Paraxial waves, Gaussian Beams

Notes for Phys 100c
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based on Photonics, Saleh Teich

$$\nabla^2 \bar{E} = \frac{1}{c^2}\partial_t^2 \bar{E} \qquad (i)$$

Let's consider one vector component of the electric field with complex amplitud $U(\bar{r},t) = U(\bar{r})e^{i\omega t}$ (ii) $\left(\text{e.g. } \bar{E} = U(\bar{r},t)\hat{n}\right)$

(ii) in (i) $\Rightarrow$ $\boxed{\nabla^2 U + k^2 U = 0}$ Helmholtz eqn
where k is defined as $k \equiv \frac{\omega}{c}$

Fresnel approx. of the spherical wave. The Paraboloidal wave.

spherical wave is soln. to Helmholtz eqn.

$$U(\bar{r}) = \frac{A_0}{r}e^{-ikr} \qquad (iii)$$

consider wave close to origin in x & y $\Rightarrow \sqrt{x^2+y^2} << z$

paraxial approx.: $\theta^2 = \frac{x^2+y^2}{z^2} << 1$

$$\Rightarrow r = \sqrt{x^2+y^2+z^2} = z\sqrt{1+\theta^2} = z\left(1+\frac{\theta^2}{2}-\frac{\theta^4}{8}+\cdots\right)$$

$$\simeq z\left(1+\frac{\theta^2}{2}\right) = z + \frac{x^2+y^2}{2z}$$

(iii) is then $U(\bar{r}) \simeq \frac{A_0}{z}e^{-ikz}e^{-ik\frac{x^2+y^2}{2z}}$

$= A(\bar{r})e^{-ikz}$ where $A(\bar{r}) = \frac{A_0}{r}e^{-ik\frac{x^2+y^2}{2z}}$

which satisfies wave eqn. too.

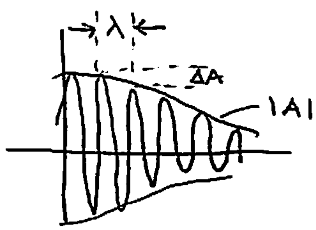
Paraxial waves

$$U(\bar{r},t) = A(\bar{r})e^{i(\omega t - kz)}, \quad U(\bar{r}) = A(\bar{r})e^{-ikz} \qquad (iv)$$

Paraxial wave envelope $A(\bar{r})$ must satisfy another condition from Helmholtz eqn.

$A(\bar{r})$ varies slowly wrt z means within a distance $\Delta z = \lambda$, $\Delta A << A$

slowly varying envelope approx.



$$\Delta A = \partial_z A \Delta z = \partial_z A\, \lambda$$

$$\Rightarrow \partial_z A << \frac{A}{\lambda} = \frac{Ak}{2\pi} \Rightarrow \partial_z A << kA \Rightarrow \partial_z^2 A << k^2 A$$

substituting (iv) in Helmholtz eqn.
& neglecting $\partial_z^2 A$ in comparison w/ $k\partial_z A$ or $k^2 A$

$$\Rightarrow \boxed{\nabla_T^2 A - i2k\partial_z A = 0} \quad \text{Paraxial Helmholtz eqn.}$$

Gaussian beams are a soln to this as well as Hermite-Gauss, Laguerre-Gauss, Bessel-, etc.

$$A(\bar{r}) = \frac{A_1}{q(z)}e^{-ik\frac{x^2+y^2}{2q(z)}} \quad \text{where} \quad q(z) = z + iz_0$$

z_0: constant. Depth of focus

or $\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}$

Beamwaist $w(z) = w_0\sqrt{1+\left(\frac{z}{z_0}\right)^2}$

Radius of curvature $R(z) = z\left(1+\left(\frac{z_0}{z}\right)^2\right)$

$\zeta(z) = \tan^{-1}\frac{z}{z_0}$ Guoy phase

$$\boxed{U(\bar{r}) = A_0\frac{w_0}{w(z)}e^{-\frac{\rho^2}{w^2(z)}}e^{-ikz-ik\frac{\rho^2}{2R(z)}+i\zeta(z)}}$$

$$\boxed{w_0^2 = \frac{\lambda z_0}{\pi}}$$

Gaussian beam determined by w_0 & λ