PHYSICS 100C: ELECTROMAGNETISM SOLUTIONS HOMEWORK #5

Prof. Julio Barreiro
TA: Sebastian Diaz

Problem I

(a) The retarded electrical potential is computed from

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r'},t_r)}{|\boldsymbol{r}-\boldsymbol{r'}|}d^3\boldsymbol{r'}, \tag{1}$$

where the retarded time is $t_r = t - |\boldsymbol{r}-\boldsymbol{r'}|/c$. The charges of this electric dipole are located at $\boldsymbol{r}_\pm = \pm\frac{s}{2}\hat{\boldsymbol{z}}$, so the corresponding retarded times are $t_r^\pm = t - |\boldsymbol{r}-\boldsymbol{r}_\pm|/c$. With the aid of the three-dimensional Dirac delta function we can write the charge density as

$$\rho(\boldsymbol{r'},t_r) = \text{Re}\left\{Q(t_r^+)\delta^3(\boldsymbol{r'}-\boldsymbol{r}_+) - Q(t_r^-)\delta^3(\boldsymbol{r'}-\boldsymbol{r}_-)\right\}, \tag{2}$$

with $Q(t_r) = Q_m e^{i\omega t_r}$. Plugging this charge density in (1) we obtain

$$V(\boldsymbol{r},t) = \text{Re}\left\{\frac{1}{4\pi\epsilon_0}\left[\frac{Q(t_r^+)}{|\boldsymbol{r}-\boldsymbol{r}_+|} - \frac{Q(t_r^-)}{|\boldsymbol{r}-\boldsymbol{r}_-|}\right]\right\}. \tag{3}$$

We now need to compute $|\boldsymbol{r}-\boldsymbol{r}_\pm|$ as these appear in the denominators and in the retarded times in the expression for the potential. We then get

$$\begin{aligned}
\left|\boldsymbol{r}\pm\tfrac{s}{2}\hat{\boldsymbol{z}}\right| &= \left[(\boldsymbol{r}\pm\tfrac{s}{2}\hat{\boldsymbol{z}})\cdot(\boldsymbol{r}\pm\tfrac{s}{2}\hat{\boldsymbol{z}})\right]^{1/2} && (4)\\
&= \left[r^2 \pm sr\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{z}} + (\tfrac{s}{2})^2\right]^{1/2} \quad \wedge \quad \hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{z}} = \cos\theta && (5)\\
&= r\left[1 \pm \tfrac{s}{r}\cos\theta + (\tfrac{s}{2r})^2\right]^{1/2} \quad \wedge \quad s \ll r \iff \tfrac{s}{r} \ll 1 && (6)\\
&\approx r\left[1 \pm \tfrac{s}{2r}\cos\theta\right]. && (7)
\end{aligned}$$

Thus, the denominators can be approximated as

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_\pm|} = \frac{1}{\left|\boldsymbol{r}\mp\frac{s}{2}\hat{\boldsymbol{z}}\right|} \approx \frac{1}{r\left[1\mp\frac{s}{2r}\cos\theta\right]} \approx \frac{1}{r}\left[1\pm\frac{s}{2r}\cos\theta\right], \tag{8}$$

while the exponents of the complex exponentials, using $\lambda \equiv c/\omega$, become

$$\omega t_r^\pm = \omega t - \frac{\omega}{c}|\boldsymbol{r}-\boldsymbol{r}_\pm| \approx \omega t - \frac{\omega}{c}r\left[1\mp\frac{s}{2r}\cos\theta\right] = \omega\left(t-\frac{r}{c}\right) \pm \frac{s}{2\lambda}\cos\theta. \tag{9}$$

With these results, the potential takes the form

$$V(\boldsymbol{r},t) \approx \text{Re}\left\{\frac{Q_m e^{i\omega(t-\frac{r}{c})}}{4\pi\epsilon_0 r}\left[(1+\tfrac{s}{2r}\cos\theta)e^{i\frac{s}{2\lambda}\cos\theta} - (1-\tfrac{s}{2r}\cos\theta)e^{-i\frac{s}{2\lambda}\cos\theta}\right]\right\}. \tag{10}$$

Assuming $s \ll \lambda \iff \frac{s}{\lambda} \ll 1$, then $e^{\pm i\frac{s}{2\lambda}\cos\theta} \approx 1 \pm i\frac{s}{2\lambda}\cos\theta$. Using this approximation for the complex exponentials, after some straightforward algebra we finally arrive at

$$V(\boldsymbol{r},t) \approx \text{Re}\left\{\frac{Q_m e^{i\omega(t-\frac{r}{c})}s\cos\theta}{4\pi\epsilon_0\lambda r}\left[\frac{\lambda}{r}+i\right]\right\} = \text{Re}\left\{\frac{[p^*]\cos\theta}{4\pi\epsilon_0\lambda r}\left[\frac{\lambda}{r}+i\right]\right\}, \tag{11}$$

where $[p^*] \equiv sQ_m e^{i\omega(t-\frac{r}{c})}$ is the magnitude of the complex electric dipole moment evaluated at the retarded time $t-\frac{r}{c}$.

(b) The expression for the retarded vector potential is

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3\boldsymbol{r}'. \tag{12}$$

One can write the current density as

$$\boldsymbol{J}(\boldsymbol{r}',t_r) = \begin{cases} \hat{\boldsymbol{z}} I(t_r)\delta(x')\delta(y') & , \quad z' \in [-\frac{s}{2},\frac{s}{2}] \\ \boldsymbol{0} & , \quad z' \in (-\infty,-\frac{s}{2}) \cup (\frac{s}{2},\infty) \end{cases} , \tag{13}$$

where $I(t) = \mathrm{Re}\left\{\frac{d}{dt}Q(t)\right\} = \mathrm{Re}\left\{\frac{d}{dt}Q_m e^{i\omega t}\right\} = \mathrm{Re}\left\{i\omega Q_m e^{i\omega t}\right\}$. Substituting in (12) yields

$$\boldsymbol{A}(\boldsymbol{r},t) = \mathrm{Re}\left\{\hat{\boldsymbol{z}}\frac{\mu_0}{4\pi}i\omega Q_m \int_{-s/2}^{s/2} \frac{e^{i\omega(t-|\boldsymbol{r}-z'\hat{\boldsymbol{z}}|/c)}}{|\boldsymbol{r}-z'\hat{\boldsymbol{z}}|}dz'\right\}. \tag{14}$$

Since $z' \in [-\frac{s}{2},\frac{s}{2}] \Longrightarrow |z'| \le \frac{s}{2}$, and recalling that $s \ll r$ we conclude $\frac{|z'|}{r} \ll 1$. So we can repeat the approximations presented in part (a) to obtain

$$|\boldsymbol{r}-z'\hat{\boldsymbol{z}}| \approx r\left[1-\tfrac{z'}{r}\cos\theta\right], \tag{15}$$

$$\frac{1}{|\boldsymbol{r}-z'\hat{\boldsymbol{z}}|} \approx \frac{1}{r}\left[1+\tfrac{z'}{r}\cos\theta\right], \tag{16}$$

$$\omega\left(t-\frac{|\boldsymbol{r}-z'\hat{\boldsymbol{z}}|}{c}\right) \approx \omega\left(t-\frac{r}{c}\right)+\frac{z'}{\lambda\!\!\!^{-}}\cos\theta. \tag{17}$$

Using these approximations we get the following vector potential

$$\boldsymbol{A}(\boldsymbol{r},t) \approx \mathrm{Re}\left\{\hat{\boldsymbol{z}}\frac{\mu_0}{4\pi r}i\omega Q_m e^{i\omega(t-\frac{r}{c})}\int_{-s/2}^{s/2} e^{i\frac{z'}{\lambda\!\!\!^{-}}\cos\theta}\left[1+\tfrac{z'}{r}\cos\theta\right]dz'\right\}. \tag{18}$$

The already shown result $|z'| \le \frac{s}{2}$ together with $s \ll \lambda\!\!\!^{-}$ imply $\frac{|z'|}{\lambda\!\!\!^{-}} \ll 1$, hence $e^{i\frac{z'}{\lambda\!\!\!^{-}}\cos\theta} \approx 1+i\frac{z'}{\lambda\!\!\!^{-}}\cos\theta$. Thus the integral to be computed simplifies to

$$\int_{-s/2}^{s/2}\left[1+i\frac{z'}{\lambda\!\!\!^{-}}\cos\theta\right]\left[1+\frac{z'}{r}\cos\theta\right]dz' \approx \int_{-s/2}^{s/2}\left[1+z'\left(\frac{i}{\lambda\!\!\!^{-}}+\frac{1}{r}\right)\cos\theta\right]dz'. \tag{19}$$

But $\int_{-s/2}^{s/2} z'dz' = 0$, so the integral can be simply approximated as s. Therefore, the retarded vector potential is given by

$$\boldsymbol{A}(\boldsymbol{r},t) \approx \mathrm{Re}\left\{\frac{1}{4\pi\epsilon_0 c\lambda\!\!\!^{-} r}isQ_m e^{i\omega(t-\frac{r}{c})}(\cos\theta\hat{\boldsymbol{r}}-\sin\theta\hat{\boldsymbol{\theta}})\right\} = \mathrm{Re}\left\{\frac{i[p^*]}{4\pi\epsilon_0 c\lambda\!\!\!^{-} r}(\cos\theta\hat{\boldsymbol{r}}-\sin\theta\hat{\boldsymbol{\theta}})\right\}, \tag{20}$$

where we used that $\mu_0\omega = \frac{1}{\epsilon_0 c\lambda\!\!\!^{-}}$ and $\hat{\boldsymbol{z}} = \cos\theta\hat{\boldsymbol{r}}-\sin\theta\hat{\boldsymbol{\theta}}$.

(c) By direct differentiation—using $\frac{\partial}{\partial t}[p^*] = i\omega[p^*]$ and $\frac{\partial}{\partial r}[p^*] = -\frac{i}{\lambda\!\!\!^{-}}[p^*]$—one can readily show

$$\epsilon_0\mu_0\frac{\partial V}{\partial t} = \mathrm{Re}\left\{\frac{i\omega[p^*]\cos\theta}{4\pi\epsilon_0 c^2\lambda\!\!\!^{-} r}\left[\frac{\lambda\!\!\!^{-}}{r}+i\right]\right\} = -\nabla\cdot\boldsymbol{A}, \tag{21}$$

hence $\epsilon_0\mu_0\frac{\partial V}{\partial t}+\nabla\cdot\boldsymbol{A} = 0$. Therefore, the potentials computed in (a) and (b) do indeed satisfy the Lorentz condition.

Problem II

(a) The retarded vector potential generated by a current $I(t) = \mathrm{Re}\left\{I_m e^{i\omega t}\right\}$ flowing in a circular loop of radius a in the xy plane reads

$$\boldsymbol{A}(\boldsymbol{r},t) = \mathrm{Re}\left\{\frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{I_m e^{i\omega(t-r'/c)}}{r'}a\hat{\boldsymbol{\phi}}d\phi\right\}, \tag{22}$$

where ϕ is the angle used to parameterize the loop and $r' = |\boldsymbol{r}'| = |\boldsymbol{r} - a\hat{\boldsymbol{\rho}}(\phi)|$. For those interested, the above expression results from the current density $\boldsymbol{J}(\boldsymbol{r},t) = I(t)\delta(\rho - a)\delta(z)\hat{\boldsymbol{\phi}}$.

In spherical coordinates (r, θ, φ), the point at which we are computing $\boldsymbol{A}(\boldsymbol{r},t)$ is

$$\boldsymbol{r} = r[\sin\theta\cos\varphi\hat{\boldsymbol{x}} + \sin\theta\sin\varphi\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{z}}] = r[\sin\theta\hat{\boldsymbol{\rho}}(\varphi) + \cos\theta\hat{\boldsymbol{z}}], \tag{23}$$

so we have $\boldsymbol{r}\cdot\hat{\boldsymbol{\rho}}(\phi) = r\sin\theta\cos(\phi-\varphi)$. Thus,

$$\begin{aligned}
r' &= \left[(\boldsymbol{r} - a\hat{\boldsymbol{\rho}}(\phi))\cdot(\boldsymbol{r} - a\hat{\boldsymbol{\rho}}(\phi))\right]^{1/2} && (24)\\
&= \left[r^2 - 2a\boldsymbol{r}\cdot\hat{\boldsymbol{\rho}}(\phi) + a^2\right]^{1/2} && (25)\\
&= \left[r^2 - 2ar\sin\theta\cos(\phi-\varphi) + a^2\right]^{1/2} && (26)\\
&= r\left[1 - 2\tfrac{a}{r}\sin\theta\cos(\phi-\varphi) + \left(\tfrac{a}{r}\right)^2\right]^{1/2} \quad \wedge \quad \tfrac{a}{r} \ll 1 && (27)\\
&\approx r\left[1 - \tfrac{a}{r}\sin\theta\cos(\phi-\varphi)\right]. && (28)
\end{aligned}$$

Using this approximate expression for r' we obtain

$$\frac{1}{r'} \approx \frac{1}{r}\left[1 + \frac{a}{r}\sin\theta\cos(\phi-\varphi)\right] \quad \text{and} \quad \omega\left(t - \frac{r'}{c}\right) \approx \omega\left(t - \frac{r}{c}\right) + \frac{a}{\lambda\!\!\!^-}\sin\theta\cos(\phi-\varphi). \tag{29}$$

The exponential in the integrand then becomes

$$e^{i\omega(t-r'/c)} \approx e^{i\omega(t-r/c)}e^{i\frac{a}{\lambda\!\!\!^-}\sin\theta\cos(\phi-\varphi)} \approx e^{i\omega(t-r/c)}\left(1 + i\frac{a}{\lambda\!\!\!^-}\sin\theta\cos(\phi-\varphi)\right) \tag{30}$$

where we used $\frac{a}{\lambda\!\!\!^-} \ll 1$ to get the last expression. Plugging all these approximations in (22) yields

$$\boldsymbol{A}(\boldsymbol{r},t) \approx \mathrm{Re}\left\{\frac{\mu_0 I_m}{4\pi r}e^{i\omega(t-r/c)}\int_0^{2\pi}\left(1 + a\sin\theta\cos(\phi-\varphi)\left[\frac{i}{\lambda\!\!\!^-} + \frac{1}{r}\right]\right)a\hat{\boldsymbol{\phi}}d\phi\right\}. \tag{31}$$

Recalling that $\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}}$ and $\cos(\phi-\varphi) = \cos\phi\cos\varphi + \sin\phi\sin\varphi$, one can compute

$$\begin{aligned}
\int_0^{2\pi}\hat{\boldsymbol{\phi}}d\phi &= \boldsymbol{0}, && (32)\\
\int_0^{2\pi}\cos(\phi-\varphi)\hat{\boldsymbol{\phi}}d\phi &= \pi\hat{\boldsymbol{\varphi}}, && (33)
\end{aligned}$$

where $\hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\boldsymbol{x}} + \cos\varphi\hat{\boldsymbol{y}}$. Therefore, we get

$$\boldsymbol{A}(\boldsymbol{r},t) \approx \mathrm{Re}\left\{\frac{\mu_0\pi a^2 I_m}{4\pi\lambda\!\!\!^- r}e^{i\omega(t-r/c)}\left[\frac{\lambda\!\!\!^-}{r} + i\right]\sin\theta\hat{\boldsymbol{\varphi}}\right\}. \tag{34}$$

Introducing $m_m \equiv \pi a^2 I_m$, $[t] \equiv t - \frac{r}{c}$, $[m^*] = m_m e^{i\omega[t]}$, and $[\boldsymbol{m}^*] = [m^*]\hat{\boldsymbol{z}}$ we finally obtain

$$\boldsymbol{A}(\boldsymbol{r}, t) \approx \text{Re}\left\{ \frac{\mu_0 m_m}{4\pi \lambda\!\!\!^{-} r} e^{i\omega[t]} \left[\frac{\lambda\!\!\!^{-}}{r} + i \right] \sin\theta \hat{\boldsymbol{\varphi}} \right\} = \text{Re}\left\{ \frac{i\mu_0 [\boldsymbol{m}^*] \times \hat{\boldsymbol{r}}}{4\pi \lambda\!\!\!^{-} r} \left[1 - i\frac{\lambda\!\!\!^{-}}{r} \right] \right\}. \tag{35}$$

Note that to get the last expression we used $\hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}} = \sin\theta \hat{\boldsymbol{\varphi}}$. It's worth mentioning the similarity in the structure between the electrical potential we computed in Problem I (a) and the present result for the vector potential. By introducing $[\boldsymbol{p}^*] \equiv [p^*]\hat{\boldsymbol{z}}$, we can rewrite V as

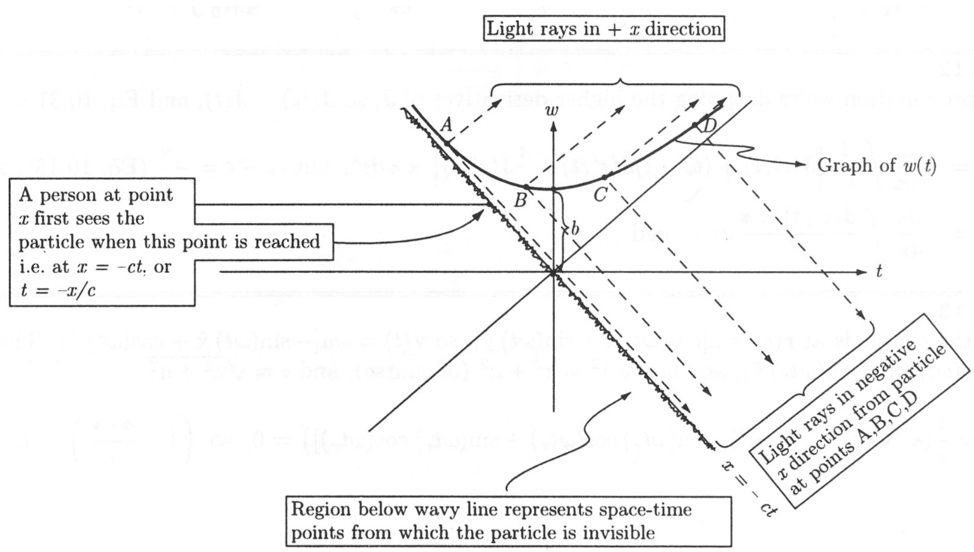
$$V(\boldsymbol{r}, t) \approx \text{Re}\left\{ \frac{i[\boldsymbol{p}^*] \cdot \hat{\boldsymbol{r}}}{4\pi\epsilon_0 \lambda\!\!\!^{-} r} \left[1 - i\frac{\lambda\!\!\!^{-}}{r} \right] \right\}. \tag{36}$$

They both exhibit the same r dependence. Their angular dependence is determined by $[\boldsymbol{p}^*] \cdot \hat{\boldsymbol{r}}$ and $[\boldsymbol{m}^*] \times \hat{\boldsymbol{r}}$, respectively.

(b) The magnitude of the vector potential is $|\boldsymbol{A}(\boldsymbol{r}, t)| = \frac{\mu_0 m_m}{4\pi \lambda\!\!\!^{-} r^2} |[\lambda\!\!\!^{-} \cos(\omega[t]) - r\sin(\omega[t])] \sin\theta|$. It vanishes at either $\theta = 0, \pi$ or $\tan(\omega[t]) = \lambda\!\!\!^{-}/r$. At fixed time t, $|\boldsymbol{A}(\boldsymbol{r}, t)|$ is maximum at $\theta = \frac{\pi}{2}$.

Problem 10.17

Once seen, from a given point x, the particle will forever remain in view—to disappear it would have to travel faster than light.



Problem 10.19

From Eq. 10.44, $c(t-t_r)=\mathscr{r} \Rightarrow c^2(t-t_r)^2=\mathscr{r}^2=\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}$. Differentiate with respect to t:
$2c^2(t-t_r)\left(1-\frac{\partial t_r}{\partial t}\right)=2\boldsymbol{\mathscr{r}}\cdot\frac{\partial\boldsymbol{\mathscr{r}}}{\partial t}$, or $c\mathscr{r}\left(1-\frac{\partial t_r}{\partial t}\right)=\boldsymbol{\mathscr{r}}\cdot\frac{\partial\boldsymbol{\mathscr{r}}}{\partial t}$. Now $\boldsymbol{\mathscr{r}}=\mathbf{r}-\mathbf{w}(t_r)$, so
$\frac{\partial\boldsymbol{\mathscr{r}}}{\partial t}=-\frac{\partial\mathbf{w}}{\partial t}=-\frac{\partial\mathbf{w}}{\partial t_r}\frac{\partial t_r}{\partial t}=-\mathbf{v}\frac{\partial t_r}{\partial t}$; $\quad c\mathscr{r}\left(1-\frac{\partial t_r}{\partial t}\right)=-\boldsymbol{\mathscr{r}}\cdot\mathbf{v}\frac{\partial t_r}{\partial t}$; $\quad c\mathscr{r}=\frac{\partial t_r}{\partial t}(c\mathscr{r}-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})=\frac{\partial t_r}{\partial t}(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})$ (Eq. 10.71), and hence $\frac{\partial t_r}{\partial t}=\frac{c\mathscr{r}}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}$. qed

Now Eq. 10.47 says $\mathbf{A}(\mathbf{r},t)=\frac{\mathbf{v}}{c^2}V(\mathbf{r},t)$, so

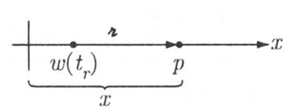
$$\begin{aligned}
\frac{\partial\mathbf{A}}{\partial t}&=\frac{1}{c^2}\left(\frac{\partial\mathbf{v}}{\partial t}V+\mathbf{v}\frac{\partial V}{\partial t}\right)=\frac{1}{c^2}\left(\frac{\partial\mathbf{v}}{\partial t_r}\frac{\partial t_r}{\partial t}V+\mathbf{v}\frac{\partial V}{\partial t}\right)\\
&=\frac{1}{c^2}\left[\mathbf{a}\frac{\partial t_r}{\partial t}\frac{1}{4\pi\epsilon_0}\frac{qc}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}+\mathbf{v}\frac{1}{4\pi\epsilon_0}\frac{-qc}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\frac{\partial}{\partial t}(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})\right]\\
&=\frac{1}{c^2}\frac{qc}{4\pi\epsilon_0}\left[\frac{\mathbf{a}}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}\frac{\partial t_r}{\partial t}-\frac{\mathbf{v}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\left(c\frac{\partial\mathscr{r}}{\partial t}-\frac{\partial\boldsymbol{\mathscr{r}}}{\partial t}\cdot\mathbf{v}-\boldsymbol{\mathscr{r}}\cdot\frac{\partial\mathbf{v}}{\partial t}\right)\right].
\end{aligned}$$

But $\mathscr{r}=c(t-t_r)\Rightarrow\frac{\partial\mathscr{r}}{\partial t}=c\left(1-\frac{\partial t_r}{\partial t}\right)$, $\boldsymbol{\mathscr{r}}=\mathbf{r}-\mathbf{w}(t_r)\Rightarrow\frac{\partial\boldsymbol{\mathscr{r}}}{\partial t}=-\mathbf{v}\frac{\partial t_r}{\partial t}$ (as above), and
$\frac{\partial\mathbf{v}}{\partial t}=\frac{\partial\mathbf{v}}{\partial t_r}\frac{\partial t_r}{\partial t}=\mathbf{a}\frac{\partial t_r}{\partial t}$.

$$\begin{aligned}
&=\frac{q}{4\pi\epsilon_0 c(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\left\{\mathbf{a}(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\frac{\partial t_r}{\partial t}-\mathbf{v}\left[c^2\left(1-\frac{\partial t_r}{\partial t}\right)+v^2\frac{\partial t_r}{\partial t}-\boldsymbol{\mathscr{r}}\cdot\mathbf{a}\frac{\partial t_r}{\partial t}\right]\right\}\\
&=\frac{q}{4\pi\epsilon_0 c(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\left\{-c^2\mathbf{v}+\left[(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a}+(c^2-v^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\right]\frac{\partial t_r}{\partial t}\right\}\\
&=\frac{q}{4\pi\epsilon_0 c(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\left\{-c^2\mathbf{v}+\left[(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a}+(c^2-v^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\right]\frac{c\mathscr{r}}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}\right\}\\
&=\frac{q}{4\pi\epsilon_0 c(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[-c^2\mathbf{v}(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})+c\mathscr{r}(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a}+c\mathscr{r}(c^2-v^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\right]\\
&=\frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})^3}\left[(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})\left(-\mathbf{v}+\frac{\mathscr{r}}{c}\mathbf{a}\right)+\frac{\mathscr{r}}{c}(c^2-v^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\right].\quad\text{qed}
\end{aligned}$$

Problem 10.20

$\mathbf{E}=\frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[(c^2-v^2)\mathbf{u}+\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})\right]$. Here
$\mathbf{v}=v\,\hat{\mathbf{x}}$, $\mathbf{a}=a\,\hat{\mathbf{x}}$, and, for points to the *right*, $\hat{\boldsymbol{\mathscr{r}}}=\hat{\mathbf{x}}$.
So $\mathbf{u}=(c-v)\,\hat{\mathbf{x}}$, $\mathbf{u}\times\mathbf{a}=\mathbf{0}$, and $\boldsymbol{\mathscr{r}}\cdot\mathbf{u}=\mathscr{r}(c-v)$.



$$\mathbf{E}=\frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{\mathscr{r}^3(c-v)^3}(c^2-v^2)(c-v)\,\hat{\mathbf{x}}=\frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\frac{(c+v)(c-v)^2}{(c-v)^3}\,\hat{\mathbf{x}}=\frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c+v}{c-v}\right)\hat{\mathbf{x}};$$

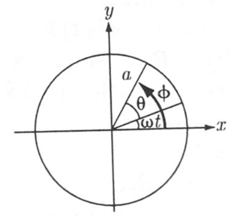
$$\mathbf{B}=\frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E}=\mathbf{0}.\quad\text{qed}$$

For field points to the *left*, $\hat{\boldsymbol{\mathscr{r}}}=-\hat{\mathbf{x}}$ and $\mathbf{u}=-(c+v)\,\hat{\mathbf{x}}$, so $\boldsymbol{\mathscr{r}}\cdot\mathbf{u}=\mathscr{r}(c+v)$, and

$$\mathbf{E}=-\frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{\mathscr{r}^3(c+v)^3}(c^2-v^2)(c+v)\,\hat{\mathbf{x}}=\boxed{\frac{-q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c-v}{c+v}\right)\hat{\mathbf{x}};\ \mathbf{B}=\mathbf{0}.}$$

Problem 10.24

$\lambda(\phi, t) = \lambda_0|\sin(\theta/2)|$, where $\theta = \phi - \omega t$. So the (retarded) scalar potential at the center is (Eq. 10.26)



$$\begin{aligned} V(t) &= \frac{1}{4\pi\epsilon_0}\int \frac{\lambda}{\mathscr{r}}\,dl' = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\lambda_0\,|\sin[(\phi-\omega t_r)/2]|}{a}a\,d\phi \\ &= \frac{\lambda_0}{4\pi\epsilon_0}\int_0^{2\pi}\sin(\theta/2)\,d\theta = \frac{\lambda_0}{4\pi\epsilon_0}\left[-2\cos(\theta/2)\right]\Big|_0^{2\pi} \\ &= \frac{\lambda_0}{4\pi\epsilon_0}\left[2-(-2)\right] = \boxed{\frac{\lambda_0}{\pi\epsilon_0}}. \end{aligned}$$

(Note: at fixed t_r, $d\phi = d\theta$, and it goes through one full cycle of ϕ *or* θ.)

Meanwhile $\mathbf{I}(\phi, t) = \lambda\mathbf{v} = \lambda_0\omega a\,|\sin[(\phi-\omega t)/2]|\;\hat{\boldsymbol{\phi}}$. From Eq. 10.26 (again)

$$\mathbf{A}(t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{I}}{\mathscr{r}}\,dl' = \frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{\lambda_0\omega a\,|\sin[(\phi-\omega t_r)/2]|\;\hat{\boldsymbol{\phi}}}{a}\,a\,d\phi.$$

But $t_r = t - a/c$ is again constant, for the ϕ integration, and $\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$.

$$= \frac{\mu_0\lambda_0\omega a}{4\pi}\int_0^{2\pi}|\sin[(\phi-\omega t_r)/2]|\,(-\sin\phi\,\hat{\mathbf{x}}+\cos\phi\,\hat{\mathbf{y}})\,d\phi.$$

Again, switch variables to $\theta = \phi - \omega t_r$, and integrate from $\theta = 0$ to $\theta = 2\pi$ (so we don't have to worry about the absolute value).

$$= \frac{\mu_0\lambda_0\omega a}{4\pi}\int_0^{2\pi}\sin(\theta/2)\left[-\sin(\theta+\omega t_r)\,\hat{\mathbf{x}}+\cos(\theta+\omega t_r)\,\hat{\mathbf{y}}\right]d\theta.$$

Now

$$\begin{aligned} \int_0^{2\pi}\sin(\theta/2)\sin(\theta+\omega t_r)\,d\theta &= \frac{1}{2}\int_0^{2\pi}\left[\cos(\theta/2+\omega t_r)-\cos(3\theta/2+\omega t_r)\right]d\theta \\ &= \frac{1}{2}\left[2\sin(\theta/2+\omega t_r)-\frac{2}{3}\sin(3\theta/2+\omega t_r)\right]\Big|_0^{2\pi} \\ &= \sin(\pi+\omega t_r)-\sin(\omega t_r)-\frac{1}{3}\sin(3\pi+\omega t_r)+\frac{1}{3}\sin(\omega t_r) \\ &= -2\sin(\omega t_r)+\frac{2}{3}\sin(\omega t_r) = -\frac{4}{3}\sin(\omega t_r). \\ \int_0^{2\pi}\sin(\theta/2)\cos(\theta+\omega t_r)\,d\theta &= \frac{1}{2}\int_0^{2\pi}\left[-\sin(\theta/2+\omega t_r)+\sin(3\theta/2+\omega t_r)\right]d\theta \\ &= \frac{1}{2}\left[2\cos(\theta/2+\omega t_r)-\frac{2}{3}\cos(3\theta/2+\omega t_r)\right]\Big|_0^{2\pi} \\ &= \cos(\pi+\omega t_r)-\cos(\omega t_r)-\frac{1}{3}\cos(3\pi+\omega t_r)+\frac{1}{3}\cos(\omega t_r) \\ &= -2\cos(\omega t_r)+\frac{2}{3}\cos(\omega t_r) = -\frac{4}{3}\cos(\omega t_r). \end{aligned}$$

So

$$\mathbf{A}(t) = \frac{\mu_0\lambda_0\omega a}{4\pi}\left(\frac{4}{3}\right)\left[\sin(\omega t_r)\,\hat{\mathbf{x}}-\cos(\omega t_r)\,\hat{\mathbf{y}}\right] = \boxed{\frac{\mu_0\lambda_0\omega a}{3\pi}\left\{\sin[\omega(t-a/c)]\,\hat{\mathbf{x}}-\cos[\omega(t-a/c)]\,\hat{\mathbf{y}}\right\}.}$$