5 Quantum Statistics: Summary

- **Second-quantized Hamiltonians:** A noninteracting quantum system is described by a Hamiltonian \( \hat{H} = \sum_\alpha \varepsilon_\alpha \hat{n}_\alpha \) where \( \varepsilon_\alpha \) is the energy eigenvalue for the single particle state \( \psi_\alpha \) (possibly degenerate), and \( \hat{n}_\alpha \) is the number operator. Many-body eigenstates \( |\vec{n}\rangle \) are labeled by the set of occupancies \( \vec{n} = \{ n_\alpha \} \), with \( \hat{n}_\alpha |\vec{n}\rangle = n_\alpha |\vec{n}\rangle \). Thus, \( \hat{H} |\vec{n}\rangle = E_{\vec{n}} |\vec{n}\rangle \), where \( E_{\vec{n}} = \sum_\alpha n_\alpha \varepsilon_\alpha \).

- **Bosons and fermions:** The allowed values for \( n_\alpha \) are \( n_\alpha \in \{0, 1, 2, \ldots, \infty\} \) for bosons and \( n_\alpha \in \{0, 1\} \) for fermions.

- **Grand canonical ensemble:** Because of the constraint \( \sum_\alpha n_\alpha = N \), the ordinary canonical ensemble is inconvenient. Rather, we use the grand canonical ensemble, in which case
  \[
  \Omega(T, V, \mu) = \pm k_B T \sum_\alpha \ln \left( 1 \mp e^{-(\varepsilon_\alpha - \mu)/k_B T} \right),
  \]
  where the upper sign corresponds to bosons and the lower sign to fermions. The average number of particles occupying the single particle state \( \psi_\alpha \) is then
  \[
  \langle \hat{n}_\alpha \rangle = \frac{\partial \Omega}{\partial \varepsilon_\alpha} = \frac{1}{e^{(\varepsilon_\alpha - \mu)/k_B T} \mp 1}.
  \]
  In the Maxwell-Boltzmann limit, \( \mu \ll -k_B T \) and \( \langle n_\alpha \rangle = z e^{-\varepsilon_\alpha/k_B T} \), where \( z = e^{\mu/k_B T} \) is the fugacity. Note that this low-density limit is common to both bosons and fermions.

- **Single particle density of states:** The single particle density of states per unit volume is defined to be
  \[
  g(\varepsilon) = \frac{1}{V} \text{Tr} \delta(\varepsilon - \hat{h}) = \frac{1}{V} \sum_\alpha \delta(\varepsilon - \varepsilon_\alpha),
  \]
  where \( \hat{h} \) is the one-body Hamiltonian. If \( \hat{h} \) is isotropic, then \( \varepsilon = \varepsilon(k) \), where \( k = |k| \) is the magnitude of the wavevector, and
  \[
  g(\varepsilon) = \frac{g O_d}{(2\pi)^d} \frac{k^{d-1}}{d\varepsilon/dk},
  \]
  where \( g \) is the degeneracy of each single particle energy state (due to spin, for example).

- **Quantum virial expansion:** From \( \Omega = -p V \), we have
  \[
  n(T, z) = \int_{-\infty}^{\infty} d\varepsilon \frac{g(\varepsilon)}{z^{-1} e^{\varepsilon/k_B T} \mp 1} = \sum_{j=1}^{\infty} (\pm 1)^{j-1} z^j C_j(T),
  \]
  \[
  \frac{p(T, z)}{k_B T} = \mp \int_{-\infty}^{\infty} d\varepsilon \frac{g(\varepsilon)}{z} \ln(1 \mp z e^{-\varepsilon/k_B T}) = \sum_{j=1}^{\infty} (\pm 1)^{j-1} \frac{z^j}{j} C_j(T),
  \]
where

\[ C_j(T) = \int_{-\infty}^{\infty} d\varepsilon \, g(\varepsilon) e^{-j\varepsilon/k_B T}. \]

One now inverts \( n = n(T, z) \) to obtain \( z = z(T, n) \), then substitutes this into \( p = p(T, z) \) to obtain a series expansion for the equation of state,

\[ p(T, n) = nk_B T \left( 1 + B_2(T) n + B_3(T) n^2 + \ldots \right). \]

The coefficients \( B_j(T) \) are the virial coefficients. One finds

\[ B_2 = \frac{C_2}{2C_1^2}, \quad B_3 = \frac{C_3^2}{2C_1^3}. \]

\section*{Photon statistics}

Photons are bosonic excitations whose number is not conserved, hence \( \mu = 0 \). The number distribution for photon statistics is then \( n(\varepsilon) = 1/(e^{\beta \varepsilon} - 1) \). Examples of particles obeying photon statistics include phonons (lattice vibrations), magnons (spin waves), and of course photons themselves, for which \( \varepsilon(k) = \hbar c k \) with \( g = 2 \). The pressure and number density for the photon gas obey \( p(T) = A_d T^{d+1} \) and \( n(T) = A'_d T^d \), where \( d \) is the dimension of space and \( A_d \) and \( A'_d \) are constants.

\section*{Blackbody radiation}

The energy density per unit frequency of a three-dimensional blackbody is given by

\[ \varepsilon(\nu, T) = \frac{8\pi \hbar^3}{c^3} \cdot \frac{\nu^3}{e^{\hbar \nu/k_B T} - 1}. \]

The total power emitted per unit area of a blackbody is \( \frac{dP}{dA} = \sigma T^4 \), where \( \sigma = \pi^2 k_B^4/60\hbar^3 c^2 = 5.67 \times 10^{-8} \text{ W/m}^2 \text{K}^4 \) is Stefan’s constant.

\section*{Ideal Bose gas}

For Bose systems, we must have \( \varepsilon_\alpha > \mu \) for all single particle states. The number density is

\[ n(T, \mu) = \int_{-\infty}^{\infty} d\varepsilon \frac{g(\varepsilon)}{e^{\beta(\varepsilon-\mu)} - 1}. \]

This is an increasing function of \( \mu \) and an increasing function of \( T \). For fixed \( T \), the largest value \( n(T, \mu) \) can attain is \( n(T, \varepsilon_0) \), where \( \varepsilon_0 \) is the lowest possible single particle energy, for which \( g(\varepsilon) = 0 \) for \( \varepsilon < \varepsilon_0 \). If \( n_c(T) \equiv n(T, \varepsilon_0) < \infty \), this establishes a critical density above which there is Bose condensation into the energy \( \varepsilon_0 \) state. Conversely, for a given density \( n \) there is a critical temperature \( T_c(n) \) such that \( n_0 \) is finite for \( T < T_c \). For \( T < T_c \), \( n = n_0 + n_c(T) \), with \( \mu = \varepsilon_0 \). For \( T > T_c \), \( n(T, \mu) \) is given by the integral formula above, with \( n_0 = 0 \). For a ballistic dispersion \( \varepsilon(k) = \hbar^2 k^2/2m \), one finds \( n \lambda_T^d = g \zeta(d/2) \), i.e. \( k_B T_c = \frac{2\pi \hbar^2}{m} \left( n/g \zeta(d/2) \right)^{2/d} \). For \( T < T_c(n) \), one has \( n_0 = n - g \zeta(1/2) \lambda_T^{-d} \) and \( p = g \zeta(1 + 1/2d) k_B T \lambda_T^{-d} \). For \( T > T_c(n) \), one has \( n = g \text{Li}_{d+1}(z) \lambda_T^{-d} \) and \( p = g \text{Li}_{d+1}(z) k_B T \lambda_T^{-d} \), where

\[ \text{Li}_q(z) \equiv \sum_{n=1}^{\infty} \frac{z^n}{n^q}. \]
• **Ideal Fermi gas:** The Fermi distribution is \( n(\varepsilon) = f(\varepsilon - \mu) = 1/(e^{(\varepsilon - \mu)/k_B T} + 1) \). At \( T = 0 \), this is a step function: \( n(\varepsilon) = \Theta(\mu - \varepsilon) \), and \( n = \int_{-\infty}^{\mu} d\varepsilon g(\varepsilon) \). The chemical potential at \( T = 0 \) is called the *Fermi energy*: \( \mu(T = 0, n) = \varepsilon_F(n) \). If the dispersion is \( \varepsilon(k) \), the locus of \( k \) values satisfying \( \varepsilon(k) = \varepsilon_F \) is called the *Fermi surface*. For an isotropic and monotonic dispersion \( \varepsilon(k) \), the Fermi surface is a sphere of radius \( k_F \), the *Fermi wavevector*. For isotropic three-dimensional systems, \( k_F = (6\pi^2 n/g)^{1/3} \).

• **Sommerfeld expansion:** Let \( \phi(\varepsilon) = \frac{d\Phi}{d\varepsilon} \). Then

\[
\int_{-\infty}^{\infty} d\varepsilon \ f(\varepsilon - \mu) \ \phi(\varepsilon) = \pi D \csc(\pi D) \Phi(\mu) = \left\{ 1 + \frac{\pi^2}{6} (k_B T)^2 \frac{d^2}{d\mu^2} + \frac{7\pi^4}{360} (k_B T)^4 \frac{d^4}{d\mu^4} + \ldots \right\} \Phi(\mu),
\]

where \( D = k_B T \frac{d}{d\mu} \). One then finds, for example, \( C_V = \gamma VT \) with \( \gamma = \frac{1}{3} \pi^2 k_B^2 g(\varepsilon_F) \).