Quantum Mechanics PHYS 212B

Problem Set 4

Due Tuesday, February 9, 2016

Exercise 4.1 Consider a potential

$$V \cos \omega t = \frac{V}{2}\left(e^{-i\omega t} + e^{i\omega t}\right)$$

turned on slowly with a factor $e^{\alpha 5}$. Use the perturbation expansion developed in class and take $\alpha \to 0$ to calculate the first-order transition probability between initial state $|0\rangle$ and $|n|\rangle$, where as before will take $|n|\rangle$ to be a continuum of states.

Solution 4.1 The first-order perturbation

$$\begin{aligned}
\langle n|\psi_t|\rangle &= \frac{1}{i\hbar}\int_{-\infty}^{t} dt' e^{i(\epsilon_n-\epsilon_0)t'/\hbar}\frac{1}{2}\left(e^{-i\omega t'} + e^{i\omega t'}\right)e^{\alpha t'}\langle n|V|0\rangle \\
&= \frac{\langle n|V|0\rangle}{2i\hbar}\int_{-\infty}^{t} dt'\left[e^{\frac{it'}{\hbar}(\epsilon_n-\epsilon_0-\hbar\omega)+\alpha t} + e^{\frac{it'}{\hbar}(\epsilon_n-\epsilon_0+\hbar\omega)+\alpha t}\right] \\
&= \frac{e^{\alpha t}\langle n|V|0\rangle}{2}\left[\frac{e^{\frac{it}{\hbar}(\epsilon_n-\epsilon_0-\hbar\omega)}}{\epsilon_0-\epsilon_n+\hbar\omega+i\alpha\hbar} + \frac{e^{\frac{it}{\hbar}(\epsilon_n-\epsilon_0+\hbar\omega)}}{\epsilon_0-\epsilon_n-\hbar\omega+i\alpha\hbar}\right]
\end{aligned}$$

Let $\alpha = 0$ and the probability is

$$P(|n\rangle = |\langle n|\psi_t|\rangle|^2 = \frac{|\langle n|V|0\rangle|^2}{4}\left[\frac{1}{(\epsilon_0-\epsilon_n+\hbar\omega)^2} + \frac{1}{(\epsilon_0-\epsilon_n-\hbar\omega)^2} + \frac{2\cos(2\omega t)}{(\epsilon_0-\epsilon_n)^2-\hbar^2\omega^2}\right]$$

Exercise 4.2 We can regard the vector potential as an operator at some spacetime point $(\mathbf{r}t)$:

$$\mathbf{A}(\mathbf{r}t) = \sum_{\mathbf{k}\hat{\epsilon}}\left[A_{\mathbf{k}\hat{\epsilon}}\hat{\epsilon}\frac{e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}}{\sqrt{\mathrm{Vol}}} + A^{\dagger}_{\mathbf{k}\hat{\epsilon}}\hat{\epsilon}^{*}\frac{e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}}{\sqrt{\mathrm{Vol}}}\right]$$

where $\mathbf{k}$ is the wavenumber and $\hat{\epsilon}$ is the polarization vector and we can regard the coefficient $A_{\mathbf{k}\hat{\epsilon}}$, etc. as operators. Show that $[A_{\mathbf{k}\hat{\epsilon}}, A_{\mathbf{k}'\hat{\epsilon}'}] = 0$, $\left[A^{\dagger}_{\mathbf{k}\hat{\epsilon}}, A^{\dagger}_{\mathbf{k}'\hat{\epsilon}'}\right] = 0$, $\left[A_{\mathbf{k}\hat{\epsilon}}, A^{\dagger}_{\mathbf{k}'\hat{\epsilon}'}\right] = \frac{2\pi\hbar c}{|\mathbf{k}|}\delta_{\mathbf{k}\mathbf{k}'}\hat{\epsilon}\hat{\epsilon}'^{*}$.

Solution 4.2 The canonical momentum of free EM field is (Derivation can be found here)

$$\begin{aligned}
\mathbf{p}(\mathbf{r}t) &= -\frac{1}{4\pi c}\mathbf{E}(\mathbf{r}t) = -\frac{1}{4\pi c}\left(-\frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r}t)}{\partial t}\right) \\
&= -\frac{i\omega}{4\pi c^2}\sum_{\mathbf{k}\hat{\epsilon}}\left[A_{\mathbf{k}\hat{\epsilon}}\hat{\epsilon}\frac{e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}}{\sqrt{\mathrm{Vol}}} - A^{\dagger}_{\mathbf{k}\hat{\epsilon}}\hat{\epsilon}^{*}\frac{e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}}{\sqrt{\mathrm{Vol}}}\right]
\end{aligned}$$

Based on the fundamental commutation relation

$$[A(\mathbf{r}t), \mathbf{p}(\mathbf{r}'t)] = i\hbar\delta(\mathbf{r}-\mathbf{r}'), \tag{1}$$

we have

$$[A(\mathbf{r}t),\mathbf{p}(\mathbf{r}'t)] = \left[\sum_{\mathbf{k}\hat{\epsilon}}\left[A_{\mathbf{k}\hat{\epsilon}}\hat{\epsilon}\frac{e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}}{\sqrt{\text{Vol}}}+A^\dagger_{\mathbf{k}\hat{\epsilon}}\hat{\epsilon}^*\frac{e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}}{\sqrt{\text{Vol}}}\right], -\frac{i\omega}{4\pi c^2}\sum_{\mathbf{k}'\hat{\epsilon}'}\left[A_{\mathbf{k}'\hat{\epsilon}'}\hat{\epsilon}'\frac{e^{i\mathbf{k}'\cdot\mathbf{r}'-i\omega t}}{\sqrt{\text{Vol}}}-A^\dagger_{\mathbf{k}'\hat{\epsilon}'}\hat{\epsilon}'^*\frac{e^{-i\mathbf{k}'\cdot\mathbf{r}'+i\omega t}}{\sqrt{\text{Vol}}}\right]\right]$$

$$= -\frac{i\omega}{4\pi c^2\ \text{Vol}}\sum_{\mathbf{k}\hat{\epsilon}}\sum_{\mathbf{k}'\hat{\epsilon}'}\Big\{[A_{\mathbf{k}\hat{\epsilon}},A_{\mathbf{k}'\hat{\epsilon}'}]\,\hat{\epsilon}\hat{\epsilon}'e^{i\mathbf{k}\cdot\mathbf{r}+i\mathbf{k}'\cdot\mathbf{r}'-2i\omega t}-\left[A^\dagger_{\mathbf{k}\hat{\epsilon}},A^\dagger_{\mathbf{k}'\hat{\epsilon}'}\right]\hat{\epsilon}^*\hat{\epsilon}'^*e^{-i\mathbf{k}\cdot\mathbf{r}-i\mathbf{k}'\cdot\mathbf{r}'+2i\omega t}$$

$$-\left[A_{\mathbf{k}\hat{\epsilon}},A^\dagger_{\mathbf{k}'\hat{\epsilon}'}\right]\hat{\epsilon}\hat{\epsilon}'^*e^{i\mathbf{k}\cdot\mathbf{r}-i\mathbf{k}'\cdot\mathbf{r}'}+\left[A^\dagger_{\mathbf{k}\hat{\epsilon}},A_{\mathbf{k}'\hat{\epsilon}'}\right]\hat{\epsilon}^*\hat{\epsilon}'e^{-i\mathbf{k}\cdot\mathbf{r}+i\mathbf{k}'\cdot\mathbf{r}'}\Big\}$$

$$= -\frac{i\omega}{4\pi c^2\ \text{Vol}}\sum_{\mathbf{k}\hat{\epsilon}}\sum_{\mathbf{k}'\hat{\epsilon}'}\Big\{[A_{\mathbf{k}\hat{\epsilon}},A_{\mathbf{k}'\hat{\epsilon}'}]\,\hat{\epsilon}\hat{\epsilon}'e^{i\mathbf{k}\cdot\mathbf{r}+i\mathbf{k}'\cdot\mathbf{r}'-2i\omega t}-\left[A^\dagger_{\mathbf{k}\hat{\epsilon}},A^\dagger_{\mathbf{k}'\hat{\epsilon}'}\right]\hat{\epsilon}^*\hat{\epsilon}'^*e^{-i\mathbf{k}\cdot\mathbf{r}-i\mathbf{k}'\cdot\mathbf{r}'+2i\omega t}$$

$$-2\left[A_{\mathbf{k}\hat{\epsilon}},A^\dagger_{\mathbf{k}'\hat{\epsilon}'}\right]\hat{\epsilon}\hat{\epsilon}'^*e^{i\mathbf{k}\cdot\mathbf{r}-i\mathbf{k}'\cdot\mathbf{r}'}\Big\}$$

The RHS of Eq.(1) doesn't contains t, so the coefficients of $e^{-2i\omega t}$ should be zero,

$$[A_{\mathbf{k}\hat{\epsilon}},A_{\mathbf{k}'\hat{\epsilon}'}]=0,$$

$$\left[A^\dagger_{\mathbf{k}\hat{\epsilon}},A^\dagger_{\mathbf{k}'\hat{\epsilon}'}\right]=0.$$

We know

$$\delta(\mathbf{r}-\mathbf{r}')=\frac{1}{\text{Vol}}\sum_{\mathbf{k}}e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')},$$

substitute it into Eq.(1), we have

$$\left[A_{\mathbf{k}\hat{\epsilon}},A^\dagger_{\mathbf{k}'\hat{\epsilon}'}\right]=\frac{2\pi\hbar c^2}{\omega}\delta_{\mathbf{k}\mathbf{k}'}\hat{\epsilon}\hat{\epsilon}'^*$$

Replace $\omega=c|\mathbf{k}|$, we have the comutation relation we need,

$$\left[A_{\mathbf{k}\hat{\epsilon}},A^\dagger_{\mathbf{k}'\hat{\epsilon}'}\right]=\frac{2\pi\hbar c}{|\mathbf{k}|}\delta_{\mathbf{k}\mathbf{k}'}\hat{\epsilon}\hat{\epsilon}'^*$$

Exercise 4.3 A hydrogen atom in its ground state is placed between the plates of a capacitor and at time $t=0$ a time-dependent, but spatially uniform, electric filed $\mathbf{E}=\mathbf{E}_0e^{-t/\tau}$ is applied. Take $\mathbf{E}_0$ to be in the positive z-direction. What is the probability for the atom to be found in each of the three $2p$ states when $t\gg\tau$?

Solution 4.3 The perturbation Hamiltonian is

$$H'=e\mathbf{r}\cdot\mathbf{E}=ezE_0e^{-t/\tau}$$

We know $[z,L_z]=0$, so

$$\begin{aligned}0&=\langle n_f,l_f,m_f|[z,L_z]|n_i,l_i,m_i\rangle\\&=(m_i-m_f)\hbar\langle n_f,l_f,m_f|z|n_i,l_i,m_i\rangle\end{aligned}$$

Therefore, the selection rule for z operator is $\Delta m=0$,

$$\langle 2,1,\pm1|H'|100\rangle=0$$

and

$$\begin{aligned}
\langle 210|H'|100\rangle &= \int d^3\mathbf{r}\psi_{210}H'\psi_{100} \\
&= eE_0e^{-t/\tau}\frac{1}{4\sqrt{2}\pi a_0^4}\int_0^{+\infty}dr\int_0^{\pi}d\theta\int_0^{2\pi}d\phi e^{-3r/2a_0}r^4\cos^2\theta\sin\theta \\
&= \frac{eE_0e^{-t/\tau}}{4\sqrt{2}\pi a_0^4}\frac{4\pi}{3}\frac{4!}{\left(\frac{3}{2a_0}\right)^5} \\
&= \frac{2^{15/2}a_0e}{3^5}E_0e^{-t/\tau}
\end{aligned}$$

where

$$\int_0^{\pi}d\theta\cos^2\theta\sin\theta = \frac{2}{3}, \qquad \int_0^{+\infty}dre^{-3r/2a_0}r^4 = \frac{4!}{\left(\frac{3}{2a_0}\right)^5}.$$

Then

$$\begin{aligned}
\langle 210|\psi_t\rangle &= \frac{1}{i\hbar}\int_0^{+\infty}\langle 210|H'|100\rangle e^{\frac{it}{\hbar}(E_2-E_1)}dt \\
&= \frac{2^{15/2}a_0eE_0}{3^5i\hbar}\int_0^{+\infty}\exp\left[-\frac{t}{\tau}+\frac{i(E_2-E-1)}{\hbar}t\right]dt \\
&= \frac{2^{15/2}a_0eE_0}{3^5i\hbar}\frac{1}{\frac{1}{\tau}-i\frac{E_2-E_1}{\hbar}}
\end{aligned}$$

The probability to find the $2p$ states

$$\begin{aligned}
P(|2,1,\pm1\rangle) &= 0 \\
P(|210\rangle) = |\langle 210|\psi_t\rangle|^2 &= \frac{2^{15}a_0^2e^2E_0^2}{3^{10}\hbar^2}\frac{1}{\frac{1}{\tau^2}+\left(\frac{E_2-E_1}{\hbar}\right)^2}
\end{aligned}$$

where $(E_2-E_1)/\hbar = 3e^2/8a_0\hbar$.