z,1}, y' = y/L_{z,2}, z' = z/L_{z,3}$. We will omit primes in the following. In order to calculate $h$ we have to find the divergence rate of two arbitrarily close particle trajectories. Equations for the differences in position $x = x_2 - x_1$ and $Y = y_2 - y_1$ can be easily written using Eq. (5):

$$dx/dz = (\partial b/\partial x)X + (\partial b/\partial y)Y, \quad dY/dz = X. \quad (6)$$

It is convenient to put $X = \Lambda Y$ so that

$$d\Lambda/dz = -\Lambda^2 + (\partial b/\partial x)\Lambda + \partial b/\partial y; \quad (7)$$

$$dY/dz = \Lambda Y.$$  

Since it will turn out later that the characteristic width of all functions with respect to $\Lambda$ is $\Lambda = 1 - h \sim b^{2/3}$, we may neglect the term $\Lambda b b/\partial x$ if $h \ll 1$. The same condition allows us to neglect the contribution from the term $\delta B_\phi$ in comparison with the shear term in Eq. (6). The case of zero shear has been recently considered by Kadomtsev and Pogutse. The basic limitation of our theory is determined by the condition $h \ll 1$. We note that we use here dimensionless $h$ appropriate to Eq. (5).

Consider a number of test particles which are distributed initially with the probability $P(x_1,0)$ (two-point distribution function which is assumed to be a smooth function) in a cross section $z = 0$, $x_1$ is a vector with the components $(x, y, \Lambda, Y)$. If these particles are moving freely along the field lines remaining in the same cross section $z$, then the evolution of their distribution function $P(x_1,z)$ is described by the continuity equation

$$\partial P/\partial z + 4 \sum_{i=1}^{4} \partial (v_i P)/\partial x_i = 0, \quad (8)$$

where $v_i$ is a generalized velocity with components $(\partial, x, -\Lambda^2 + \partial b/\partial y, \Lambda Y)$ and $z$ coordinate plays the role of time. With the use of a statistical description, the Kolmogorov entropy can be defined by the formula

$$h = (\partial/\partial x)\left[\frac{1}{4} \int (X^2 + Y^2) (P/\partial dY)\right], \quad (9)$$

where $\langle P \rangle$, which is a function only of $\Lambda$, $Y$, and $z$, is a probability averaged over $x$ and $y$ which varies slowly in $z$. It satisfies the equation

$$\frac{\partial \langle P \rangle}{\partial z} - \frac{\partial}{\partial \Lambda}(\Lambda^2 \langle P \rangle) + \frac{\partial}{\partial Y}(\Lambda Y \langle P \rangle) = 0 \quad (10)$$

We derive this equation by averaging Eq. (6). The fluctuating part $\hat{P} = P - \langle P \rangle$ is small and satisfies a first-order equation

$$\frac{\partial \hat{P}}{\partial z} + \frac{\partial}{\partial x}(\hat{P} \langle P \rangle) = - \frac{\partial}{\partial \Lambda}(\Lambda^2 \hat{P}) + \frac{\partial}{\partial Y}(\Lambda Y \hat{P})$$

$$= - \frac{\partial}{\partial x} \langle P \rangle - \frac{\partial}{\partial \Lambda}(\Lambda^2 \langle P \rangle). \quad (11)$$

We may neglect the third and fourth terms in this equation because $\Lambda \ll 1$. We can now solve Eq. (11) taking $\partial \langle P \rangle / \partial \Lambda$ to be slowly varying because of the smallness of $b$ and substitute $\hat{P}$ into (10) to get

$$\frac{\partial \langle P \rangle}{\partial z} - \frac{\partial}{\partial \Lambda}(\Lambda^2 \langle P \rangle) + \frac{\partial}{\partial Y}(\Lambda Y \langle P \rangle)$$

$$= - D_{ef} \frac{\partial}{\partial \Lambda} \langle P \rangle + \frac{\partial}{\partial \Lambda} \langle P \rangle = 0, \quad (12)$$

$$D_{ef} = \frac{R L_{z,1}}{a^2} \sum_{m,n} \langle \mu b_{mn}(x) [\partial \delta (mx/R + n)] \rangle. \quad (13)$$

These calculations are very similar to the derivation of the magnetic diffusion coefficient in quasilinear theory. Notice that the first term on the right-hand side of Eq. (11) averages to zero. We will deal later only with $\langle P \rangle$ and suppress the angular brackets hereafter.

A general initial-value solution for $P(\Lambda, Y, z)$ is impractical to obtain. It is sufficient to look for moments. Let us define

$$P_0(\Lambda, z) = \int_{-\infty}^{\infty} P \ dY$$

and

$$P_\epsilon(\Lambda, z) = \frac{1}{4} \int_{-\infty}^{\infty} (\Lambda Y^2) P \ dY;$$

then

$$\frac{\partial P_0}{\partial z} - \frac{\partial}{\partial \Lambda}(\Lambda^2 P_0) = D_{ef} \frac{\partial}{\partial \Lambda} P_0 = 0 \quad (14)$$
and
\[
\frac{d}{dz} \left( \frac{\partial P_x}{\partial \Lambda} \right) - \frac{\partial}{\partial \Lambda} \left( \frac{\Lambda^2 P_x}{\Lambda^2} \right) - D_{ef} \frac{\partial^2 P_x}{\partial \Lambda^2} = \Lambda P_x, \tag{15}
\]

The steady-state solution of Eq. (14) valid for large \( z \), and well behaved at \( \Lambda = \pm \infty \), is
\[
P_x(\Lambda) = C \exp(-\Lambda^2/3D_{ef})
\quad \times \int_\Lambda^\infty \exp(\Lambda'^3/3D_{ef}) d\Lambda',
\tag{16}
\]

where \( C \) is a normalization constant determined from the condition \( \int P_x d\Lambda = 1 \). We may now solve for
\[
P_x = h^2 P_0(\Lambda) + Q(\Lambda),
\tag{17}
\]

where \( Q \) obeys
\[
\frac{d}{d\Lambda} \left( \Lambda^2 Q \right) + D_{ef} \frac{\partial^2 Q}{\partial \Lambda^2} = (h - \Lambda) P_0,
\tag{18}
\]

with the boundary conditions \( Q(\pm \infty) = 0 \). We note that the constant \( h \) is just the Kolmogorov entropy as defined by Eq. (9). It can be determined by the solvability condition for Eq. (18). This may be obtained by integrating Eq. (18) over \( \Lambda \) from \( -\infty \) to \( +\infty \). The left-hand side of Eq. (18) is a full derivative and gives zero. Hence we deduce that \( h \) is given by
\[
h = \frac{1}{3} \left( \frac{2}{3} \right)^{1/3} (3D_{ef})^{1/3}.
\tag{19}
\]

We should mention here that in all previous formulas we have to use a cutoff parameter \( \pm \Lambda_0 \) \((1 \gg \Lambda_0 \gg h)\) instead of \( \pm \infty \). But because all integrals are rapidly converging within region of \(|\Lambda| \leq h\) we can formally extend the integration to \( \pm \infty \).

Going back to dimensional units we can write \( L_c = L_0/h \). The basic approximation in our theory can also be written as \( L_x/L_c \ll 1 \).

Let us turn now to the results of our numerical computations. We used the following dimensionless units: \( 2\pi a = 1, 2\pi R = 1, L_s = 1 \), and took \( b_{mn} = e \exp(i2\pi m \varphi_{np})/2i, m = 1, \ldots, M, p = n \mod N, n = 0 \pm 1, \pm 2, \ldots \), where phases \( \varphi_{np} \) are randomly chosen numbers between 0 and 1. Summing over \( n \) with fixed \( p \) in Eq. (3) gives us 6 functions, allowing differential Eq. (5) to be reduced to a simple mapping:
\[
y_{i+1} = y_i + x_i/N,
\]
\[
x_{i+1} = x_i + f(y_{i+1})/N.
\tag{20}
\]

Here
\[
f(y_{i+1}) = \epsilon = \sum_{\mu=1}^{M} \sum_{p=0}^{1-1} \sin[2\pi m(y_{i+1} + \varphi_{np}) - (i + 1)p/N]
\]

and one step corresponds to \( \Delta z = 1/N \). Equations (13) and (19), applied to this specific model, give
\[
D_{ef} = \pi^2 \epsilon^2 \sum_{m=1}^{M} m^2 \approx \pi^2 \epsilon^2 M^3/3, M \gg 1,
\]

and
\[
h_{\text{th}} = 0.54M \epsilon^{2/3}.
\tag{21}
\]

On the other hand, \( h \) has been calculated directly using Eqs. (20) and the numerical method described by Casartelli et al.\(^6\) To obtain reasonable accuracy approximately \( 10^4 \) steps were necessary. The results of these computations are presented in Fig. 1. One can see a remarkable agreement with the theoretical formula (21). We have found that \( h \) is equal to zero below the stochastic transition (where there exist good magnetic surfaces) and has a well-defined positive value above the stochasticity threshold. This value is independent of initial conditions, that is, it appears numerically that \( h \) has the same value for almost all trajectories. Stochastic transition takes place at \( \epsilon_t = 2/M^3 \), in good agreement with the resonance overlapping criterion.\(^1,3\) The results do not depend on the step size \( \Delta z = 1/N \) in the region \( h < N \), but \( h \) does depend on \( N \) when \( h > N \), and scales in a completely different way with \( \epsilon \) and \( M \) in this region: \( h \propto \ln(M^3 \epsilon^2) \). In order to derive a formula for \( h \) in this case, let us consider eigenvalues of tangential transformation for the mapping (20),\(^3\)
\[
\lambda^* = 1 + K/2 \pm (K^2/4 + K)^{1/2},
\tag{22}
\]

where \( K = f'(y_{i+1})/N^2 \). If \( |K| \gg 1 \) then \( \lambda^* = K \) and

![FIG. 1. The numerically obtained Kolmogorov entropy. The solid curve is the theoretical result. Also shown are the stochastic transition points \( \epsilon_M \).](image-url)
divergence takes place preferentially in the x direction. The Kolmogorov entropy can be easily written in this case:

\[
h = \frac{1}{2} N \ln \langle K'^2 \rangle_{\text{trajectory}},
\]

(23)

The angular brackets here mean averaging along the trajectory. Because \( M \gg 1 \) and \( \theta_{mf} \) are random we can make use of a central-limit theorem and perform averaging in Eq. (23) with the Gaussian distribution

\[
P(K) = \exp\left(-K^2/2\langle K'^2 \rangle_{\text{trajectory}}\right)/(2\pi \langle K'^2 \rangle_{\text{trajectory}})^{1/2},
\]

where \( \langle K'^2 \rangle_{\text{trajectory}} = (2/3)\pi^2 e^2 (M/N)^3 \). After a simple calculation we can get from (23)

\[
h = \frac{1}{2} N \ln \left[ \frac{3}{2} \langle \theta^2 \rangle e^{-c} (M/N)^3 e^2 \right].
\]

(24)

Here \( c = 0.577 \) is the Euler constant. This formula is also in very good agreement with computer calculations. Using Eq. (21), we can rewrite Eq. (24) as

\[
h = \frac{3}{2} N \ln(2.3h_e/N),
\]

applicable when \( h > N \). But in the region \( h < N \) we have to use Eq. (21).

We believe that our results have some general interest: They illustrate how a statistical description can be introduced in a deterministic but stochastically unstable system.

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7. More detailed results of these computations will be published elsewhere.