PHYS 201 Mathematical Physics, Fall 2017, Homework 3

Due date: Tuesday, October 24th, 2017

1. Find the Taylor series expansions around the indicated points z_0. Where the function is multi-valued, give the results for at least two branches.

  i. $z^{1/2}$; $z_0 = 1, i\pi$

  ii. $(z - \pi)/(\sin z)$; $z_0 = \pi$

2. Find the Laurent series expansion of

$$f(z) = \frac{1}{z(z-1)(z-2)}$$

in the three regions, $|z| < 1$, $1 < |z| < 2$ and $|z| > 2$.

3. In this exercise, we will find the solution (i.e., the complex potential Ω) to a Dirichlet problem inside the unit circle $|z| < 1$ with $\mathrm{Re}(\Omega) \equiv \phi = 0$ on the upper semicircle Γ_1 of the domain ($|z| = 1, \mathrm{Im}(z) > 0$) and $\phi = k$ (with k real) on the lower semicircle Γ_2.

  i. Recall the mapping $\zeta = f(z)$ from the unit circle to the infinite horizontal strip from Homework 1. Where do Γ_1 and Γ_2 lie after the transformation to ζ-space?

  ii. Find the solution to the Dirichlet problem in the ζ-space. Now, find the solution in the original z space by substituting $\zeta = f(z)$. (**Hint:** If stuck, see Example 1 in Chapter 4 of Carrier et al)

  iii. Verify that this solution is the same as the one obtained using Poisson's formula.

4. Suppose $f(z)$ is analytic inside and on a simple curve C except for a finite number of poles inside C and is non-zero on C. Consider the contour integral

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz.$$

Argue that the integrand of the above integral has singularities around the zeros and poles in C. Suppose the integral is evaluated by modifying the contour to enclose all the zeros and poles in C; show that it evaluates to $N - P$, where N is the number of zeros and P is the number of poles inside C. Since the integrand is also equal to $d \log f/dz$, and since $\log f = \log|f| + i\arg f$, we have the result that the change in argument of f (in units of 2π) resulting from a traversal of C is equal to $N - P$. Outline a proof for why this implies the fundamental theorem of algebra: a polynomial of degree $n \geq 1$ with complex coefficients has n complex roots.