PHYS 201 Mathematical Physics, Fall 2017, Homework 5

Due date: Thursday, November 16th, 2017

1. Use integration by parts and show that the asymptotic expansion as \( x \to \infty \) of the upper incomplete Gamma function \( \Gamma(1/2, x) \) is

\[
\Gamma(1/2, x) = \int_0^x t^{-1/2}e^{-t}dt \\
\sim \Gamma(1/2) - \frac{e^{-x}}{\sqrt{x}} \left[ 1 - \frac{1}{2x} + \ldots \right]
\]

2.

a. Use Laplace’s method to verify the first two or three terms of the asymptotic expansion as \( x \to \infty \) of the modified Bessel function \( K_0(x) \)

\[
K_0(x) = \int_1^\infty (s^2 - 1)^{-1/2}e^{-xs}ds \\
\sim \frac{e^{-x}}{\sqrt{x}} \sum_{n=0}^\infty (-1)^n \frac{[\Gamma(n + \frac{1}{2})]^2}{2^{n+1/2}n!\Gamma(1/2)x^n}
\]

This is an application of Watson’s lemma where the integral \( \int_0^{a>0} h(t)e^{-xt}dt \) is approximated as \( x \to \infty \) by expanding \( h(t) \) in a power series about 0. (Hint: First shift the limits of the integral to 0 and \( \infty \).)

b. Use similar ideas as part (a) to find the first two terms of the asymptotic expansion as \( x \to \infty \) of

\[
f(x) = \int_0^{\pi/2} e^{-x\sin^2 t}dt
\]

(1)

c. Show the following leading behavior as \( x \to \infty \) using Laplace’s method:

\[
f(x) = \int_0^\infty \exp \left( -t - \frac{x}{\sqrt{t}} \right)dt \\
\sim \pi^{1/2}2^{2/3}3^{-1/2}x^{1/3}e^{-3(x/2)^{2/3}}
\]

(Hint: Direct application of Laplace’s method does not work (why?) Use the transformation \( t = sx^{2/3} \) and then apply Laplace’s method.)

3. Use the method of stationary phase to find the leading behavior of the following integrals as \( x \to \infty \)
i. $\int_0^1 e^{ixt} \cosh t^2 dt$

ii. $\int_{-1}^1 \sin[x(t - \sin t)] \sinh(t) dt$

4. Use the method of steepest descent to find the first two terms of the asymptotic expansion for the Airy function $I(x)$ as $x \to \infty$ where

$$I(x) = \int_{-\infty}^{\infty} \exp \left[ i x \left( \frac{t^3}{3} + t \right) \right] dt$$

See the full problem description in Exercise 2 of Section 6-3 of Carrier et al for hints.