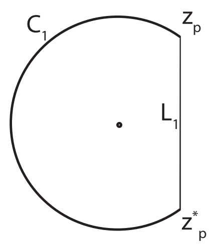
PHYS 201 Mathematical Physics, Fall 2016, Midterm

Due date: Thursday, November 17th, 2016

Rules: Open book and without help from another person.

1. In this problem, we will transform the truncated disk (as shown in the figure) into



the upper half-plane.

You may follow these steps (you can try out other ways as well):

i. Show that any bilinear mapping of the form $f(z) = \frac{z-z_1}{z-z_2}$, where $z_1, z_2 \equiv e^{i\theta_1}, e^{i\theta_2}$ lie on the unit circle, transforms a unit circle into a straight line passing through the origin. What angle does the transformed straight line make with the positive real axis?

ii. Apply the same transformation to the truncated disk shown in the figure such that $z_p \equiv e^{i\alpha}$ maps to the origin and its conjugate z_p^* maps to ∞. Show that the line L_1 and the arc C_1 both map to straight lines and find the angle between them. What region does the truncated disk map to?

iii. Finally, rotate the transformed version of the truncated disk as needed and apply another map to transform it into the upper half-plane.

2. A function $f(z)$ is analytic in the strip $|\text{Im } z| < \alpha$. Show that in the region $|\text{Im } z| < \beta$

$$f(z) = f_+(z) - f_-(z)$$

where

$$f_+ = \frac{1}{2\pi i}\int_{-\infty-i\beta}^{\infty-i\beta} \frac{f(t)}{t-z}dt, \qquad f_- = \frac{1}{2\pi i}\int_{-\infty+i\beta}^{\infty+i\beta} \frac{f(t)}{t-z}dt$$

with $0 < \beta < \alpha$ and $|f(x+iy)| < C|x|^{-p}, p > 0$ as $|x| \to \infty$. By using Morera's theorem, show that f_- is analytic for $\text{Im } z < \beta$ while f_+ is analytic for $\text{Im } z > -\beta$.

3. Use contour integration in the following problems:

i. Show that, for $a^2 > b^2 + c^2$,

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta + c\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2 - c^2}}$$

ii. For $-1 < \alpha < 3$, evaluate

$$\int_0^{\infty} \frac{x^\alpha}{(1+x^2)^2} dx$$

Explain why we need the conditions on α.

iii. *For $a > b > 0$, show that

$$\int_0^{2\pi} \ln(a + b\cos\theta) d\theta = 2\pi \ln\left(\frac{a + \sqrt{a^2 - b^2}}{2}\right)$$

(**Hint:** You may proceed using the usual way of solving trigonometric integrals.)
*The problem is challenging and is worth bonus points.