Solutions - Midterm

1.) a.) $\nabla \cdot \underline{V}_{\Gamma} = 0$ from

$$\frac{\partial}{\partial q}\dot{q} + \frac{\partial}{\partial p}\dot{p} = \frac{\partial}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial}{\partial p}\frac{\partial H}{\partial q} = 0$$

Local $\rho(q, p, t)$ conserved

i.e. $$\frac{d\rho}{dt} = -\rho\, \cancel{\nabla \cdot \underline{V}_{\Gamma}} = 0$$

b.) $\partial L/\partial t = 0$
$\partial L/\partial \underline{x} = 0$
$\partial L/\partial \underline{\phi} = 0$

respectively. see L/L

i.e. LEOM $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \cancel{\frac{\partial L}{\partial x}}^{\,0}$$

$$\frac{d}{dt} P_x = 0$$

c.) $$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m}\nabla^2\psi + V\psi$$

$$\psi = A e^{iS/\hbar}$$

$$\Rightarrow -\frac{\partial S}{\partial t} = \frac{1}{2m}(\nabla S)^2 + V$$

Hamilton-Jacobi eqn.

d.) Virial thm seeks to relate E, $\langle T\rangle$, $\langle V\rangle$ by factors. Only possible if V homogeneous, so

$$\langle \underline{x}\cdot \partial V/\partial \underline{x}\rangle = \alpha \langle V\rangle .$$

see L/L

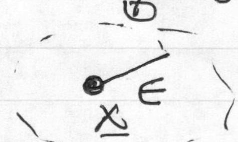
e.) $$p_1 = \frac{\partial L}{\partial \dot{x}_1} = \dot{x}_1 + \frac{\dot{x}_2}{2}$$

$$p_2 = \partial L/\partial \dot{x}_2 = \dot{x}_2 + \frac{\dot{x}_1}{2}$$

$$[\underline{p}] = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

<u>yes</u> $\det M \neq 0$.

f.) Trajectory fills surface.
Poincare recurrence thm - finite system
yes
see above. ϵ ball around $\underline{x}$ → arbitrarily small

Recurrence if  then if

Motion iterates ball, <u>some</u> point in ball will return within ϵ of $\underline{x}$, eventually.

g.) P-C : $\oint_{\text{exact}} p\cdot dq = I_{PC}$

Adiabatic Inv : $\oint_{E,\lambda} p\cdot dq = I$ at fixed E, λ

h.) No.

Attractor is sink $\rightarrow$ $\underline{\nabla} \cdot \underline{V} \neq 0$ at sink.

i.) $$\tau = \int_{x_1, y_1}^{x_2, y_2} d\ell \, n(y)$$

$$= \int_{x_1, y_1}^{x_2, y_2} dx \, \underbrace{\left((1 + y'^2)^{1/2} n(y) \right)}_{L}$$

$$\delta \tau = 0 \Rightarrow \frac{d}{dx}\left(\frac{\partial L}{\partial y'} \right) - \frac{\partial L}{\partial y} = 0$$

etc.

j.) holonomic $\rightarrow$ can express as

$$F(\underline{x}) = 0$$

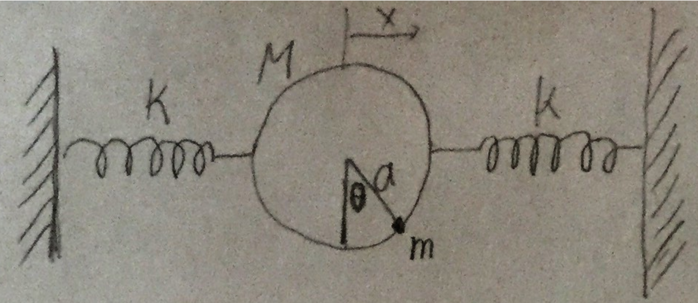
so

$$L \rightarrow L + \lambda F(\underline{x})$$

i.e. ball on circular rim

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy + \lambda (x^2 + y^2 - R^2)$$

2.



(a)

$$V = kx^2 + mga(1-\cos\theta)$$

$$K = K_M + K_m = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m[\underbrace{(\dot{x} + a\dot{\theta}\cos\theta)^2}_{\text{horizontal}} + \underbrace{a^2\dot{\theta}^2\sin^2\theta}_{\text{vertical}}]$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + a^2\dot{\theta}^2 + 2a\cos\theta\,\dot{\theta}\dot{x})$$

$$\boxed{\mathcal{L} = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ma^2\dot{\theta}^2 + ma\cos\theta\,\dot{\theta}\dot{x} - kx^2 - mga(1-\cos\theta)}$$

(b) $x|$ $\frac{\partial \mathcal{L}}{\partial x} = -2kx$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = (M+m)\dot{x} + ma\cos\theta\,\dot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0 \Rightarrow -2kx - (m+M)\ddot{x} - ma\cos\theta\,\ddot{\theta} + ma\sin\theta\,\dot{\theta}^2 = 0$$

$$\Rightarrow \boxed{(m+M)\ddot{x} + ma\cos\theta\,\ddot{\theta} = -2kx + ma\sin\theta\,\dot{\theta}^2} \quad \text{—— ①}$$

$\theta|$ $\frac{\partial \mathcal{L}}{\partial \theta} = -ma\sin\theta\,\dot{x}\dot{\theta} - mga\sin\theta$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ma^2\dot{\theta} + ma\cos\theta\,\dot{x}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0 \Rightarrow -ma\sin\theta\,\dot{x}\dot{\theta} - mga\sin\theta - ma^2\ddot{\theta} + ma\dot{x}\sin\theta - ma\cos\theta\,\ddot{x} = 0$$

$$\Rightarrow \boxed{ma\cos\theta\,\ddot{x} + ma^2\ddot{\theta} = -mga\sin\theta} \quad \text{—— ②}$$

By setting $\ddot{x} = \ddot{\theta} = \dot{x} = \dot{\theta} = 0$, one finds the equilibrium points

$(x, \theta) = (0, 0)$ and $(x, \theta) = (0, \pi)$

We can linearize EOM near $\theta=0$ and $\theta=\pi$.

Let $\theta = 0 + \varepsilon$

$$\text{EOM} \Rightarrow \begin{cases} (m+M)\ddot{x} + ma\ddot{\varepsilon} = -2kx \\ ma\ddot{x} + ma^2\ddot{\varepsilon} = -mga\,\varepsilon \end{cases} \Rightarrow \boxed{\begin{cases} (1+\gamma)\ddot{x} + \gamma a\ddot{\varepsilon} = -\Omega^2 x \\ \ddot{x} + a\Delta\ddot{\theta} = -\nu^2 a\varepsilon \end{cases}} \text{ — ③}$$

$$\gamma = \frac{m}{M},\ \Omega^2 = \frac{2k}{M},\ \nu^2 = \frac{g}{a}$$

Let $\theta = \pi + \Delta\theta$

$$\text{EOM} \Rightarrow \begin{cases} (m+M)\ddot{x} - ma\ddot{\varepsilon} = -2kx \\ -ma\ddot{x} + ma^2\ddot{\varepsilon} = -mga\,\varepsilon \end{cases} \Rightarrow \boxed{\begin{cases} (1+\gamma)\ddot{x} - \gamma a\ddot{\varepsilon} = -\Omega^2 x \\ -\ddot{x} - a\ddot{\varepsilon} = -\nu^2 a\varepsilon \end{cases}} \text{ — ④}$$

(c) Let's set $S = a\varepsilon$. We can rewrite ③ and ④ in the matrix form:

$$M_\pm \begin{pmatrix} \ddot{x} \\ \ddot{s} \end{pmatrix} = \begin{pmatrix} -\Omega^2 & 0 \\ 0 & -\nu^2 \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix}, \quad M_\pm = \begin{pmatrix} 1+\gamma & \pm\gamma \\ 1 & \pm 1 \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{pmatrix} \ddot{x} \\ \ddot{s} \end{pmatrix} = -A_\pm \begin{pmatrix} x \\ s \end{pmatrix}}, \quad A_\pm = M_\pm^{-1} \cdot \begin{pmatrix} \Omega^2 & 0 \\ 0 & \nu^2 \end{pmatrix} = \begin{pmatrix} \Omega^2 & \mp\gamma\nu^2 \\ \mp\Omega^2 & \pm(1+\gamma)\nu^2 \end{pmatrix}$$

⑤

The eigenvalues of $A_\pm$ are the squares of the eigen-frequencies.

For A_+, $$\omega_+^2 = \frac{\Omega^2 + (1+\gamma)\nu^2 \pm \sqrt{[\Omega^2 + (1+\gamma)\nu^2]^2 - 4\Omega^2\nu^2}}{2} \text{ — ⑥}$$

For A_-, $$\omega_-^2 = \frac{\Omega^2 - (1+\gamma)\nu^2 \pm \sqrt{[\Omega^2 - (1+\gamma)\nu^2]^2 + 4\Omega^2\nu^2}}{2} \text{ — ⑦}$$

Eq ⑥ is for $\theta \simeq 0$. The RHS is always positive, so there are always two oscillation modes.

Eq ⑦ is for $\theta \simeq \pi$. The RHS is positive if $\Omega^2 > (1+\gamma)\nu^2$. Only one possible mode exists with $\omega_- = \left[\frac{\Omega^2 - (1+\gamma)\nu^2 + \sqrt{[\Omega^2 - (1+\gamma)\nu^2]^2 + 4\Omega^2\nu^2}}{2}\right]^{\frac{1}{2}}$.

3.(a) Let the phase be $\Phi = \int \underline{k}\cdot d\underline{x} - \omega dt$.

The stationarity of Φ gives the ray equation

$$\delta \Phi = 0$$

$$\Rightarrow \delta \int [\underline{k}\cdot d\underline{x} - \omega dt]$$

$$= \delta \int [\underline{k}\cdot \dot{\underline{x}} - \omega] dt$$

$$= \int [\delta \underline{k}\cdot \dot{\underline{x}} + \underline{k}\cdot \delta\dot{\underline{x}} - \frac{\partial \omega}{\partial \underline{x}}\cdot \delta \underline{x} - \frac{\partial \omega}{\partial \underline{k}}\cdot \delta k] dt = 0$$

Using $\delta \dot{x} = \frac{d}{dt}\delta \underline{x}$ and integration by part

$$\delta \Phi = \cancel{k\cdot \delta x \Big|_{t_1}^{t_2}} + \int [\delta \underline{k}\cdot \dot{x} - \frac{dk}{dt}\cdot \delta x - \frac{\partial \omega}{\partial x}\delta x - \frac{\partial \omega}{\partial k}\delta k] = 0$$

$$\Rightarrow \begin{cases} \dfrac{d\underline{x}}{dt} = \dfrac{\partial \omega}{\partial \underline{k}} \\ \dfrac{d\underline{k}}{dt} = -\dfrac{\partial \omega}{\partial \underline{x}} \end{cases}$$

(b) For acoustic waves, $\omega^2 = c_s^2 k^2$

$$\Rightarrow \omega \partial \omega = c_s^2 \underline{k}\cdot \partial \underline{k}$$

$$\Rightarrow \partial \omega = \hat{k}\cdot \partial \underline{k}\, c_s(\underline{x}) \Rightarrow \frac{\partial \omega}{\partial \underline{k}} = c_s(\underline{x})\hat{k}$$

$$\frac{\partial \omega}{\partial \underline{x}} = \frac{\partial}{\partial x}[c_s^2(x)\underline{k}^2]^{\frac{1}{2}} = k\frac{\partial c(\underline{x})}{\partial \underline{x}}, \text{ so } \boxed{\begin{aligned} \frac{d\underline{x}}{dt} &= c_s(x)\hat{k} \\ \frac{d\underline{k}}{dt} &= -k\frac{\partial c_s(\underline{x})}{\partial \underline{x}} \end{aligned}}$$

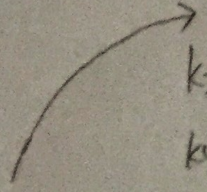
If $c_s(\underline{x}) = c_s(y)$

$$\frac{dk_x}{dt} = 0 \Rightarrow k_x = \text{constant}$$

$$\frac{dk_y}{dt} = -\sqrt{k_x^2 + k_y^2}\,\frac{d c_s(y)}{dy}$$

If $\frac{d(c_s(y))}{dy} > 0$, k_y decreases with time

$\uparrow \hat{y}$

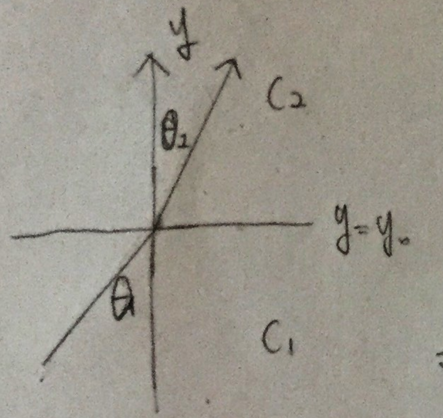


k_x conserves while k_y decreases.

If $\frac{d(c_s(y))}{dy} < 0$, k_y increases with time

$\uparrow \hat{y}$

k_x conserves while k_y increases



$C_1 > C_2$

$k_{x_1} = k_{x_2} \Rightarrow k_1 \sin\theta_1 = k_2 \sin\theta_2$

$\Rightarrow \frac{w}{C_1}\sin\theta_1 = \frac{w}{C_2}\sin\theta_2 \Rightarrow \frac{\sin\theta_1}{C_1} = \frac{\sin\theta_2}{C_2}$

(c) $\nabla^2 \psi + \frac{w^2}{c^2}\psi = 0 \leftarrow \psi = Ae^{i\frac{\phi}{\epsilon}}$

$\Rightarrow \left[-\left(\nabla\frac{\phi}{\epsilon}\right)^2 A + i\frac{\nabla^2\phi}{\epsilon}A + 2i\frac{\nabla A\cdot\nabla\phi}{\epsilon} + \nabla^2 A\right]e^{i\phi} = -\frac{w^2}{c_s^2(x)}Ae^{i\phi}$

$\epsilon \to 0 \Rightarrow \left(\nabla\frac{\phi}{\epsilon}\right)^2 = \frac{w^2}{c_s^2} \xrightarrow{\text{Absorb } \epsilon \text{ to } \phi} (\nabla\phi)^2 = \frac{w^2}{c_s^2} \Rightarrow \boxed{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 = \frac{w^2}{c_s^2(x,y)}}$

(d) From 2D eikonal equation

$\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 = \frac{w^2}{c_s^2(x,y)}$,

by observing, if $\boxed{\frac{1}{c_s^2(x,y)} = \frac{1}{c_1^2(x)} + \frac{1}{c_2^2(y)}}$ then

$\left[\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{w^2}{c_1^2(x)}\right] + \left[\left(\frac{\partial\phi}{\partial y}\right)^2 - \frac{w^2}{c_2^2(y)}\right] = 0$ is separable.

Let $\phi = \phi_1(x) + \phi_2(y)$, we obtain two 1st order ODEs $\begin{cases} \left(\frac{\partial\phi_1(x)}{\partial x}\right)^2 = \frac{w^2}{c_1^2(x)} \\ \left(\frac{\partial\phi_2(y)}{\partial y}\right)^2 = \frac{w^2}{c_2^2(X)} \end{cases}$

4.) a.) $\dot{k}/k < (k/m)^{1/2}$

b.) See WKB analysis, Class Notes on Adiabatic Invariants, pgs 4-7. With $\omega \equiv k(\epsilon t)/m$, problem is same.

c.) $I \sim E/\omega \sim \text{const}$

$$2 \cdot \tfrac{1}{2} k x^2 / \sqrt{k/m} \sim \text{const}$$

so

$$x^2 \sim 1/\sqrt{k}$$

$$x \sim k^{-1/4}$$

As k decreases, x increases $\sim k^{-1/4}$