Quantum Mechanics PHYS 212B

Problem Set 5

Due Tuesday, February 16, 2016

Exercise 5.1 Consider a heavy particle of mass M at rest in a particular frame. This particle decays into two particles, both of which can be regarded as moving near the speed of light c so that the energy of the first particle produced in the decay is $E_1 \approx cp_1$, with p_1 the magnitude of this particle's momentum and, likewise, for the quantities related to the second particle, $E_2 \approx cp_2$.

In Part (b) below use Fermi's Golden Rule and assume that the relevant matrix element-squared of the potential which mediates this decay is $|\langle\psi_f|\hat{H}_{\text{int}}|\psi_i\rangle|^2$ and can be taken as constant, i.e. independent of E_1 and E_2.

(a) In the frame in which the initial particle of M is at rest, what are E_1 and E_2? (Hint: use momentum conservation.)

(b) What is the total decay rate of this particle? Assume that the two particles in the final state are in plane wave state.

(c) If we double the mass of the initial particle what happens to the total decay rate? What happens to the energies E_1 and E_2?

Solution 5.1

(a) According to momentum conservation, we have

$$0 = \mathbf{p}_1 + \mathbf{p}_2.$$

Then we know $p_1 = p_2$, energy conservation,

$$E_1 + E_2 = c(p_1 + p_2) = Mc^2$$

therefore

$$E_1 = E_2 = \frac{1}{2}Mc^2$$

(b) Fermi-Golden rule gives

$$\Gamma = \frac{2\pi}{\hbar}|\langle\psi_f|\hat{H}_{\text{int}}|\psi_i\rangle|^2\delta(cp_1 + cp_2 - Mc^2)$$

Summation over all possible $\mathbf{p}_1$ and $\mathbf{p}_2$ direction

$$\begin{aligned}\Gamma_{\text{total}} &\propto \int \Gamma\delta(\mathbf{p}_1 + \mathbf{p}_2)d^3\mathbf{p}_1 d^3\mathbf{p}_2 \\ &\propto \int \delta(2cp_1 - Mc^2)4\pi p_1^2 dp_1 \\ &\propto (Mc^2)^2\end{aligned}$$

(c) if $M' = 2M$, then $\Gamma'_{\text{total}} = 4\Gamma_{\text{total}}$, and $E'_1 = 2E_1$, $E'_2 = 2E_2$

Exercise 5.2 Consider the absorption of light by a hydrogen atom initially in its $1s$ ground state, making a transition to the $2p$ first excited state. The lifetime of a hydrogen atom in the $2p$ state in vacuum is$\tau = \hbar/\Gamma \approx 1.6 \times 10^{-9}$sec . Here Γ is the width.

(a) Let's say that the atom in its ground state is immersed in an isotropic radiation filed, roughly the situation that might obtain in the deep interior of a star. To calculate the absorption rate we are going to need to evaluate the matrix element-squared of the dipole operator, $|\langle 2p|\mathbf{r}\cdot\mathbf{e}|1s\rangle|^2$, where $\mathbf{e}$ is the (unit) photo polarization vector. Show that averaging over the isotropic radiation field gives

$$\left\langle |\langle 2p|\mathbf{r}\cdot\mathbf{e}|1s\rangle|^2 \right\rangle = \frac{1}{3}|\langle 2p|\mathbf{r}|1s\rangle|^2$$

(b) In this radiation field, and using the above dipole operator, what is the $1s \to 2p$ absorption cross section as a function of frequency ω? What is the peak value of this cross section?

The normalized radial wave functions for the hydrogen atom are

$$R_{1s} = R_{10} = 2a_0^{-3/2}e^{-r/a_0},$$
$$R_{21} = \frac{1}{2\sqrt{6}}a_0^{-5/2}re^{-r/2a_0}.$$

Remember that

$$n! = \int_0^\infty x^n e^{-x}dx,$$

and the Bohr radius is $a_0 = \hbar^2/(m_e e^2) \approx 0.529 \times 10^{-8}$ cm, with the electron rest mass $m_e c^2 \approx 0.511$MeV.

Solution 5.1

(a) Take $\mathbf{e}$ as z-direction, with the coordinate system of $\mathbf{e}$,

$$\langle 2p|\mathbf{r}\cdot\mathbf{e}|1s\rangle = \langle 2p|r\cos\theta|1s\rangle = \langle 2p|r|1s\rangle\cos\theta$$

Average over all possible direction

$$\begin{aligned}\left\langle |\langle 2p|\mathbf{r}\cdot\mathbf{e}|1s\rangle|^2 \right\rangle &= \frac{\int_0^{2\pi} d\phi \int_0^\pi |\langle 2p|r|1s\rangle|^2\cos^2\theta\sin\theta d\theta}{\int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta} \\ &= |\langle 2p|r|1s\rangle|^2 \frac{\int_0^{2\pi} d\phi \int_0^\pi \cos^2\theta\sin\theta d\theta}{\int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta} \\ &= \frac{1}{3}|\langle 2p|\mathbf{r}|1s\rangle|^2\end{aligned}$$

(b) The cross section (see class notes)

$$\sigma(\omega) = \frac{4\pi^2\alpha}{m^2\omega}|\langle 2p|\mathbf{r}\cdot\mathbf{e}|1s\rangle|^2\delta(\omega_{fi}-\omega)$$

Average all possible directions

$$\langle\sigma(\omega)\rangle = \frac{4\pi^2\alpha}{m^2\omega}\frac{1}{3}|\langle 2p|r|1s\rangle|^2\delta(\omega_{fi}-\omega)$$

Considering the with Γ, the $\delta(\omega_{fi}-\omega)$ can be approximated by Breit-Wigner shape

$$\delta(\omega_{fi}-\omega) = \frac{\Gamma/2\pi\hbar}{(\omega-\omega_{fi})^2 + (\Gamma/2\hbar)^2}$$

Evaluation of $\langle 2p|r|1s\rangle$,

$$\begin{aligned}
\langle 2p|r|1s\rangle &= \int_0^\infty dr \int_0^{2\pi} d\phi \int_0^\pi 2a_0^{-3/2} e^{-r/a_0} \frac{1}{2\sqrt{6}} a^{-5/2} r e^{-4/2a_0} r r^2 \sin\theta \\
&= \frac{4\pi a_0^{-4}}{\sqrt{6}} \int_0^\infty r^4 e^{-3r/2a_0} dr \\
&= \frac{2^{19/2}}{3^{9/2}} \pi a_0
\end{aligned}$$

Finally,

$$\langle \sigma(\omega)\rangle = \frac{4\pi^2\alpha}{3m^2\omega} \frac{2^{19}}{3^9} \pi^2 a_0^2 \frac{\Gamma/2\pi\hbar}{(\omega-\omega_{fi})^2 + (\Gamma/2\hbar)^2}$$

It reaches it maximum when

$$\omega = \omega_{fi} = \frac{E_{2p} - E_{1s}}{\hbar} = \frac{3e^2}{8a_0\hbar}$$