PHYS 273, Winter 2016, Homework 2

Due date: Thursday, January 21st, 2016

1. *Sequence length.* How much information does the length of a sequence give about the content of a sequence? Suppose that we consider a Bernoulli (1/2) process $\{X_i\}$ i.e., for every i, X_i is 1 with probability 1/2 and 0 with probability 1/2. Stop the process when the first 1 appears. Let N designate this stopping time. Thus, X^N is an element of the set of all finite-length binary sequences $\{0,1\}^* = \{0, 1, 00, 01, 10, 11, 000, \dots\}$. Find $I(N; X^N), H(X^N|N), H(X^N)$.

Now consider a different stopping time. For this part, again assume that $X_i \sim$ Bernoulli(1/2) but stop at time $N = 6$ with probability 1/3 and stop at time $N = 12$ with probability 2/3. Let this stopping time be independent of the sequence $X_1 X_2 \dots X_{12}$. Find $I(N; X^N), H(X^N|N), H(X^N)$.

2. *The value of a question.* Let $X \sim p(x), x = 1, 2, \dots, m$. We are given a set $S \subseteq \{1, 2, \dots, m\}$. We ask whether $X \in S$ and receive the answer

$$Y = \begin{cases} 1 & \text{if } X \in S \\ 0 & \text{if } X \notin S \end{cases} \tag{1}$$

Suppose that $\Pr(X \in S) = \alpha$. Find the decrease in uncertainty $H(X) - H(X|Y)$.

3. *Noisy channel.* Consider three random variables X, Y, Z which can each take values 0 or 1; x and y are independent with $\Pr(X = 0) = p$ and $\Pr(Y = 0) = q$ and

$$z = (x + y) \mod 2$$

a. If $q = 1/2$, what is $\Pr(Z = 0)$? What is $I(Z; X)$?

b. For general p and q, what is $\Pr(Z = 0)$? What is $I(Z; X)$? This is an example of a single-bit noisy channel with $x =$ input, $y =$ noise and $z =$ output.

4. *Mutual information of heads and tails.*

a. Consider a fair coin flip. What is the mutual information between the top and bottom sides of the coin?

b. A six-sided fair die is rolled. What is the mutual information between the top side and the front face (the side most facing you)?