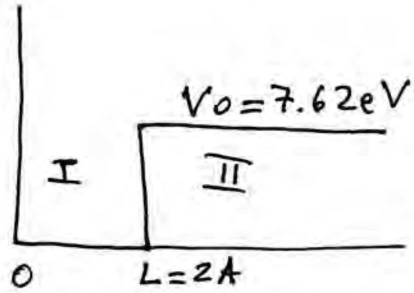
Physics 4E **Quiz 5 Solutions** Spring 2016

Problem 1



$\psi(x) = A\sin(kx)$ for $0 < x < L$. $k = \sqrt{\frac{2mE}{\hbar^2}}$

$\psi(x) = Be^{-\alpha x}$ for $x > L$ $\alpha = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$

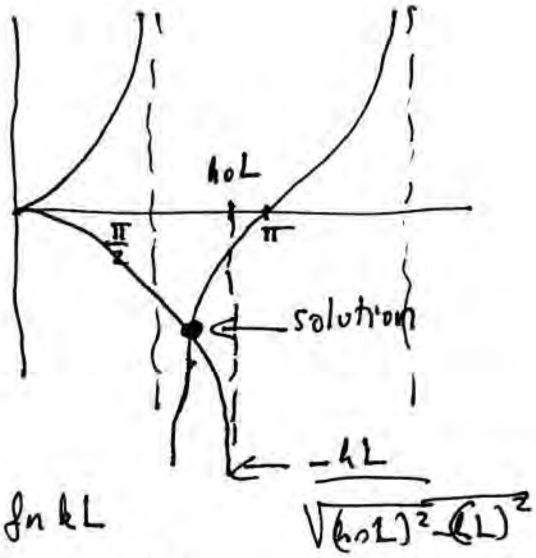
Continuity:

$$A\sin(kL) = Be^{-\alpha L}$$
$$kA\cos(kL) = -\alpha Be^{-\alpha L}$$

divide: $\tan(kL) = -\frac{k}{\alpha} = \frac{-kL}{\sqrt{(k_0L)^2 - (kL)^2}}$

$\alpha = \sqrt{k_0^2 - k^2}$, $k_0 = \sqrt{\frac{2mV_0}{\hbar^2}} = \sqrt{2}$

$k_0L = \sqrt{\frac{7.62}{3.81}} \cdot 2 = \sqrt{2} \cdot 2 = 2.828$



Solution is somewhere between $\frac{\pi}{2} = 1.57$ and 2.82 for kL
$\Rightarrow 0.79 < k < 1.41$

Check numerically on calculator to find it

k	$-\tan(kL)$	$\frac{+kL}{\sqrt{(k_0L)^2 - (kL)^2}} = +\frac{k}{\sqrt{2 - k^2}}$
1	2.18	2
1.1	1.37	1.24
1.11	1.318	1.267
1.115	1.2906	1.2817
1.117	1.2800	1.2878
1.116	1.2853	$\boxed{1.2848}$

$\boxed{\alpha = \sqrt{k_0^2 - k^2} = 0.8686}$

$\boxed{k = 1.116\,Å^{-1}}$

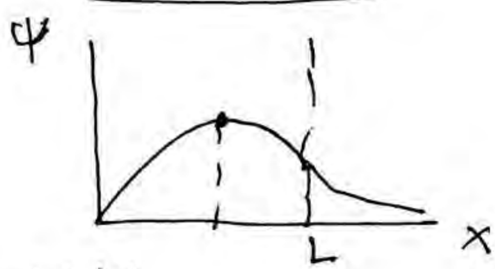
$E = \frac{\hbar^2 k^2}{2m} = 3.81 \times 1.116^2\,eV$

$\Rightarrow \boxed{E = 4.75\,eV}$ (4.745)

$\psi = A\sin kx$

$\psi' = kA\cos(kx) = 0 \Rightarrow$

$kx = \frac{\pi}{2} \Rightarrow \boxed{x = \frac{\pi}{2k} = 1.41\,Å}$ most likely to be found



(c) From continuity: $B = A e^{\alpha L} \sin(kL)$

I: $\psi(x) = A \sin(kx)$

II: $\psi(x) = A \sin(kL)\, e^{\alpha(L-x)}$

Need to find A from: $\int_0^\infty dx\, |\psi(x)|^2 = 1$

I: $\int_0^L dx \sin^2 kx = \int_0^L dx \frac{1-\cos(2kx)}{2} = \frac{L}{2} - \frac{\sin(2kx)}{4k}\Big]_0^L = \frac{L}{2} - \frac{\sin(2kL)}{4k}$

$= 1 - \frac{\sin(4.464)}{4.464} = 1.217$

II: $\int_L^\infty dx\, e^{-2\alpha x} = -\frac{e^{-2\alpha x}}{2\alpha}\Big]_L^\infty = \frac{e^{-2\alpha L}}{2\alpha}$ $\quad (\alpha = 0.8686)$

So:

$1 = \int_0^\infty dx\, |\psi(x)|^2 dx = A^2 \int_0^L dx \sin^2 kx + A^2 \sin^2(kL) e^{2\alpha L} \int_L^\infty dx\, e^{-2\alpha x} =$

$= A^2\left[1.217 + \frac{\sin^2(2.232)}{1.7372}\right] = A^2[1.217 + 0.359] = A^2 \cdot 1.576$

$\Rightarrow \boxed{A = 0.797} \boxed{\approx 0.8}$ $\quad \boxed{A \cdot \sin(kL) = 0.629}$

So the complete wavefunction including normalization is:

$$\boxed{\begin{array}{ll} \text{I}: \ \psi(x) = 0.8 \sin(kx) & k = 1.116 \\ \text{II}: \ \psi(x) = 0.63\, e^{\alpha(2-x)} & \alpha = 0.869 \end{array}}$$

Check continuity: $0.8 \sin(2k) = 0.63$

Check continuity of derivative: $0.8\, k \cos(2k) = -0.55$

$-\alpha \cdot 0.63 = -0.55$

Probability to find electron in region $0 < x < L$:

$$\int_0^L dx\, |\psi(x)|^2 = A^2 \int_0^L dx\, \sin^2(kx) = A^2 \cdot 1.217 = 0.77 = 1 - 0.23$$

So if we check 1000 times, we will find $\boxed{\text{230 times}}$ that the box is empty

(e)

$$\int_0^{L/2} dx\, \sin^2(kx) = \frac{L}{4} - \frac{\sin(kL)}{4k} = 0.5 - \frac{\sin(2.232)}{4.464} = 0.323$$

$$\int_{L/2}^{L} dx\, \sin^2(kx) = \frac{L}{4} - \frac{\sin(2kx)}{4k}\Bigg]_{L/2}^{L} = \frac{L}{4} - \frac{\sin(2kL) - \sin(kL)}{4k} =$$

$$= 0.5 - \frac{\sin(4.464) - \sin(2.232)}{4.464} = 0.894$$

$$\frac{0.894}{0.323} = 2.77$$

$\Rightarrow$ it is 2.77 times more likely to find electron in region $1 < x < 2$ than in region $0 < x < 1$

Problem 2

$$V(x,y,z) = \alpha\,(x^2+y^2) \quad \text{for } 0<z<a$$
$$= \infty \quad \text{for } z<0 \cap z>a$$

we can try $\quad \Psi(x,y,z) = \Psi_x(x)\,\Psi_y(y)\,\Psi_z(z)$

Schrödinger equation in region $0<z<a$ is:

$$-\frac{\hbar^2}{2m}\left[\Psi_x(x)\Psi_y(y)\frac{\partial^2\Psi_z(z)}{\partial z^2}+\cdots\right]+\alpha\,(x^2+y^2)\,\Psi = E\,\Psi$$

Clearly it separates in to 3 independent equations.

For x, y it is a harmonic oscillator with

$$\alpha = \frac{1}{2}m\omega^2 \Rightarrow \omega = \sqrt{\frac{2\alpha}{m}} \Rightarrow \hbar\omega = \sqrt{\frac{2\alpha\hbar^2}{m}} = \sqrt{8\times 3.61}\ \mathrm{eV}$$

$\Rightarrow \boxed{\hbar\omega = 5.52\,\mathrm{eV}} \qquad E_n = \hbar\omega(n+\frac{1}{2})$ for harmonic oscillator.

For z, it's box of length $a = 5\,\text{Å}$. $\quad E = \frac{\hbar^2\pi^2}{2ma^2}n^2 = 1.504\,n^2\,\mathrm{eV}$

So total energy is:

$$\boxed{E_{n_1 n_2 n_3} = \hbar\omega(n_1+\tfrac{1}{2}) + \hbar\omega(n_2+\tfrac{1}{2}) + \frac{\hbar^2\pi^2}{2ma^2}n_3^2}$$

Ground state: $n_1 = n_2 = 0$, $n_3 = 1$

$$E_{001} = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{2} + \frac{\hbar^2\pi^2}{2ma^2} = 5.52\,\mathrm{eV} + 1.504\,\mathrm{eV} = \boxed{7.02\,\mathrm{eV}}$$

Ground state wave function for harmonic oscillator:

$$\Psi_0(x) = A_0\,e^{-\frac{m\omega}{2\hbar}x^2} \qquad m\omega = \sqrt{2\alpha m} \Rightarrow \frac{m\omega}{2\hbar} = \sqrt{\frac{\alpha m}{2\hbar^2}}$$

For square well: $\quad \Psi_1(z) = B\sin(k_1 z) \quad k_1 = \frac{\pi}{a}$

$$\Rightarrow \boxed{\Psi_{001}(x,y,z) = A\,e^{-\sqrt{\frac{\alpha m}{2\hbar^2}}(x^2+y^2)}\sin\left(\frac{\pi z}{a}\right)}$$

Probability for electron to be at $(0, 0, 2.5 = \frac{a}{2})$

$$|\psi(0,0,\tfrac{a}{2})|^2 = A^2 \sin^2\frac{\pi}{2} = A^2$$

Probability for electron to be at $(1, 1, 1.25 = \frac{a}{4})$

$$|\psi(1,1,\tfrac{a}{4})|^2 = A^2 e^{-2\sqrt{\frac{\alpha m}{2\hbar^2}}(1+1)} \sin^2\frac{\pi}{4}$$

$$\sqrt{\frac{\alpha m}{2\hbar^2}} = \sqrt{\frac{\alpha}{4}\frac{2m}{\hbar^2}} = \sqrt{\frac{1}{2\times 3.81}} = 0.362 \; ; \quad \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\Rightarrow |\psi(1,1,\tfrac{a}{4})|^2 = A^2 e^{-4\times 0.362} \times \frac{1}{2} = A^2 \times 0.117 = \frac{A^2}{8.5}$$

$\Rightarrow$ $\boxed{\text{it is 8.5 times more likely to find electron at } (0,0,\tfrac{a}{2}) \text{ than at } (1,1,\tfrac{a}{4})}$

(d) Find lowest energy states:

Harmonic oscillator: $n = 0 \quad 1 \quad 2$

$E_n = 2.76 \quad 8.28 \quad 13.8$

Box: $n = 1 \quad 2 \quad 3$

$E_n = 1.504 \quad 6.016 \quad 13.54$

so lowest states are:

n_1	n_2	n_3	$E_{n_1 n_2 n_3}$
0	0	1	7.02
1	0	1	12.54
0	1	1	12.54
0	0	2	11.54
1	0	2	17.06
0	1	2	17.06
1	1	1	18.06
0	0	3	19.06

lowest states

(1,0,2) (0,1,2)	↑↓	17.06
(1,0,1) (0,1,1)	↑↓ ↑↓	12.54
(0,0,2)	↑↓	11.54
(0,0,1)	↑↓	7.02

2 electrons per state

$$\boxed{E_{tot} = 2\times[7.02 + 11.54 + 2\times 12.54 + 17.06] = 121.4\,\text{eV}}$$

(e) α-particles are bosons, so we can put all in the lowest energy state.

$$\frac{m_\alpha}{m_e} = \frac{3727}{0.511} = 7294$$

since $\omega = \sqrt{\frac{2\alpha}{m}}$, need to divide $\hbar\omega$ by $\sqrt{7294}$

$$\Rightarrow \quad \hbar\omega = \frac{5.52}{\sqrt{7294}}\, eV = 0.065\, eV$$

For box, need to divide by 7294 $\Rightarrow$

$$E_1 = \frac{1.504\, eV}{7294} = 0.0002\, eV$$

the ground state energy for one α-particle is then

$$E_{001} = 0.065\, eV + 0.0002\, eV \simeq 0.065\, eV$$

$\Rightarrow$ for 10 α-particles

$$\boxed{E = 0.65\, eV}$$

(f) for extra credit:

the first excited state of the harmonic oscillator is

$$\psi_1(x) = A\,x\,e^{-\frac{m\omega}{2\hbar}x^2} = A\,x\,e^{-\sqrt{\frac{\alpha m}{2\hbar^2}}\,x^2}$$

In polar coordinates, $\begin{matrix} x = r\cos\phi \\ y = r\sin\phi \end{matrix} \Rightarrow x + iy = re^{i\phi}$

The operator L_z is $L_{z,op} = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$, so $x+iy$ is eigenfunction with eigenvalue $\hbar$.

So the wavefunction we are looking for is

$$\psi(x,y,z) = C\left[\psi_1(x)\,\psi_0(y) + i\,\psi_0(x)\,\psi_1(y)\right]\sin\left(\frac{\pi z}{a}\right)$$

or $$\boxed{\psi(x,y,z) = C\,(x+iy)\,e^{-\sqrt{\frac{\alpha m}{2\hbar^2}}\,(x^2+y^2)}\sin\left(\frac{\pi z}{a}\right)}$$

its energy is $$\boxed{E = \frac{\hbar\omega}{2} + \frac{3}{2}\hbar\omega + \frac{\hbar^2\pi^2}{2ma^2} = 12.54\,eV}$$