1. Find the Taylor series expansions around the indicated points $z_0$. Where the function is multi-valued, give the results for at least two branches.
   
   i. $z^{1/2}; z_0 = 1, i\pi$
   
   ii. $(z - \pi)/(\sin z); z_0 = \pi$

2. Find the Laurent series expansion of
   
   $$f(z) = \frac{1}{z(z-1)(z-2)}$$

   in the three regions, $|z| < 1$, $1 < |z| < 2$ and $|z| > 2$.

3. In this exercise, we will find the solution (i.e., the complex potential $\Omega$) to a Dirichlet problem inside the unit circle $|z| < 1$ with $\text{Re}(\Omega) \equiv \phi = 0$ on the upper semicircle $\Gamma_1$ of the domain ($|z| = 1, \text{Im}(z) > 0$) and $\phi = k$ (with $k$ real) on the lower semicircle $\Gamma_2$.
   
   i. Recall the mapping $\zeta = f(z)$ from the unit circle to the infinite horizontal strip from Homework 1. Where do $\Gamma_1$ and $\Gamma_2$ lie after the transformation to $\zeta$-space?

   ii. Find the solution to the Dirichlet problem in the $\zeta$-space. Now, find the solution in the original $z$ space by substituting $\zeta = f(z)$. (Hint: If stuck, see Example 1 in Chapter 4 of Carrier et al)

   iii. Verify that this solution is the same as the one obtained using Poisson’s formula.

4. Suppose $f(z)$ is analytic inside and on a simple curve $C$ except for a finite number of poles inside $C$ and is non-zero on $C$. Consider the contour integral
   
   $$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz.$$

   Argue that the integrand of the above integral has singularities around the zeros and poles in $C$. Suppose the integral is evaluated by modifying the contour to enclose all the zeros and poles in $C$; show that it evaluates to $N - P$, where $N$ is the number of zeros and $P$ is the number of poles inside $C$. Since the integrand is also equal to $d \log f/dz$, and since $\log f = \log |f| + i \arg f$, we have the result that the change in argument of $f$ (in units of $2\pi$) resulting from a traversal of $C$ is equal to $N - P$. Outline a proof for why this implies the fundamental theorem of algebra: a polynomial of degree $n \geq 1$ with complex coefficients has $n$ complex roots.