PHYS 201 Mathematical Physics, Fall 2017, Homework 5

Due date: Thursday, November 16th, 2017

1. Use integration by parts and show that the asymptotic expansion as $x \to \infty$ of the upper incomplete Gamma function $\Gamma(1/2, x)$ is

$$\begin{aligned}\Gamma(1/2, x) &= \int_0^x t^{-1/2} e^{-t} dt \\ &\sim \Gamma(1/2) - \frac{e^{-x}}{\sqrt{x}} \left[1 - \frac{1}{2x} + \dots \right]\end{aligned}$$

2.

a. Use Laplace's method to verify the first two or three terms of the asymptotic expansion as $x \to \infty$ of the modified Bessel function $K_0(x)$

$$\begin{aligned}K_0(x) &= \int_1^\infty (s^2 - 1)^{-1/2} e^{-xs} ds \\ &\sim \frac{e^{-x}}{\sqrt{x}} \sum_{n=0}^{\infty} (-1)^n \frac{[\Gamma(n + \frac{1}{2})]^2}{2^{n+1/2} n! \Gamma(\frac{1}{2}) x^n}\end{aligned}$$

This is an application of Watson's lemma where the integral $\int_0^{a>0} h(t) e^{-xt} dt$ is approximated as $x \to \infty$ by expanding $h(t)$ in a power series about 0. (**Hint:** First shift the limits of the integral to 0 and ∞.)

b. Use similar ideas as part (a) to find the first two terms of the asymptotic expansion as $x \to \infty$ of

$$f(x) = \int_0^{\pi/2} e^{-x \sin^2 t} dt \tag{1}$$

c. Show the following leading behavior as $x \to \infty$ using Laplace's method:

$$\begin{aligned}f(x) &= \int_0^\infty \exp\big(- t - \frac{x}{\sqrt{t}} \big) dt \\ &\sim \pi^{1/2} 2^{2/3} 3^{-1/2} x^{1/3} e^{-3(x/2)^{2/3}}\end{aligned}$$

(**Hint:** Direct application of Laplace's method does not work (why?) Use the transformation $t = s x^{2/3}$ and then apply Laplace's method.)

3. Use the method of stationary phase to find the leading behavior of the following integrals as $x \to \infty$

i. $\int_0^1 e^{ixt^2} \cosh t^2 dt$

ii. $\int_{-1}^1 \sin[x(t - \sin t)] \sinh(t) dt$

4. Use the method of steepest descent to find the first two terms of the asymptotic expansion for the Airy function $I(x)$ as $x \to \infty$ where

$$I(x) = \int_{-\infty}^{\infty} \exp\left[ix\left(\frac{t^3}{3} + t\right)\right] dt$$

See the full problem description in Exercise 2 of Section 6-3 of Carrier et al for hints.