Chapter 34

Models of Antennas

34.1 Simplified Model

We have been discussing a small system, in which the time delay effects are not great. For an example of the opposite situation, let us consider an oversimplified model of an antenna. In this model we have a wire of length l, and of negligible cross section, carrying a current density flowing in the z direction:

$$J_z = I\delta(x)\delta(y)\sin\omega t, \quad -\frac{l}{2} \leq z \leq \frac{l}{2}, \tag{34.1}$$

which has the property

$$\boldsymbol{\nabla}\cdot\mathbf{J} = \frac{\partial}{\partial z}J_z = \begin{cases} 0 & \text{for} \quad -\frac{l}{2} < z < \frac{l}{2}, \\ \neq 0 & \text{for} \quad z = \pm\frac{l}{2}. \end{cases} \tag{34.2}$$

From the local charge conservation condition (31.5),

$$0 = \frac{\partial}{\partial t}\rho + \frac{\partial}{\partial z}J_z, \tag{34.3}$$

we see that (34.2) implies that there is an oscillating charge density at both ends of the antenna. In any realistic model, J_z will depend on z. We lack this dependence since our model assumes that the antenna is fed at every point along its length. Even though this model is oversimplified, it possesses many of the significant characteristics of a real antenna. To compute the power radiated, (32.18), we evaluate the integral

$$\begin{aligned}
&\int (d\mathbf{r}')\frac{1}{c}\frac{\partial}{\partial t}J_z\left(\mathbf{r}', t - \frac{r}{c} + \frac{1}{c}\mathbf{n}\cdot\mathbf{r}'\right) \\
&= \frac{\omega}{c}I\int_{-l/2}^{l/2} dz' \cos\omega\left(t - \frac{r}{c} + \frac{1}{c}z'\cos\theta\right) \\
&= \frac{\omega}{c}I\int_{-l/2}^{l/2} dz' \cos\omega\left(t - \frac{r}{c}\right)\cos\left(\frac{\omega z'}{c}\cos\theta\right)
\end{aligned}$$

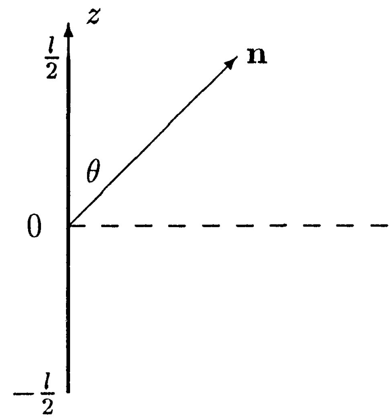

Figure 34.1: Geometry of linear antenna.

$$= \frac{2\frac{\omega}{c} I \cos\omega\left(t-\frac{r}{c}\right)\sin\left(\frac{\omega l}{2c}\cos\theta\right)}{\frac{\omega}{c}\cos\theta}, \tag{34.4}$$

where, as indicated in Fig. 34.1, θ denotes the angle between the direction of observation and the antenna. The angular distribution of the radiated power, at the observation time t, is then

$$\frac{dP(t)}{d\Omega} = \frac{1}{\pi c}\sin^2\theta \frac{I^2\cos^2\omega\left(t-\frac{r}{c}\right)\sin^2\left(\frac{\omega l}{2c}\cos\theta\right)}{\cos^2\theta}, \tag{34.5}$$

which, when averaged over one cycle of oscillation, becomes

$$\frac{dP}{d\Omega} = \frac{I^2}{2\pi c}\frac{\sin^2\theta\sin^2\left(\frac{\omega l}{2c}\cos\theta\right)}{\cos^2\theta}. \tag{34.6}$$

To rewrite (34.6) in terms of more convenient parameters, we recognize that, far from the antenna, the fields oscillate periodically both in space and in time:

$$E, H \sim \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\left(\omega t - \frac{\omega}{c}r\right) = \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\left(2\pi\nu t - \frac{2\pi\nu}{c}r\right), \tag{34.7}$$

so we identify the relation between the frequency ν (Hertz) and wavelength λ to be

$$\lambda\nu = c, \tag{34.8}$$

or

$$\frac{\omega}{c} = \frac{2\pi}{\lambda}. \tag{34.9}$$

Therefore the radiation of the system is characterized by two parameters: the length of the antenna, l, and the wavelength of the radiation, λ. The combination that appears in the expression for the power, (34.6), is

$$\frac{\omega}{c}\frac{l}{2} = \frac{\pi l}{\lambda}, \tag{34.10}$$

in terms of which the angular distribution is

$$\frac{dP}{d\Omega} = \frac{I^2}{2\pi c}\frac{\sin^2\theta\sin^2\left(\frac{\pi l}{\lambda}\cos\theta\right)}{\cos^2\theta}. \tag{34.11}$$

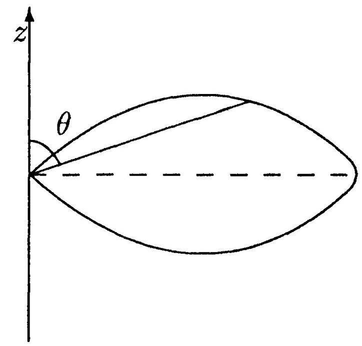

Figure 34.2: Radiation pattern produced by a short antenna, $l < \lambda$. In this and the following figures axial symmetry about the z-axis is to be understood.

In particular, in the direction perpendicular to the antenna, $\theta = \frac{\pi}{2}$, the radiated power is proportional to the square of the length of the antenna,

$$\left.\frac{dP}{d\Omega}\right|_{\theta=\pi/2} = \frac{I^2}{2\pi c}\left(\frac{\pi l}{\lambda}\right)^2 . \tag{34.12}$$

To appreciate the characteristic features of this radiation, we will consider the application of this general formula, (34.11), to three special circumstances:

1. $\lambda \gg l$.

   For a short antenna, $l \ll \lambda$, the approximation

   $$\frac{\sin^2\left(\frac{\pi l}{\lambda}\cos\theta\right)}{\cos^2\theta} \approx \left(\frac{\pi l}{\lambda}\right)^2 \tag{34.13}$$

   holds true for all angles. The resulting radiation pattern may be alternatively derived from the dipole radiation formula, (32.29), which is appropriate to a small system:

   $$\frac{dP}{d\Omega} \approx \frac{I^2}{2\pi c}\left(\frac{\pi l}{\lambda}\right)^2 \sin^2\theta. \tag{34.14}$$

2. $\lambda > l$.

   When $\frac{\pi l}{\lambda} < \pi$, the argument of the factor $\sin^2\left(\frac{\pi l}{\lambda}\cos\theta\right)$ goes from 0 to something less than π when the angle θ varies from $\frac{\pi}{2}$ to 0. Therefore, the only angles at which the power radiated vanishes are 0 and π, so the radiation pattern has a single lobe. (See Fig. 34.2.)

3. $\lambda < l < 2\lambda$.

   When $\pi < \frac{\pi l}{\lambda} < 2\pi$ there is an additional zero in the radiated power at the angle $\theta = \cos^{-1}\left(\frac{\lambda}{l}\right)$. Consequently the radiation pattern exhibits both a main lobe and two side lobes. See Fig. 34.3. Evidently, as l/λ increases, more and more side lobes appear.

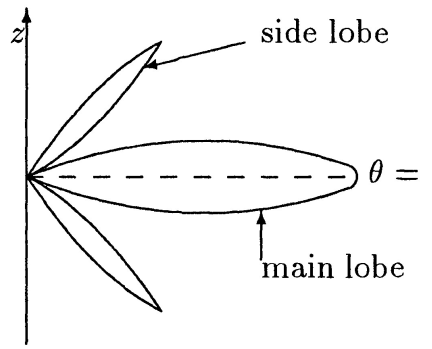

Figure 34.3: Radiation from an antenna of intermediate length, $\lambda < l < 2\lambda$. This diagram is meant to be understood schematically only. Actually, the side lobes are far smaller than indicated.

The total power radiated by this antenna may be obtained by integrating (34.11) over all angles:

$$\begin{aligned} P &= \int_0^\pi 2\pi \sin\theta \, d\theta \frac{I^2}{2\pi c} \frac{\sin^2\theta \sin^2\left(\frac{\pi l}{\lambda}\cos\theta\right)}{\cos^2\theta} \\ &= \frac{I^2}{c} \int_{-\pi/2}^{\pi/2} \cos\chi \, d\chi \cos^2\chi \frac{\sin^2\left(\frac{\pi l}{\lambda}\sin\chi\right)}{\sin^2\chi} \\ &= \frac{2I^2}{c}\left(\frac{\pi l}{\lambda}\right) \int_0^{\pi l/\lambda} dz \frac{\sin^2 z}{z^2}\left[1 - \frac{z^2}{(\pi l/\lambda)^2}\right], \end{aligned} \tag{34.15}$$

where we have made the successive changes of variables,

$$\chi = \frac{\pi}{2} - \theta, \quad z = \frac{\pi l}{\lambda}\sin\chi. \tag{34.16}$$

If $\frac{\pi l}{\lambda} \gg 1$, the second term in the square brackets in (34.15) is negligible compared with the first for the significant values of z, so we find

$$P \approx \frac{2I^2}{c}\frac{\pi l}{\lambda}\int_0^\infty dz \frac{\sin^2 z}{z^2} = \frac{\pi^2 I^2}{c}\frac{l}{\lambda}, \tag{34.17}$$

where we have used the integral

$$\int_0^\infty dz \frac{\sin^2 z}{z^2} = \int_0^\infty dz \frac{\sin 2z}{z} = \int_0^\infty dt \frac{\sin t}{t} = \frac{\pi}{2}. \tag{34.18}$$

Here we observe that the total radiated power, (34.17), increases linearly with l, while the power radiated in the direction perpendicular to the antenna, (34.12), is proportional to l^2; that is, as the length of the antenna increases, a larger and larger fraction of the radiated power is concentrated near $\theta = \pi/2$.

To see how much energy is radiated into a very small angular range near $\chi = 0$ (or $\theta = \pi/2$), we consider the power radiated into the main lobe by a long antenna

$$0 < \chi < \frac{\lambda}{l} \ll 1. \tag{34.19}$$

Following the same procedure used to obtain the total power radiated, (34.15), we find for the total power radiated into the main lobe

$$P_{\text{main lobe}} \approx \frac{2I^2}{c}\int_0^{\lambda/l} d\chi \frac{\sin^2\left(\frac{\pi l}{\lambda}\chi\right)}{\chi^2}$$
$$= \frac{2I^2}{c}\frac{\pi l}{\lambda}\int_0^{\pi} dz \frac{\sin^2 z}{z^2}. \tag{34.20}$$

The fraction of the energy radiated into the main lobe is obtained by taking the ratio of (34.20) to (34.17):

$$\frac{2}{\pi}\int_0^{\pi} dz\frac{\sin^2 z}{z^2} = \frac{2}{\pi}\int_0^{2\pi} dt\frac{\sin t}{t}$$
$$= \frac{2}{\pi}\int_0^{\infty} dt\frac{\sin t}{t} - \frac{2}{\pi}\int_{2\pi}^{\infty} dt\frac{\sin t}{t}$$
$$= 1 - \frac{1}{\pi^2} + \frac{1}{2\pi^4} - \ldots = 0.9028, \tag{34.21}$$

where the infinite series is derived by integrating by parts repeatedly. Over 90% of the power is radiated into the main lobe, which has angular width λ/l, implying that the radiation from the antenna is highly directional, a feature characteristic of large systems. In contrast, small systems, for which the dipole approximation is valid, typically have angular distributions proportional to $\sin^2\theta$.

34.2 Center-Fed Antenna

The previous model is greatly oversimplified, of course, and the results (see Problem 34.2) greatly de-emphasize the importance of radiation into the side lobes. Therefore, we turn to the consideration of what might seem to be a somewhat more realistic model, that of a *center fed, linear antenna*, described by

$$J_z = I\sin\left(\frac{kl}{2} - k|z|\right)\delta(x)\delta(y)\sin\omega t, \quad |z| < l/2, \tag{34.22}$$

where now we have introduced the wavenumber $k = \omega/c = 2\pi/\lambda$. Now, when we repeat the steps in (34.4) we find

$$\int (d\mathbf{r}')\frac{1}{c}\frac{\partial}{\partial t}J_z(\mathbf{r}', t - \frac{r}{c} + \frac{1}{c}\mathbf{n}\cdot\mathbf{r}')$$
$$= kI\int_{-l/2}^{l/2} dz'\sin\left(\frac{kl}{2} - k|z'|\right)\cos\omega\left(t - \frac{r}{c} + \frac{z'}{c}\cos\theta\right)$$
$$= 2kI\int_0^{l/2} dz'\sin\left(\frac{kl}{2} - kz'\right)\cos\left(kz'\cos\theta\right)\cos\omega\left(t - \frac{r}{c}\right)$$

$$= kI \int_0^{l/2} dz' \left[\sin k \left(z'(\cos\theta - 1) + \frac{l}{2} \right) - \sin k \left(z'(\cos\theta + 1) - \frac{l}{2} \right) \right]$$
$$\times \cos\omega \left(t - \frac{r}{c} \right)$$
$$= \frac{2I \cos\omega \left(t - \frac{r}{c} \right)}{\sin^2\theta} \left[\cos\left(\frac{kl}{2} \cos\theta \right) - \cos\frac{kl}{2} \right] . \tag{34.23}$$

The power radiated into a given solid angle is then given by (32.18), or

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c} \sin^2\theta \frac{4I^2 \cos^2\omega \left(t - \frac{r}{c} \right)}{\sin^4\theta} \left[\cos\left(\frac{kl}{2} \cos\theta \right) - \cos\frac{kl}{2} \right]^2 , \tag{34.24}$$

which, when averaged over one cycle, gives

$$\frac{dP}{d\Omega} = \frac{I^2}{2\pi c} \frac{1}{\sin^2\theta} \left[\cos\left(\frac{kl}{2} \cos\theta \right) - \cos\frac{kl}{2} \right]^2 . \tag{34.25}$$

Notice for a short antenna, $\lambda \gg l$, we recover the dipole formula:

$$\frac{dP}{d\Omega} = \frac{(I/2)^2}{2\pi c} \left(\frac{\pi l}{\lambda} \right)^4 \sin^2\theta, \tag{34.26}$$

which may alternatively be derived from the dipole radiation formula, (32.29). See Problem 34.1.

The angular distribution of the power radiated, $dP/d\Omega$, vanishes whenever

$$\cos\theta = \pm \left(1 - \frac{4\pi n}{kl} \right) , \quad n = 1, 2, \ldots . \tag{34.27}$$

Thus side lobes appear in the radiation pattern whenever kl passes through a multiple of 2π. Plots of $dP/d\Omega$ are given in Figures 34.4 and 34.5. It will be noted that now as $l/\lambda \to \infty$, substantial energy is radiated into the extreme side lobes, increasingly near the z axis, the antenna direction. See Problem 34.5. This is very similar to the radiation pattern produced by impulsive scattering—see Chapter 37. Therefore, such an antenna is most useful in the half- or full-wave regime.

34.3 Problems for Chapter 34.

1. Derive (34.14) and (34.26) from the dipole radiation formula, (32.29).

2. Using Mathematica, Maple, or any other computer program of your choice, plot accurately the radiation pattern produced by the simplified antenna for $l = \lambda/2$, $l = 3\lambda/2$, $l = 5\lambda$, and $l = 13\lambda/2$. Calculate the fraction of power in the main lobe and check how closely the limit (34.21) is approached.

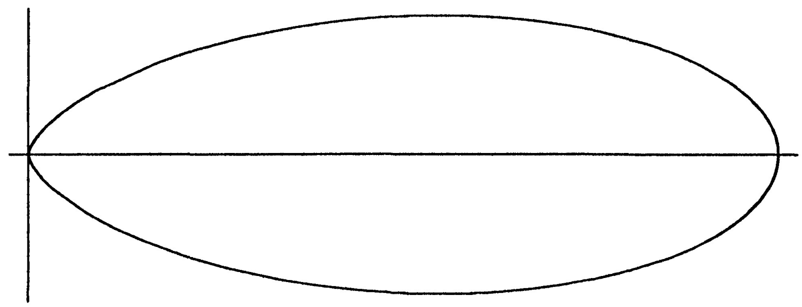

Figure 34.4: Radiation pattern produced by center-fed antenna for $kl = \pi$. This is called a half-wave antenna because $l = \lambda/2$.

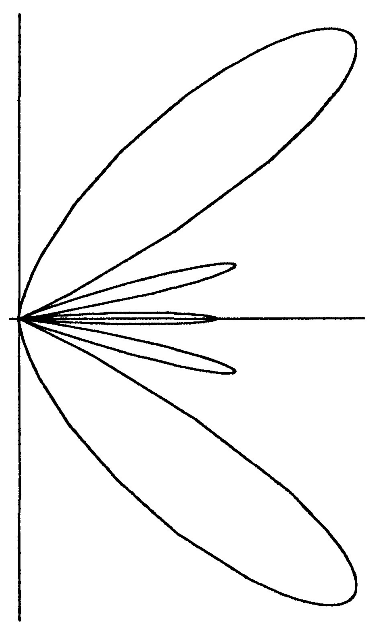

Figure 34.5: Radiation pattern produced by center-fed antenna for $kl = 5\pi$.

3. Verify the result (34.21) by computing a sufficient number of terms in the series, and by direct numerical integration. Give an error estimate for your answer.

4. Obtain formulas for the angular distribution of radiated power for a half-wave center fed antenna, $kl = \pi$, and for a full-wave antenna. Plot the latter, and compare with Figure 34.4.

5. Consider (34.25) for $kl \gg 1$. Compute the total power radiated in this limit, and compare with the power radiated in the two extreme side lobes, corresponding to $n = [l/\lambda]$ and $n = 1$ in (34.27), where $[x]$ denotes the largest integer less than or equal to x.

6. A straight wire of negligible thickness and infinite length carries a current that varies in time and with distance z along the wire, according to the relation
$$I(z,t) = I_0 \cos(\kappa z - \omega t).$$
Prove that this current does not radiate if the propagation constant κ exceeds k, the intrinsic wavenumber of the external medium, supposed to be uniform and infinitely extended. Verify that radiation does occur if $\kappa < k$, and that the time-averaged power radiated per unit length of the wire is given by
$$\frac{k\pi}{2c} I_0^2 \left(1 - \frac{\kappa^2}{k^2}\right).$$
Note that this agrees with (34.17) if $\kappa = 0$.