PHYSICS 100 C Lecture 1

Wave Eq^n: $\dfrac{\partial^2 f}{\partial z^2} = \dfrac{1}{v^2}\dfrac{\partial^2 f}{\partial t^2}$ (in one dimension - along z)

General Sol^n: $f = g(z - vt)$; g any function.

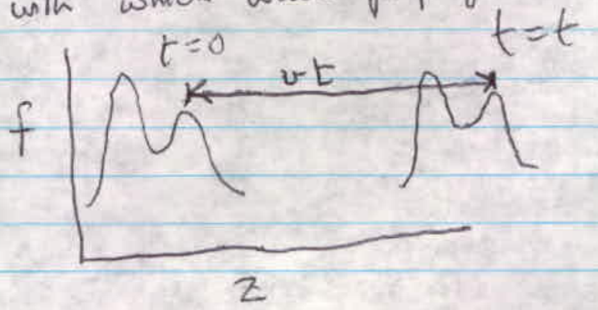
Proof: Let $u \equiv z - vt$

then $\dfrac{\partial f}{\partial z} = \dfrac{dg}{du}\dfrac{\partial u}{\partial z} = \dfrac{dg}{du}$; $\dfrac{\partial^2 f}{\partial z^2} = \dfrac{\partial}{\partial z}\left(\dfrac{dg}{du}\right) = \dfrac{d^2 g}{du^2}\left(\dfrac{\partial u}{\partial z}\right) = \dfrac{d^2 g}{dz^2}$

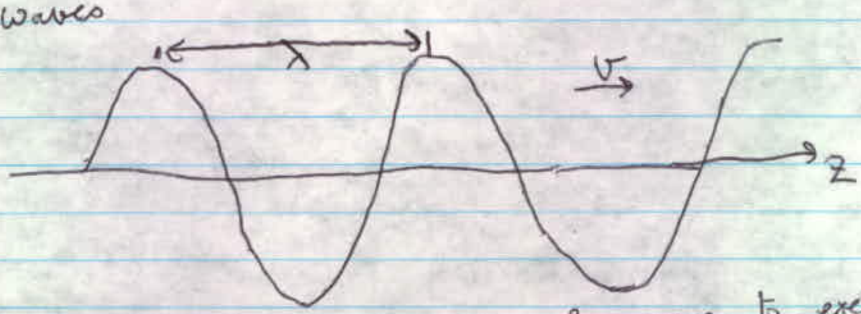
$\dfrac{\partial f}{\partial t} = \dfrac{dg}{du}\left(\dfrac{\partial u}{\partial t}\right) = -v\dfrac{dg}{du}$; $\dfrac{\partial^2 f}{\partial t^2} = \left(\dfrac{d^2 g}{du^2}\right)(-v)\left(\dfrac{\partial u}{\partial t}\right) = v^2\dfrac{d^2 g}{dz^2}$

$\therefore \dfrac{\partial^2 f}{\partial z^2} = \dfrac{1}{v^2}\dfrac{\partial^2 f}{\partial t^2}$

v is velocity with which wave propagates along z



Properties of waves



$k = \dfrac{2\pi}{\lambda}$ T = Period = Time taken for wave to execute 1 complete oscillation at a fixed point in z = $\dfrac{\lambda}{v}$

$\omega = \dfrac{2\pi}{T} = \dfrac{2\pi v}{\lambda} = kv$

Notation $\tilde{f}$ denotes a complex variable, whose real part is f.

$$\tilde{f} = \tilde{A} e^{i(kz-\omega t)} \longrightarrow \text{Complex representation of wave}$$

$\tilde{A}$ is a complex amplitude $= A e^{-i\delta}$ (A, δ are real)

$$\therefore \quad \tilde{f} = A e^{i(kz-\omega t-\delta)}$$

$$\therefore \quad f = \mathrm{Re}(\tilde{f}) = A\cos(kz-\omega t-\delta)$$

In 3D, polarization of wave means amplitude is a 3D vector, so is $\vec{k}$

Polarization can be longitudinal (~~Polarization~~ Amplitude $\parallel \vec{k}$)

or transverse (amplitude $\perp \vec{k}$)

or general.

Consider Maxwell's Equations in Free Space

$$\boxed{\begin{array}{ll} \nabla\cdot\vec{E} = 0 & \nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \nabla\cdot\vec{B} = 0 & \nabla\times\vec{B} = \mu_0\epsilon_0 \dfrac{\partial\vec{E}}{\partial t} \end{array}}$$

From second Eqn., $\nabla\times(\nabla\times\vec{E}) = \nabla(\underbrace{\cancel{\nabla\cdot\vec{E}}}_{\text{by 1st Eqn}}) - \nabla^2\vec{E} = -\dfrac{\partial}{\partial t}(\nabla\times\vec{B})$

$$= -\mu_0\epsilon_0 \frac{\partial^2\vec{E}}{\partial t^2}$$

from 4th Eqn.

$$\boxed{\begin{array}{l} \therefore \quad \nabla^2\vec{E} = \mu_0\epsilon_0 \dfrac{\partial^2\vec{E}}{\partial t^2} \\ \text{similarly } \nabla^2\vec{B} = \mu_0\epsilon_0 \dfrac{\partial^2\vec{B}}{\partial t^2} \end{array}}$$

$\therefore$ $\vec{E}$, $\vec{B}$ fields propagate as waves in free space with velocity

$c = \dfrac{1}{\sqrt{\mu_0\epsilon_0}}$ (velocity of light 3×10^8 m/s !!!)

Each component $E_x, E_y, E_z, B_x, B_y, B_z$ obeys this wave equation

Plane wave solutions $\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{i(kz-\omega t)}$

$\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_0 e^{i(kz-\omega t)}$

$\tilde{\vec{E}}_0, \tilde{\vec{B}}_0 \rightarrow$ complex vector amplitudes

These are solutions for plane waves propagating along z-direction.

For most general plane waves in 3D

$$\tilde{\vec{E}}(\vec{r},t) = \tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \tilde{\vec{B}}(\vec{r},t) = \tilde{\vec{B}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

For <u>plane waves</u> (only) ∇ ~~behaves~~ operator acts like a vector multiplier $i\vec{k}$

Thus $\nabla\times\tilde{\vec{E}} = i\vec{k}\times\tilde{\vec{E}} \qquad \nabla\cdot\tilde{\vec{E}} = i\vec{k}\cdot\tilde{\vec{E}}$

& similarly $\nabla\times\tilde{\vec{B}} = i\vec{k}\times\tilde{\vec{B}} \qquad \nabla\cdot\tilde{\vec{B}} = i\vec{k}\cdot\tilde{\vec{B}}$

& $\frac{\partial}{\partial t}$ operator acts like the scalar multiplier $-i\omega$

so $\frac{\partial\tilde{\vec{E}}}{\partial t} = -i\omega\tilde{\vec{E}} \qquad \frac{\partial\tilde{\vec{B}}}{\partial t} = -i\omega\tilde{\vec{B}}$

so for plane waves in free space, Maxwell's Equations become

$$\boxed{\begin{array}{ll} \vec{k}\cdot\tilde{\vec{E}} = 0 & \vec{k}\times\tilde{\vec{E}} = \omega\tilde{\vec{B}} \\ \vec{k}\cdot\tilde{\vec{B}} = 0 & \vec{k}\times\tilde{\vec{B}} = -\mu_0\epsilon_0\omega\tilde{\vec{E}} \end{array}}$$

By removing common factor $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, above Eqs are also true for $\tilde{\vec{E}}_0, \tilde{\vec{B}}_0$ (amplitudes)

so from left hand equations, $\tilde{\vec{E}}_0, \tilde{\vec{B}}_0$ are both $\perp$ to $\vec{k}$

since $\vec{k}$ is direction of propagation, $\vec{E}_0, \vec{B}_0$ are <u>transversely polarized</u>.

Both right hand equations give $\tilde{\vec{B}} = \frac{\vec{k}}{\omega}\times\tilde{\vec{E}}$

$$\rightarrow \boxed{\tilde{\vec{B}} = \frac{1}{c}\hat{k}\times\tilde{\vec{E}}}$$

$\vec{E}, \vec{B}, \vec{k}$ form a right-handed orthogonal set of vectors.

In <u>magnitudes</u>, $B = \frac{1}{c}E$

Poynting Vector $\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$

measures energy flow in direction of propagation in J/m²/sec

Consider expression for energy density in free space

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) \qquad \text{(E, B magnitudes of electric, magnetic fields)}$$

But $B = \frac{1}{c}E$

$$\therefore u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0 c^2} E^2\right) = \epsilon_0 E^2 \qquad \left(\text{since } c^2 = \frac{1}{\mu_0 \epsilon_0}\right)$$

(for wave propagating along z) $\quad = \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) \qquad$ if $\hat{E}_0 = E e^{i\delta}$

We have $\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) = \frac{1}{\mu_0}\frac{1}{\omega}(\vec{E} \times \vec{k} \times \vec{E}) \qquad \left[\because \vec{B} = \frac{\vec{k} \times \vec{E}}{\omega}\right]$

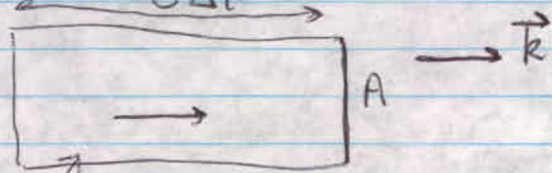
$$= \frac{1}{\mu_0 \omega}\left[\vec{k}(\vec{E}\cdot\vec{E}) - \vec{E}\,\underbrace{(\vec{E}\cdot\vec{k})}_{=0 \text{ since } \vec{E} \perp \vec{k}}\right]$$

$$= \frac{\vec{k}}{\mu_0 \omega} E^2 = \frac{\vec{k}}{\mu_0 \omega} E_0^2 \cos^2(kz - \omega t + \delta)$$

$$= \hat{k}\, c\, \epsilon_0 E^2 \cos^2(kz - \omega t + \delta)$$

$$= \hat{k}\, c\, u$$

$\therefore$ This proves that $\vec{S}$ represents energy passing through unit area normal to $\vec{k}$ per unit time



Total energy inside this volume goes through A in time Δt

But this energy $= u A (c\Delta t)$ so energy/unit area/time $= uc$

Momentum density in EM field $\quad \vec{g} = \frac{1}{c^2}\vec{S} \quad \therefore \vec{g} = \hat{k}\frac{1}{c}u$

averaging over time $\cos^2(\omega t - kz)$ yields factor $\frac{1}{2}$.

$$\therefore \langle \vec{S} \rangle = \frac{1}{2} c \epsilon_0 E_0^2 \hat{k}$$

$$\langle u \rangle = \frac{1}{2} \epsilon_0 E_0^2$$

$$\langle \vec{g} \rangle = \frac{1}{2c} \epsilon_0 E_0^2 \hat{k}$$

Intensity I of EM wave $= \langle S \rangle = \frac{1}{2} c \epsilon_0 E_0^2$

Pressure on absorbing medium normal to propagation

$=$ Momentum given up per unit area per unit time

$$\therefore \boxed{P = \frac{1}{2} \epsilon_0 E_0^2}$$

If surface is perfectly reflecting $\boxed{P = \epsilon_0 E^2}$ momentum change is twice as large

Lecture 2 100 C

EM Waves in Matter

$$\left| \begin{array}{ll} \nabla \cdot \vec{D} = 0 & \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{E} = 0 & \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \end{array} \right| \qquad \begin{array}{l} \vec{D} = \epsilon \vec{E} \\ \vec{H} = \frac{1}{\mu} \vec{B} \end{array}$$

(uniform, isotropic medium)

$$\therefore \boxed{\begin{array}{ll} \nabla \cdot \vec{E} = 0 & \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{B} = 0 & \vec{\nabla} \times \vec{B} = \mu \epsilon \frac{\partial \vec{E}}{\partial t} \end{array}}$$

gives wave Eqs. $\nabla^2 \vec{E} = \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$

$$\nabla^2 \vec{B} = \mu \epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

$\therefore$ velocity of propagation in medium $v = \frac{1}{\sqrt{\mu \epsilon}} = \frac{c}{n}$

where $n = \sqrt{\frac{\epsilon \mu}{\epsilon_0 \mu_0}}$. If $\mu \simeq \mu_0$, then $n \approx \sqrt{\epsilon_r}$ ϵ_r = dielectric constant

∴ so it follows that:

$$u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu} B^2\right)$$

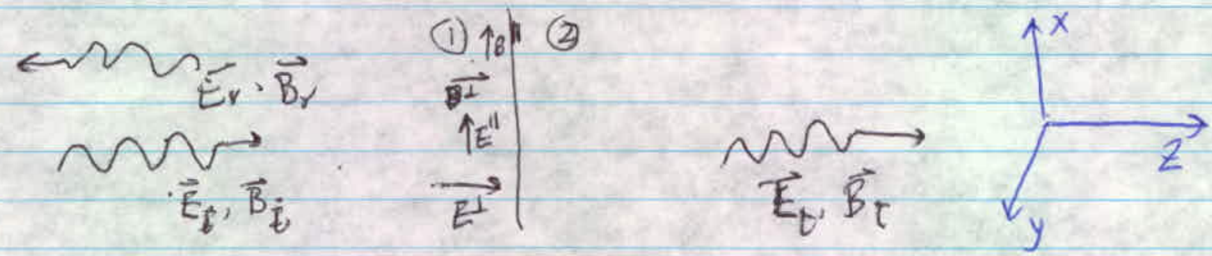
$$\vec{S} = \frac{1}{\mu}(\vec{E} \times \vec{B})$$

$$I = \frac{1}{2}\epsilon v E_0^2$$

Magnitude of $\vec{B}$, $B = \frac{1}{v} E$

Boundary Conditions at Interface between 2 media

If $E^{\perp}$, $B^{\perp}$ represent components of $\vec{E}, \vec{B}$ normal to interface
& $\vec{E}^{\parallel}$, $\vec{B}^{\parallel}$ " " " " " parallel " "



$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \qquad \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$$

$$B_1^{\perp} = B_2^{\perp} \qquad \frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}$$

Consider normal incidence: Fields are:

Incident
$$\begin{cases} \tilde{\vec{E}}_i(z,t) = \tilde{E}_{0,i}\, e^{i(k_1 z - \omega t)}\, \hat{x} \\ \tilde{\vec{B}}_i(z,t) = \frac{1}{v_1}\tilde{E}_{0i}\, e^{i(k_1 z - \omega t)}\, \hat{y} \end{cases}$$

Reflected
$$\begin{cases} \tilde{\vec{E}}_r(z,t) = \tilde{E}_{0r}\, e^{i(-k_1 z - \omega t)}\, \hat{x} \\ \tilde{\vec{B}}_r(z,t) = -\frac{1}{v_1}\tilde{E}_{0r}\, e^{i(-k_1 z - \omega t)}\, \hat{y} \end{cases}$$

Transmitted
$$\begin{cases} \tilde{\vec{E}}_t(z,t) = \tilde{E}_{0t}\, e^{i(k_2 z - \omega t)}\, \hat{x} \\ \tilde{\vec{B}}_t(z,t) = \frac{1}{v_2}\tilde{E}_{0t}\, e^{i(k_2 z - \omega t)}\, \hat{y} \end{cases}$$

applying Boundary Conditions, we obtain

$$\tilde{E}_{0i} + \tilde{E}_{0r} = \tilde{E}_{0t} \qquad ①$$

$$\frac{1}{\mu_1}\left(\frac{1}{v_1}\right)(\tilde{E}_{0i} - \tilde{E}_{0r}) = \frac{1}{\mu_2}\frac{1}{v_2}\tilde{E}_{0t} \qquad ②$$

Let $\tilde{r} = \frac{\tilde{E}_{0r}}{\tilde{E}_{0i}}$ + $\tilde{t} = \frac{\tilde{E}_{0t}}{\tilde{E}_{0i}}$

and $\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$

Solving Eqs. ① + ② above, we obtain

$$\tilde{r} = \frac{1-\beta}{1+\beta} \quad + \quad \tilde{t} = \frac{2}{1+\beta}$$

If $\mu_1 \simeq \mu_2 \simeq \mu_0$ (non-magnetic media), then $\beta \simeq \frac{v_1}{v_2} = \frac{n_2}{n_1}$ ③

$$\rightarrow \tilde{r} = \frac{v_2 - v_1}{v_2 + v_1} \qquad \tilde{t} = \frac{2v_2}{v_1 + v_2}$$

Reflected wave is <u>in phase</u> with incident wave if $v_2 > v_1$, out of phase if $v_1 < v_2$

Can also use ③ above to write

$$\tilde{r} = \frac{n_1 - n_2}{n_1 + n_2} \qquad \tilde{t} = \frac{2n_1}{n_1 + n_2}$$

Now intensities of 3 beams are given by

$$I_i = \tfrac{1}{2}\epsilon_1 v_1 E_{0i}^2 \; ; \; I_r = \tfrac{1}{2}\epsilon_1 v_1 E_{0r}^2 \; ; \; I_t = \tfrac{1}{2}\epsilon_2 v_2 E_{0t}^2$$

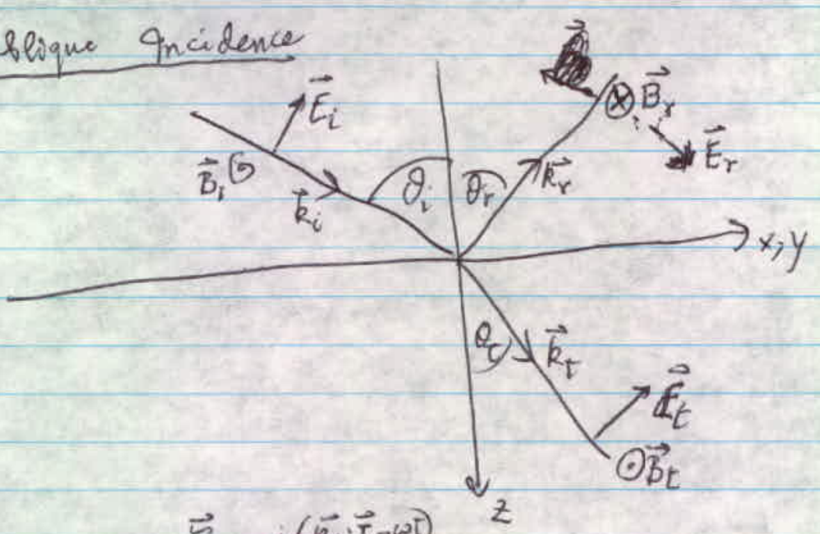
$\therefore$ <u>Reflectivity</u> $\equiv \frac{I_r}{I_i} = \left|\frac{E_{0r}}{E_{0i}}\right|^2 = |\tilde{r}|^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$

<u>Transmittivity</u> $\equiv \frac{I_t}{I_i} = \frac{\epsilon_2}{\epsilon_1}\frac{v_2}{v_1}|\tilde{t}|^2 = \frac{4n_1 n_2}{(n_1+n_2)^2}$

(using $\epsilon_1 \simeq n_1^2$; $\epsilon_2 \simeq n_2^2$ + $\frac{v_2}{v_1} = \frac{n_1}{n_2}$)

Easily show that $\boxed{R + T = 1}$ (Conservation of Energy)

Reflection at Oblique Incidence



Consider polarization ($\vec{E}$-fields) $\underline{\underline{\text{in}}}$ plane of incidence.

$$\vec{E}_i(\vec{r}, t) = \vec{E}_{0,i} e^{i(\vec{k}_i\cdot\vec{r}-\omega t)}$$
$$\vec{E}_r(\vec{r}, t) = \vec{E}_{0,r} e^{i(\vec{k}_r\cdot\vec{r}-\omega t)}$$
$$\vec{E}_t(\vec{r}, t) = \vec{E}_{0,t} e^{i(\vec{k}_t\cdot\vec{r}-\omega t)}$$

We have (From $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$)

$$\left.\begin{aligned} &E_{0,i}\cos(\vec{k}_i\cdot\vec{r}-\omega t)\cos\theta_i + E_{0,r}\cos(\vec{k}_r\cdot\vec{r}-\omega t)\cos\theta_r \\ &\qquad = E_{0,t}\cos(\vec{k}_t\cdot\vec{r}-\omega t)\cos\theta_t \end{aligned}\right\} \text{ at } z=0$$

If this is true for all $\vec{r}_\parallel$ (ie x, y) at $z=0$

then we have to have $\vec{k}_i^{\parallel} = \vec{k}_r^{\parallel} = \vec{k}_t^{\parallel}$

so $k_i \sin\theta_i = k_r \sin\theta_r$ But $k_i = k_r = \frac{\omega}{v_1}$, so $\boxed{\theta_i = \theta_r}$

also $k_i \sin\theta_i = k_t \sin\theta_t \rightarrow \boxed{\frac{\sin\theta_t}{\sin\theta_i} = \frac{k_i}{k_t} = \frac{\omega/v_1}{\omega/v_2} = \frac{n_1}{n_2}}$

ie Law of Reflection + Snell's Law of Refraction

so from above $\tilde{E}_{0i}\cos\theta_i + \tilde{E}_{0r}\cos\theta_r = \tilde{E}_{0t}\cos\theta_t$

Let $\alpha = \frac{\cos\theta_t}{\cos\theta_i} \longrightarrow \boxed{\tilde{E}_{0i} + \tilde{E}_{0r} = \alpha\tilde{E}_{0t}}$ ①

From $\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \longrightarrow \epsilon_1\left(-\tilde{E}_{0i}\sin\theta_i + \tilde{E}_{0,r}\sin\theta_r\right) = \epsilon_2\left(-\tilde{E}_{0,t}\sin\theta_t\right)$

$(\sin\theta_i = \sin\theta_r)$ This gives $-\tilde{E}_{0i} + \tilde{E}_{0,r} = -\frac{\epsilon_2}{\epsilon_1} \tilde{E}_{0,t} \frac{\sin\theta_t}{\sin\theta_i} = -\frac{\epsilon_2}{\epsilon_1}\frac{n_1}{n_2} \tilde{E}_{0t}$

$$\text{or} \quad \boxed{\tilde{E}_{0,i} - \tilde{E}_{0,r} = \beta \tilde{E}_{0,t}} \quad (2)$$

where $\beta = \frac{\epsilon_2}{\epsilon_1}\frac{n_1}{n_2} = \frac{\epsilon_2}{\epsilon_1}\sqrt{\frac{\epsilon_1\mu_1}{\epsilon_2\mu_2}} = \sqrt{\frac{\mu_1/\epsilon_1}{\mu_2/\epsilon_2}} = \frac{\mu_1}{\mu_2}\frac{n_2}{n_1}$

$$= \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}$$

Solving (1) + (2), we get $\boxed{\tilde{E}_{0,r} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{0i} \; ; \; \tilde{E}_{0t} = \frac{2}{\alpha+\beta}\tilde{E}_{0,i}}$

$\alpha = \frac{\cos\theta_t}{\cos\theta_i} = \frac{\sqrt{1-\sin^2\theta_t}}{\cos\theta_i} = \frac{\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_i\right)^2}}{\cos\theta_i}$ If $\theta_i \to 0$, $\alpha = 1$
$\theta_t = \theta_i = 0$

we get results of normal incidence.

If $\theta_i \to 90°$, $\alpha \to \infty$ $\quad R = 1$

If $\alpha = \beta$, $R = 0$ at θ_i = Brewster angle

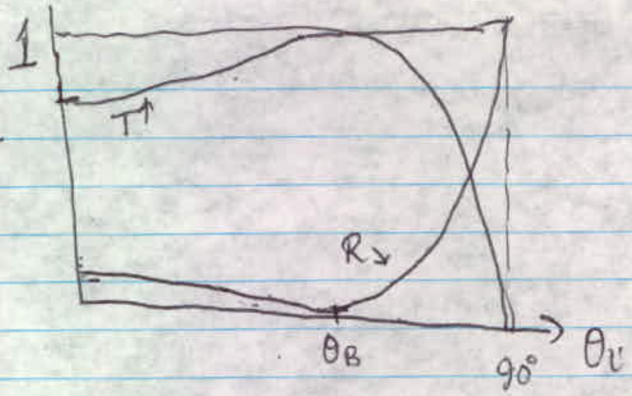
$$\frac{\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_B\right)^2}}{\cos\theta_B} = \beta$$

$$\sin^2\theta_B = \frac{1-\beta^2}{\left(\frac{n_1}{n_2}\right)^2 - \beta^2}$$

If $\mu_1 \simeq \mu_2$, $\beta \simeq \frac{n_2}{n_1} \longrightarrow \sin^2\theta_B \simeq \frac{1-\beta^2}{\left(\frac{n_1}{n_2}\right)^2-\beta^2} = \frac{1-\left(\frac{n_2}{n_1}\right)^2}{\left(\frac{n_1}{n_2}\right)^2-\left(\frac{n_2}{n_1}\right)^2}$

$$= \frac{n_2^2}{n_1^2+n_2^2} = \frac{\beta^2}{1+\beta^2}$$

$$\text{so} \quad \boxed{\tan\theta_B = \beta \simeq \frac{n_2}{n_1}}$$



$$R = \frac{I_R}{I_i} = \left|\frac{E_{0r}}{E_{0i}}\right|^2 = \left|\frac{\alpha-\beta}{\alpha+\beta}\right|^2$$

$$I_R = \tfrac{1}{2}\epsilon_1 v_1 E_{0r}^2 \cos\theta_r = \vec{S}\cdot\hat{z}$$

$$I_t = \tfrac{1}{2}\epsilon_2 v_2 E_{0t}^2 \cos\theta_t = \vec{S}\cdot\hat{z}$$

$$T = \frac{I_T}{I_i} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left|\frac{E_{0t}}{E_{0i}}\right|^2 \frac{\cos\theta_t}{\cos\theta_i}$$

$$= \alpha\beta \left|\frac{2}{\alpha+\beta}\right|^2$$

$\rightarrow R + T = 1$ still!

If $\sin\theta_i \frac{n_1}{n_2} > 1$, $\sin\theta_t > 1$ ie no solution for θ_t

$\longrightarrow$ TOTAL INTERNAL REFLECTION for $\theta_i > \theta_c$

Condition $\sin\theta_c = \frac{n_2}{n_1}$ or $\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$

only occurs for $n_2 < n_1$.

In this case α becomes imaginary $= i\alpha'$, say

$$R = \left|\frac{i\alpha' - \beta}{i\alpha' + \beta}\right|^2 = 1$$

$\vec{k}_T$ becomes purely imaginary, so wave goes as

$\tilde{E}_t(\vec{r},t) = \tilde{E}_{0t}\, e^{i(\vec{k}_\parallel\cdot\vec{r}_\parallel - \omega t)}\, e^{-\kappa z} \longrightarrow$ evanescent wave