# PHYS 201 Mathematical Physics, Fall 2016, Final

## Due date: Monday, December 12th, 2016

**Rules: Open book and without help from another person.**

1. If $\alpha$ and $\beta$ are real and positive, show that (without necessarily computing the integrals)

$$\int_0^\infty \frac{\cos \alpha x}{x+\beta} dx = \int_0^\infty \frac{x e^{-\alpha\beta x}}{1+x^2} dx$$

2. The digamma function $\Psi(z) = \Gamma'(z)/\Gamma(z)$ has the integral representation

$$\Psi(z) = \ln z - 1/2z - \int_0^\infty \left[(e^t - 1)^{-1} - t^{-1} + \frac{1}{2}\right] e^{-tz} dt$$

Use this integral representation to generate the first two terms of the asymptotic expansion of $\Psi(z) - \ln z + 1/2z$ as $z \to \infty$.

3. Find the leading contribution as $x \to \infty$ to the integral $I(x) = \int_a^b f(t) e^{ix\psi(t)} dt$ from the neighborhood of the stationary point $a$ under the following assumptions: $\psi'(a) = \cdots = \psi^{(p-1)}(a) = 0$; $\psi^{(p)}(a) > 0$; $f(t) \sim A(t-a)^\alpha (t \to a+)$ with $\alpha > -1$. For what values of $\alpha$ and $p$ is the formula valid?

4. The function $D_\nu(x) = \frac{e^{x^2/4}}{i\sqrt{2\pi}} \int_C t^\nu e^{-xt+t^2/2} dt$, where $C$ is a contour connecting $-i\infty$ to $i\infty$ on which Re $t > 0$, satisfies the parabolic cylinder equation.

a. Show that the function satisfies the initial conditions $D_\nu(0) = \pi^{1/2} 2^{\nu/2} / \Gamma(\frac{1-\nu}{2})$ and $D'_\nu(0) = -\pi^{1/2} 2^{(v+1)/2} / \Gamma(-\nu/2)$. You will have to use a result derived in a previous HW that $\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a}$.

b. Use the method of steepest descents to show that $D_\nu(x) \sim x^\nu e^{-x^2/4}$ as $x \to \infty$.