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Tunneling and energy splitting in an asymmetric double-well potential

Dae-Yup Song*

Department of Physics Education, Sunchon National University, Jungang-ro 413, Suncheon, Jeonnam 540-742, Republic of Korea

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ABSTRACT

An asymmetric double-well potential is considered, assuming that the minima of the wells are quadratic with a frequency ω and the difference of the minima is close to a multiple of $\hbar\omega$. A WKB wave function is constructed on both sides of the local maximum between the wells, by matching the WKB function to the exact wave functions near the classical turning points. The continuities of the wave function and its first derivative at the local maximum then give the energy-level splitting formula, which not only reproduces the instanton result for a symmetric potential, but also elucidates the appearance of resonances of tunneling in the asymmetric potential.

1. Introduction

The quantum tunneling in a double-well potential appears in a variety of physical cases. Well-known examples include inversion of ammonia molecule [1], for which the double-well is symmetric, and proton tunneling in hydrogen bonds, for which the two wells could be unsymmetrical [2]. For a symmetric potential, instanton method developed in [3], has been elaborated and applied to calculate energy splitting [4]. The tunneling splitting calculated from the instanton method exactly agrees with the WKB result when the quadratic connection formula is adopted, and it has been confirmed that the result is very accurate for large separation between the two wells [5–7].

Quantum tunneling in asymmetric double-well potentials has also long been considered [8,9], and it is known that calculations of the tunneling are necessary to locate the diabolic points of the magnetic molecule Fe_8, where the bottoms of the wells can be moved around by applying magnetic fields [10]. Furthermore, recent realizations of Bose–Einstein condensations (BECs) in the asymmetric (tilted) double-well potentials [11] provide the need for the theoretical analysis of the tunneling

* Fax: +82 61 750 3308.
E-mail address: dsong@sunchon.ac.kr

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[12]. Though interactions are important in the tunneling of the BECs realized so far [13], as the interactions between atoms are controllable, the quantum mechanics of a single particle is directly relevant to physics of the BECs or quantum degenerate Fermi atoms in the noninteracting limit [14].

One of the intriguing properties of the tunneling in asymmetric double-well potentials is the appearance of resonances. For example, the wave function appropriate for the false vacuum of the potential

$$V_{\mathrm{D}}(x) = \begin{cases} \frac{m\omega^2}{2}\left[(x+\alpha)^2 + (\beta^2-\alpha^2)\right] & \text{for } x < 0 \\ \frac{m\omega^2}{2}(x-\beta)^2 & \text{for } x \geqslant 0, \end{cases} \tag{1}$$

$(\beta > \alpha > 0)$ can significantly tunnel to the side of $x \geqslant 0$ only when β is tuned to satisfy the condition

$$\frac{m\omega^2}{2}(\beta^2-\alpha^2) \approx n\hbar\omega, \quad n = 0, 1, 2, 3 \cdots. \tag{2}$$

Through various numerical calculations, it has been known that the appearance of these resonances is not limited to $V_{\mathrm{D}}(x)$, but is a general property of tunneling in asymmetric potentials [8]. In this paper, in order to elucidate the analytic structure of the resonances, we construct the WKB wave functions for a class of asymmetric double-well potentials.

Specifically, we consider a smooth double-well potential $V(x)$, assuming that $V(x)$ has minima at $x = b$ and at $x = -a$ $(a, b > 0)$, and a local maximum at $x = 0$. The minima are taken to be quadratic with a frequency ω, $V(b) = 0$, and $V(-a) = (n+\epsilon)\hbar\omega$ (See Fig. 1). As in the instanton method, we are interested in the large separation between the two wells, and consider the ground and low lying excited states of energy eigenvalue E satisfying $V(0) \gg E > n\hbar\omega$. We also assume that the potential is still quadratic near the classical turning points between the wells.

Around the minima, exact solutions to the Schrödinger equation are described by the parabolic cylinder functions [15]. As anticipated in [10], on both sides of $x = 0$, a WKB wave function is constructed by matching the WKB function to the asymptotic forms of the exact solutions near the classical turning points. The continuities of the wave function and its first derivative at the local maximum then give the energy-level splitting formula. Though our method of requiring continuities is very different from the instanton or WKB method in [4], the splitting formula reduces to the known one in the symmetric case [4,16,17].

In the symmetric potential, for a given energy, an approximate solution to the Schrödinger equation localized in left(right) well implies, by the inversion symmetry, another solution localized in the right(left) well, and this fact has been conveniently used to evaluate energy splittings [18,16]. In this paper, we also show that tunneling in the asymmetric potential of $\epsilon = 0$ can be explored by

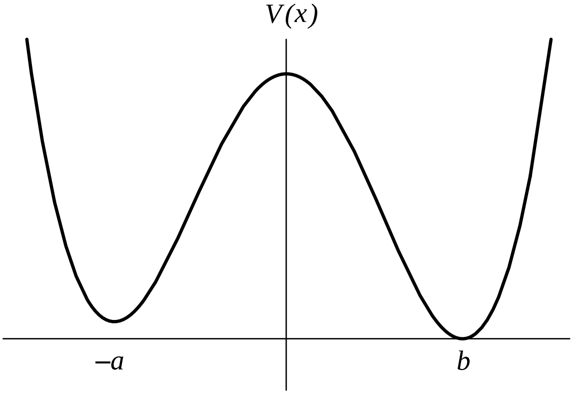

Fig. 1. An asymmetric double-well potential $V(x)$: $V(b) = 0$, $V(-a) = (n+\epsilon)\hbar\omega$, $(n = 0, 1, 2, \cdots)$. We assume that, for a given energy E, $V(x)$ is quadratic in the regions of classical motions with the frequency ω, and concentrate on the case of $\epsilon \ll 1$.

assuming the degenerate approximate solutions to the Schrödinger equation $\psi_{\mathrm{R}}(x)$ and $\psi_{\mathrm{L}}(x)$ which are localized in the right and left wells, respectively. By explicitly constructing $\psi_{\mathrm{R}}(x)$ and $\psi_{\mathrm{L}}(x)$, the splitting formula is found from these wave functions. Indeed it turns out that the WKB wave functions satisfying the continuities could be written as linear combinations of $\psi_{\mathrm{R}}(x)$ and $\psi_{\mathrm{L}}(x)$, while the linear combinations lift the degeneracy to make the splitting. Therefore, a linear combination of the time-dependent WKB wave functions gives a system which shuttles back and forth between $\psi_{\mathrm{R}}(x)$ and $\psi_{\mathrm{L}}(x)$, to clearly elucidate the resonance structure of the tunneling in the asymmetric potential of $\epsilon = 0$.

This paper is organized as follows: in Section 2, we construct the WKB wave function on both sides of the local maximum and, by requiring the continuities at the maximum, we evaluate the energy splitting. In Section 3, we find the appropriate localized wave functions $\psi_{\mathrm{R}}(x)$ and $\psi_{\mathrm{L}}(x)$, to re-obtain the energy splitting formula. We also establish a time-dependent WKB wave function of a system shuttling back and forth. In Section 4, we give some concluding remarks. Finally in Appendix A we give exact solutions for the system of $V_{\mathrm{D}}(x)$ in the limit of large separation between the two wells.

2. WKB method with continuity requirements

As the results could be easily modified to incorporate the small non-zero ϵ, we start with $\epsilon = 0$.

2.1. WKB wave function for $x \geqslant 0$

For the wave function $\psi_{\mathrm{I}}(x)$ of the energy eigenvalue $E = (v + n + \frac{1}{2})\hbar\omega$ around the right minimum, the Schrödinger equation is written as

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_{\mathrm{I}} + \frac{m\omega^2}{2}(x-b)^2\psi_{\mathrm{I}} = \hbar\omega\left(v + n + \frac{1}{2}\right)\psi_{\mathrm{I}}. \tag{3}$$

By introducing

$$l_{ho} = \sqrt{\frac{\hbar}{m\omega}}, \tag{4}$$

and $z_{\mathrm{R}} = \sqrt{2}(x-b)/l_{ho}$, we rewrite the equation as

$$\frac{\mathrm{d}^2\psi_{\mathrm{I}}}{\mathrm{d}z_{\mathrm{R}}^2} + \left(v + n + \frac{1}{2} - \frac{z_{\mathrm{R}}^2}{4}\right)\psi_{\mathrm{I}} = 0,$$

to obtain

$$\psi_{\mathrm{I}}(x) = C_{\mathrm{R}} D_{v+n}(z_{\mathrm{R}}) = C_{\mathrm{R}} D_{v+n}\left(\frac{\sqrt{2}(x-b)}{l_{ho}}\right), \tag{5}$$

where C_{R} is a constant and D_{v+n} denotes the parabolic cylinder function [15]. Bearing in mind that we wish to construct a normalizable wave function, we choose the solution in Eq. (5) so that $\int_b^\infty |\psi_{\mathrm{I}}(x)|^2 \mathrm{d}x$ is finite if the expression of $\psi_{\mathrm{I}}(x)$ is valid for $x > b$. Asymptotic expansions of the parabolic cylinder function are well-known. For large and negative z ($z \ll -1$ and $z \ll -\mid k \mid$), we have

$$D_k(z) \sim \mathrm{e}^{-\frac{z^2}{4}} z^k \left[1 - \frac{k(k-1)}{2z^2} + \cdots\right] - \frac{\sqrt{2\pi}}{\Gamma(-k)} \mathrm{e}^{k\pi \mathrm{i}} \mathrm{e}^{\frac{z^2}{4}} z^{-k-1}\left[1 + \frac{(k+1)(k+2)}{2z^2} + \cdots\right]. \tag{6}$$

For $x > 0$, a classical turning point may be written as

$$x = b_v = b - \sqrt{2v + 2n + 1}\, l_{ho}. \tag{7}$$

In the classically forbidden region of $b_v - x \gg l_{ho}$, within the WKB approximation, a solution $\psi_{\mathrm{II}}(x)$ to the Schrödinger equation is given as a linear combination of exponentially growing and decaying functions:

$$\psi_{\rm II}(x) = \frac{A_{\rm R}}{\sqrt{p(x)}}{\rm e}^{\int_0^x \frac{p(y)}{\hbar}{\rm d}y} + \frac{B_{\rm R}}{\sqrt{p(x)}}{\rm e}^{-\int_0^x \frac{p(y)}{\hbar}{\rm d}y}, \tag{8}$$

where $p(x)$ is defined as

$$p(x) = \sqrt{2m[V(x) - E]}, \tag{9}$$

and $A_{\rm R}, B_{\rm R}$ are constants. In the region of quadratic potential satisfying $b - x \gg b - b_v$, by introducing

$$\Phi_{\rm R}(x) = -\int_x^{b_v} \frac{p(y)}{\hbar}\,{\rm d}y = -\int_x^{b_v} \frac{\left[(b-y)^2 - (b-b_v)^2\right]^{1/2}}{l_{ho}^2}\,{\rm d}y, \tag{10}$$

we have [19,16]

$$\Phi_{\rm R}(x) = -\frac{(b-x)^2}{2l_{ho}^2} + \frac{1}{4}(2v+2n+1) + \frac{1}{2}(2v+2n+1)\ln\left(\frac{2(b-x)}{b-b_v}\right) + {\rm O}\left(\left(\frac{b-b_v}{b-x}\right)^2\right), \tag{11}$$

and thus

$$\begin{aligned}\psi_{\rm II}(x) \approx{}& \frac{A_{\rm R}}{\sqrt{m\omega(b-x)}}\left(\frac{2\sqrt{\rm e}(b-x)}{b-b_v}\right)^{v+n+\frac{1}{2}} \exp\left[\int_0^{b_v}\frac{p(y)}{\hbar}dy - \frac{(b-x)^2}{2l_{ho}^2}\right] \\ &+ \frac{B_{\rm R}}{\sqrt{m\omega(b-x)}}\left(\frac{b-b_v}{2\sqrt{\rm e}(b-x)}\right)^{v+n+\frac{1}{2}} \exp\left[-\int_0^{b_v}\frac{p(y)}{\hbar}\,{\rm d}y + \frac{(b-x)^2}{2l_{ho}^2}\right].\end{aligned} \tag{12}$$

As we are interested in the limit of large separation between the two wells where the energy splitting is small, we introduce δ_l with an integer $l\ (=[v])$ as

$$\delta_l = v - l, \tag{13}$$

so that

$$|\delta_l| \ll 1. \tag{14}$$

In the region of the quadratic potential the wave function is also described by $\psi_{\rm I}(x)$. Making use of the asymptotic form in (6), for $b - x \gg l_{ho}$, in the leading orders we obtain

$$\psi_{\rm I}(x) \sim C_{\rm R}\left[{\rm e}^{-\frac{(x-b)^2}{2l_{ho}^2}}\left(\frac{\sqrt{2}(x-b)}{l_{ho}}\right)^{l+n} + \delta_l {\rm e}^{\frac{(x-b)^2}{2l_{ho}^2}}\frac{\sqrt{2\pi}(l+n)!}{\left(\sqrt{2}l_{ho}^{-1}(x-b)\right)^{l+n+1}}\right]. \tag{15}$$

By matching the asymptotic form of $\psi_{\rm I}(x)$ onto that of $\psi_{\rm II}(x)$ in this overlap region, we have

$$A_{\rm R} = (-1)^{l+n}C_{\rm R}\sqrt{\frac{\hbar(l+n)!g_{l+n}}{2\sqrt{\pi}l_{ho}}}{\rm e}^{-\int_0^{b_l}\frac{p(y)}{\hbar}{\rm d}y}, \tag{16}$$

$$B_{\rm R} = (-1)^{l+n+1}C_{\rm R}\delta_l\sqrt{\frac{2\pi^{3/2}\hbar(l+n)!}{l_{ho}g_{l+n}}}{\rm e}^{\int_0^{b_l}\frac{p(y)}{\hbar}{\rm d}y}, \tag{17}$$

where

$$g_k = \frac{\sqrt{2\pi}}{k!}\left(k+\frac{1}{2}\right)^{k+\frac{1}{2}}{\rm e}^{-(k+1/2)}. \tag{18}$$

2.2. WKB wave function for $x \leqslant 0$

Since $V(x) = m\omega^2(x+a)^2/2 + n\hbar\omega$ near the minimum of the left well, a classical turning point is given as

$$x = -a_v = -a + \sqrt{2v+1}\, l_{ho}. \tag{19}$$

In the classically forbidden region of $a_v + x \gg l_{ho}$, a WKB solution may be written as

$$\psi_{\rm III}(x) = \frac{A_{\rm L}}{\sqrt{p(x)}} e^{-\int_x^0 \frac{p(y)}{\hbar} dy} + \frac{B_{\rm L}}{\sqrt{p(x)}} e^{\int_x^0 \frac{p(y)}{\hbar} dy}, \tag{20}$$

where $A_{\rm L}, B_{\rm L}$ are constants. In the region of the quadratic potential satisfying $a + x \gg a - a_v$, from the fact that

$$\Phi_{\rm L}(x) = -\int_{-a_v}^{x} \frac{p(y)}{\hbar} dy = -\int_{-a_v}^{x} \frac{\left[(a+y)^2 - (a-a_v)^2\right]^{1/2}}{l_{ho}^2} dy, \tag{21}$$

we obtain

$$\begin{aligned}\psi_{\rm III}(x) \approx\ & \frac{A_{\rm L}}{\sqrt{m\omega(a+x)}} \left(\frac{a-a_v}{2\sqrt{e}(a+x)}\right)^{l+\frac{1}{2}} \exp\left[-\int_{-a_v}^{0} \frac{p(y)}{\hbar} dy + \frac{(a+x)^2}{2l_{ho}^2}\right] \\ & + \frac{B_{\rm L}}{\sqrt{m\omega(a+x)}} \left(\frac{2\sqrt{e}(a+x)}{a-a_v}\right)^{l+\frac{1}{2}} \exp\left[\int_{-a_v}^{0} \frac{p(y)}{\hbar} dy - \frac{(a+x)^2}{2l_{ho}^2}\right].\end{aligned} \tag{22}$$

With $z_{\rm L} = \sqrt{2}(x+a)/l_{ho}$, the Schrödinger equation around the left minimum is written as

$$\frac{d^2\psi_{\rm IV}}{dz_{\rm L}^2} + \left(v + \frac{1}{2} - \frac{z_{\rm L}^2}{4}\right)\psi_{\rm IV} = 0,$$

and thus

$$\psi_{\rm IV}(x) = C_{\rm L} D_v(-z_{\rm L}) = C_{\rm L} D_v\left(-\frac{\sqrt{2}(x+a)}{l_{ho}}\right), \tag{23}$$

with a constant $C_{\rm L}$. The solution in (23) is chosen, so that $\int_{-\infty}^{-a} |\psi_{\rm IV}(x)|^2 dx$ is finite if the expression of $\psi_{\rm IV}(x)$ is valid for $x < -a$. Around the left minimum of $a + x \gg l_{ho}$, from (14) and the asymptotic form in (6), in the leading orders we have

$$\psi_{\rm IV}(x) \sim (-1)^l C_{\rm L} \left[e^{-\frac{(x+a)^2}{2l_{ho}^2}} \left(\frac{\sqrt{2}(x+a)}{l_{ho}}\right)^l - \delta_l\, e^{\frac{(x+a)^2}{2l_{ho}^2}} \frac{\sqrt{2\pi}\, l!}{\left(\sqrt{2}\, l_{ho}^{-1}(x+a)\right)^{l+1}} \right]. \tag{24}$$

By matching $\psi_{\rm III}(x)$ to $\psi_{\rm IV}(x)$ in the overlap region, we obtain

$$A_{\rm L} = (-1)^{l+1} C_{\rm L} \delta_l \sqrt{\frac{2\pi^{3/2}\hbar l!}{l_{ho} g_l}}\, e^{\int_{-a_l}^{0} \frac{p(y)}{\hbar} dy}, \tag{25}$$

$$B_{\rm L} = (-1)^l C_{\rm L} \sqrt{\frac{\hbar l! g_l}{2\sqrt{\pi} l_{ho}}}\, e^{-\int_{-a_l}^{0} \frac{p(y)}{\hbar} dy}. \tag{26}$$

2.3. Continuity and energy splitting

For a smooth potential, a wave function and its first derivative must be continuous at $x = 0$, which gives the relations

$$A_{\rm L} = A_{\rm R}, \quad B_{\rm L} = B_{\rm R}. \tag{27}$$

There are three unknowns $C_{\rm L}, C_{\rm R}, \delta_l$ in these two equations, while another equation may come from the normalization of the wave function. From $A_{\rm L}/B_{\rm L} = A_{\rm R}/B_{\rm R}$, making use of Eqs. (16), (17), (25), and (26), we obtain

$$\delta_l^2 = \frac{g_l g_{l+n}}{(2\pi)^2} \exp\left[-2\int_{-a_l}^{b_l} \frac{p(y)}{\hbar}\,\mathrm{d}y\right], \tag{28}$$

which indicates that the splitting of the energy level Δ_l is given by

$$\Delta_l = \sqrt{g_l g_{l+n}}\frac{\hbar\omega}{\pi} \exp\left[-\int_{-a_l}^{b_l} \frac{p(y)}{\hbar}\,\mathrm{d}y\right]. \tag{29}$$

For a symmetric potential of $n = 0$, (29) exactly agrees with the result in [16,17].

Expression in (29) is not easy to use, because the integrand in the exponential is close to a singularity near the limits. By introducing

$$I_a = \int_{-a}^{0} \sqrt{2m[V(y) - n\hbar\omega]}\,\mathrm{d}y, I_b = \int_0^b \sqrt{2mV(y)}\,\mathrm{d}y, \tag{30}$$

and

$$\begin{aligned} \gamma_a &= \int_0^a \left(\frac{\sqrt{m\omega^2}}{\sqrt{2[V(y-a) - n\hbar\omega]}} - \frac{1}{y}\right)\mathrm{d}y, \\ \gamma_b &= \int_0^b \left(\frac{\sqrt{m\omega^2}}{\sqrt{2V(b-y)}} - \frac{1}{y}\right)\mathrm{d}y, \end{aligned} \tag{31}$$

the splitting is written for a general potential as

$$\Delta_l = \hbar\omega \frac{\sqrt{2}\mathrm{e}^{-(I_a+I_b)/\hbar}}{\sqrt{\pi(l+n)!l!}} \left(\frac{\sqrt{2}a\mathrm{e}^{\gamma_a}}{l_{ho}}\right)^{l+1/2} \left(\frac{\sqrt{2}b\mathrm{e}^{\gamma_b}}{l_{ho}}\right)^{l+n+1/2}. \tag{32}$$

The expression in (32) can be conveniently used to find that our formula reduces to the known one in [17] for a symmetric potential.

2.4. For a non-zero ϵ

The above formalism can be modified to include non-zero ϵ, as far as $\delta_l - \epsilon \ll 1$. In this case, without a change in $\psi_{\mathrm{I}}(x)$ and $\psi_{\mathrm{II}}(x)$, the modifications of $\psi_{\mathrm{III}}(x)$ and $\psi_{\mathrm{IV}}(x)$ are obtained by replacing δ_l with $\delta_l - \epsilon$ (or, v with $v - \epsilon$). Due to the changes in Eqs. (24,25), the continuity requirements then give the relation

$$\delta_l(\delta_l - \epsilon) = \left(\frac{\Delta_l}{2\hbar\omega}\right)^2. \tag{33}$$

If Δ_l^ϵ denotes the energy splitting in the presence of ϵ, (33) implies

$$\Delta_l^\epsilon = \sqrt{\Delta_l^2 + (\hbar\omega\epsilon)^2}. \tag{34}$$

3. An alternative method with localized wave functions

For $\epsilon = 0$, if we assume two states of the normalized real wave functions $\psi_{\mathrm{R}}(x)$ and $\psi_{\mathrm{L}}(x)$ with energy E_0 as mentioned in Section 1, the Hamiltonian in this two-state subspace may be given by

$$H = \begin{pmatrix} E_0 & \Delta/2 \\ \Delta/2 & E_0 \end{pmatrix} = E_0 I + \frac{\Delta}{2}\sigma_x, \tag{35}$$

with the Pauli matrices $\sigma_i (i = 1,2,3)$, where the small tunneling splitting Δ is written as

$$\Delta = 2\left|\int_{-\infty}^{\infty} \psi_{\mathrm{L}}(x)\left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right)\psi_{\mathrm{R}}(x)\,\mathrm{d}x\right|. \tag{36}$$

The two eigenstates of the Hamiltonian are given by $(\psi_{\rm R}(x) \pm \psi_{\rm L}(x))/\sqrt{2}$. From the Schrödinger equations for these eigenstates, and the equations

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi_i(x) = E_0\psi_i(x)(i = {\rm R}, {\rm L}),$$

and from the requirements

$$\int_0^\infty (\psi_{\rm R}(x))^2 dx \approx 1, \quad \int_{-\infty}^0 (\psi_{\rm L}(x))^2 dx \approx 1, \quad \int_0^\infty \psi_{\rm L}(x)\psi_{\rm R}(x)\, dx \approx 0, \tag{37}$$

we find

$$\Delta \approx \frac{\hbar^2}{m}\left|\psi_{\rm L}(0)\psi'_{\rm R}(0) - \psi_{\rm R}(0)\psi'_{\rm L}(0)\right|, \tag{38}$$

with $\psi'_i(x) = d\psi_i(x)/dx$ ($i = {\rm R}, {\rm L}$), which is a generalization of the method used for the symmetric case [16,18].

In the classically forbidden region, we may write

$$\psi_{\rm R}(x) = \frac{N_{\rm R}}{\sqrt{p(x)}} e^{\int_0^x \frac{p(y)}{\hbar} dy}, \quad \psi_{\rm L}(x) = \frac{N_{\rm L}}{\sqrt{p(x)}} e^{-\int_0^x \frac{p(y)}{\hbar} dy}, \tag{39}$$

where $N_{\rm R}$ and $N_{\rm L}$ are constants. From these expressions of $\psi_{\rm R}(x)$ and $\psi_{\rm L}(x)$, by assuming that the validity condition for the WKB approximation

$$\left|\frac{d}{dx}\frac{\hbar}{p}\right| \ll 1 \tag{40}$$

is satisfied at $x = 0$, we obtain

$$\Delta \approx 2\frac{\hbar}{m}|N_{\rm L}N_{\rm R}|. \tag{41}$$

For $E_0 = (l + n + \frac{1}{2})\hbar\omega$, near the right-hand well, $\psi_{\rm R}(x)$ would be accurately described by the $(l + n)$th harmonic oscillator eigenfunction. This description holds well into the forbidden region, and, for $(b - x)/l_{ho} \gg 1$, we may write

$$\psi_{\rm R}(x) \approx \frac{\exp\left[-\frac{(x-b)^2}{2l_{ho}^2}\right]}{\sqrt{\sqrt{\pi}l_{ho}(l+n)!}}\left(\frac{\sqrt{2}(x-b)}{l_{ho}}\right)^{l+n}. \tag{42}$$

On the other hand, making use of (11), we can find the asymptotic expansion form of $\psi_{\rm R}(x)$ of (39) in the overlap region. By matching the asymptotic form onto the expression in (42), we obtain

$$N_{\rm R} = (-1)^{l+n}\frac{\sqrt{\hbar g_{l+n}}}{\sqrt{2\pi}l_{ho}}\exp\left[-\int_0^{b_v}\frac{p(y)}{\hbar}dy\right]. \tag{43}$$

Similarly, by matching the asymptotic form of $\psi_{\rm L}(x)$ onto that of the lth excited harmonic oscillator state near the left-hand well, we have

$$N_{\rm L} = \frac{\sqrt{\hbar g_l}}{\sqrt{2\pi}l_{ho}}\exp\left[-\int_{-a_v}^0\frac{p(y)}{\hbar}dy\right]. \tag{44}$$

By plugging these explicit forms of $N_{\rm R}$ and $N_{\rm L}$ into (41), for $E_0 = (l + n + \frac{1}{2})\hbar\omega$, we confirm that Δ reduces to Δ_l.

If (27) is satisfied, after some algebra, we find that $\psi_{\rm II}(x)$ and $\psi_{\rm III}(x)$ of $\epsilon = 0$ can be merged, in the classically forbidden region, into

$$\psi^{\pm}_{\rm WKB}(x) = \frac{\sqrt{2\pi}A_{\rm R}l_{ho}}{\sqrt{\hbar g_{l+n}}}\exp\left[\int_0^{b_v}\frac{p(y)}{\hbar}dy\right]\left[(-1)^{l+n}\psi_{\rm R}(x) \mp \psi_{\rm L}(x)\right], \tag{45}$$

namely, $\psi^{\pm}_{\rm WKB}(x)$ becomes $\psi_{\rm II}(x)$ for $x \geqslant 0$, and $\psi_{\rm III}(x)$ for $x \leqslant 0$, where $\psi^{+}_{\rm WKB}(x)$ ($\psi^{-}_{\rm WKB}(x)$) is given when we choose $\delta_l > 0$ ($\delta_l < 0$) in (28). Thus this alternative method is in fact equivalent to the WKB method of requiring the continuities. We also note that $\psi^{+}_{\rm WKB}(x)$ has a node in the classically forbidden region between the wells, while $\psi^{-}_{\rm WKB}(x)$ has no node in the same region.

A (unnormalized) time-dependent WKB solution is given as

$$
\begin{aligned}
\psi(x,t) &= \mathrm{e}^{-\mathrm{i}\omega\left(n+l+\frac{1}{2}+|\delta_l|\right)t}\psi^{+}_{\rm WKB}(x) + \mathrm{e}^{-\mathrm{i}\omega\left(n+l+\frac{1}{2}-|\delta_l|\right)t}\psi^{-}_{\rm WKB}(x) \\
&= 2\frac{\sqrt{2\pi}A_{\rm R} l_{ho}}{\sqrt{\hbar g_{l+n}}}\exp\left[-\mathrm{i}\omega\left(n+l+\frac{1}{2}\right)t + \int_0^{b_v}\frac{p(y)}{\hbar}\,\mathrm{d}y\right] \\
&\quad\times\left[(-1)^{l+n}\cos\left(\frac{\Delta_l}{2\hbar}t\right)\psi_{\rm R}(x) - \mathrm{i}\sin\left(\frac{\Delta_l}{2\hbar}t\right)\psi_{\rm L}(x)\right].
\end{aligned}
\tag{46}
$$

This last form shows clearly that the system shuttles back and forth between $\psi_{\rm R}(x)$ and $\psi_{\rm L}(x)$ with the frequency $\Delta_l/\hbar$.

In order to include ϵ, let us consider a slight modification of the potential around the left well so that $V(-a)$ changes from $n\hbar\omega$ to $(n+\epsilon)\hbar\omega$. While $\psi_{\rm R}(x)$ would still be an appropriate solution of the new system with energy E_0, we introduce $\psi^{\epsilon}_{\rm L}(x)$ as an approximate solution with energy $E_0 + \epsilon\hbar\omega$ localized in the left well. If we confine our attention on the two state subspace described by $\psi_{\rm R}(x)$ and $\psi^{\epsilon}_{\rm L}(x)$, within the approximation that $\psi^{\epsilon}_{\rm L}(x)$ is the same with $\psi_{\rm L}(x)$, the Hamiltonian of the new system is written as

$$
\left(E_0 + \frac{1}{2}\epsilon\hbar\omega\right)I + \frac{1}{2}(-\epsilon\hbar\omega\sigma_z + \Delta_l\sigma_x),
$$

which is analogous to that of a particle in a magnetic field [20]. This spin analogy can be used to derive (34) and to easily find the time-evolution of the system. If $\psi^{\epsilon}(x,t)$ is a solution in this subspace with $\psi^{\epsilon}(x,0) = \psi_{\rm R}(x)$, the maximum of the probability of $|\int_{-\infty}^{\infty}(\psi^{\epsilon}_{\rm L}(x))^{*}\psi^{\epsilon}(x,t)\,\mathrm{d}x|^2$ during the time-evolution can be evaluated to be $\Delta_l^2/[\Delta_l^2 + (\epsilon\hbar\omega)^2]$, which indicates that the resonance peaks in tunneling have the Lorentzian shape.

4. Concluding remarks

Energy splitting formula has been obtained for the asymmetric double-well potential, by assuming that the potential is quadratic near the minima. As has been well known for the symmetric case, we expect that the splitting formula given here would be very accurate for the large separation between wells, which needs to be confirmed through numerical calculations. If we could add a linear term sx to the potential $V(x)$ of $\epsilon = 0$ with a controllable constant s, $V(x) + sx$ has two minima at $x = b - s/m\omega^2$ and at $x = -a - s/m\omega^2$. Since the difference of the minima is given as $n\hbar\omega - s(a+b)$, in the light of numerical results [8], the tunneling would be significant only if s is close to a multiple of $\hbar\omega/(a+b)$. It would be of great interest to realize the asymmetric system with controllable constants, as is partially accomplished in dynamical situations [11]. For $n = 0$ and $\epsilon = 0$, we note that $\Delta_{2l}/\Delta_0 = (2ab\mathrm{e}^{\gamma_a+\gamma_b}/l_{ho}^2)^{2l}/(2l)!$ which shows the quasi-Weierstrassian nature of the tunneling spectrum [17]. In this asymmetric case, thus, the tunneling behavior of an initially squeezed wave packet is erratic, and the trajectory of the expected position of the wave packet has a fractal structure. As a final remark, though the results obtained in this paper would be exact in the limit of $a, b \gg l_{ho}$ where an energy splitting is very small, numerical calculations imply that the appearance of resonances is manifest even when a and b are only a few times larger than l_{ho} [8], which could be important in the quantum tunnelings of the BECs.

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Appendix A. For the potential $V_D(x)$ in (1), WKB method may not be applicable, since the potential is not differentiable at $x = 0$. In this case, however, exact wave functions could be written in terms of the parabolic cylinder functions on both sides of $x = 0$. From the continuities of the wave function and its first derivative at $x = 0$, assuming $\frac{m\omega^2}{2}(\beta^2 - \alpha^2) = (n + \epsilon)\hbar\omega$, we find that the eigenstate of an energy eigenvalue $(v + n + \epsilon + \frac{1}{2})\hbar\omega$ exists if the condition

$$D_v\left(-\frac{\sqrt{2}\alpha}{l_{ho}}\right)D'_{v+n+\epsilon}\left(-\frac{\sqrt{2}\beta}{l_{ho}}\right) = -D'_v\left(-\frac{\sqrt{2}\alpha}{l_{ho}}\right)D_{v+n+\epsilon}\left(-\frac{\sqrt{2}\beta}{l_{ho}}\right) \tag{47}$$

is satisfied. (47) can be solved in the limit of $\alpha, \beta \gg l_{ho}$ and $\epsilon \ll 1$. Making use of the asymptotic expansion of (6), in this limit we obtain

$$\delta_l^2 + (r(R_l - L_l) + \epsilon)\delta_l - R_l L_l - \epsilon r L_l = 0, \tag{48}$$

where

$$R_l = \frac{(\sqrt{2}\beta/l_{ho})^{2(n+l)+1}}{\sqrt{2\pi}(l+n)!}\mathrm{e}^{-\beta^2/l_{ho}^2}, \quad L_l = \frac{(\sqrt{2}\alpha/l_{ho})^{2l+1}}{\sqrt{2\pi}l!}\mathrm{e}^{-\alpha^2/l_{ho}^2},$$
$$r = \frac{\beta - \alpha}{\alpha + \beta}. \tag{49}$$

(48) implies that the energy splitting is given as

$$\hbar\omega\sqrt{4R_l L_l + \epsilon^2 + 2\epsilon r(R_l + L_l) + r^2(R_l - L_l)^2}. \tag{50}$$

If we formally use the formulas in Eqs. (30,31) by replacing $V(x)$ with $V_D(x)$, Δ_l coincides with the splitting of (50) in the symmetric case $(n = 0, \epsilon = 0)$.

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