PHYSICS 210A : STATISTICAL PHYSICS
HW ASSIGNMENT #6

(1) Consider the one-dimensional Ising model with next-nearest neighbor interactions,

$$\hat{H} = -J\sum_n \sigma_n \sigma_{n+1} - K\sum_n \sigma_n \sigma_{n+2} \ ,$$

on a ring with N sites, where N is even. By considering consecutive pairs of sites, show that the partition function may be written in the form $Z = \mathsf{Tr}(R^{N/2})$, where R is a 4×4 transfer matrix. Find R. *Hint:* It may be useful to think of the system as a railroad trestle, depicted in Fig. 2, with Hamiltonian

$$\hat{H} = -\sum_j \Big[J\sigma_j \mu_j + J\mu_j \sigma_{j+1} + K\sigma_j \sigma_{j+1} + K\mu_j \mu_{j+1} \Big] \ .$$

Then $R = R_{(\sigma_j \mu_j),(\sigma_{j+1}\mu_{j+1})}$, with $(\sigma\mu)$ a composite index which takes one of four possible values $(++)$, $(+-)$, $(-+)$, or $(--)$.

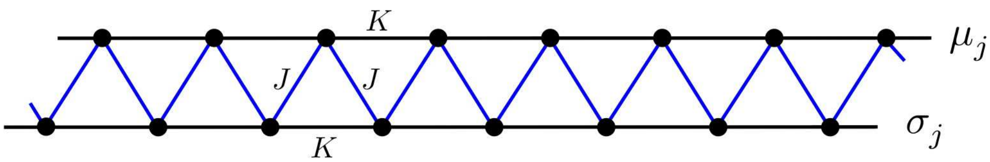

Figure 1: Railroad trestle representation of next-nearest neighbor chain.

(2) For each of the cluster diagrams in Fig. 2, find the symmetry factor s_γ and write an expression for the cluster integral b_γ.

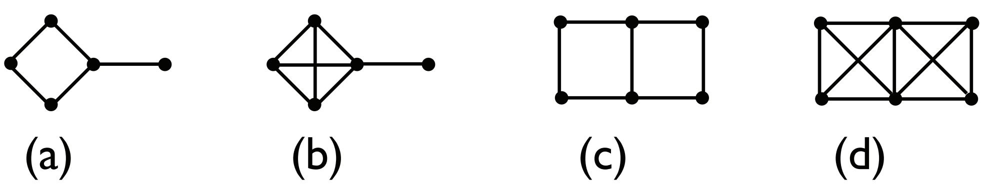

Figure 2: Cluster diagrams for problem 2.

(3) Compute the partition function for the one-dimensional Tonks gas of hard rods of length a on a ring of circumference L. This is slightly tricky, so here are some hints. Once again, assume a particular ordering so that $x_1 < x_2 < \cdots < x_N$. Due to translational invariance, we can define the positions of particles $\{2, \ldots, N\}$ relative to that of particle 1, which we initially place at $x_1 = 0$. Then periodicity means that $x_N \leq L - a$, and in general one then has

$$x_{j-1} + a \leq x_j \leq L - (N - j + 1)a \ .$$

Now integrate over $\{x_2, \ldots, x_N\}$ subject to these constraints. Finally, one does the x_1 integral, which is over the entire ring, but which must be corrected to eliminate overcounting from cyclic permutations. How many cyclic permutations are there?