→ Sound and Acoustic Emission

a.) Review

Now, consider

- compressible potential flow

$$\underline{V} = \nabla\phi, \quad \nabla\cdot\underline{V} \neq 0$$

- adiabatic eqn. state

$$\frac{\partial\rho}{\partial t} + \underline{V}\cdot\nabla\rho = -\rho\,\nabla\cdot\underline{V}$$

$$\rho\left(\frac{\partial\underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right) = -\nabla p$$

linearizing: $\rho = \rho_0 + \tilde{\rho}$

$\underline{V} = \cancel{\underline{V}_0} + \tilde{\underline{V}}$

$$\frac{\partial\tilde{\rho}}{\partial t} = -\rho_0\,\nabla\cdot\tilde{\underline{V}}$$

$$\rho_0\frac{\partial\tilde{\underline{V}}}{\partial t} = -\nabla\tilde{p}$$

$$\tilde{p} = \left(\frac{\partial p}{\partial\rho}\right)_s \tilde{\rho}$$

adiabatic

$$= c_s^2\,\tilde{\rho}$$

$\underline{V} = \nabla\tilde{\phi} \Rightarrow$

$$\tilde{p} = -\rho_0\frac{\partial\tilde{\phi}}{\partial t}$$

relates ϕ, p.

$$p = p_0\,(\rho/\rho_0)^\gamma$$

$$\therefore\ c_s^2 = \gamma p/\rho$$

$$-i\omega\tilde{\rho} = \rho_0 k^2 \tilde{\phi}$$
$$+i\omega\rho_0\tilde{\phi} = c_s^2\tilde{\rho}$$
$$\Rightarrow \omega^2 = k^2 c_s^2$$

sound / acoustic wave.

or
$$\frac{\partial\tilde{\rho}}{\partial t} = -\rho_0\nabla^2\tilde{\phi}$$
$$-\frac{\partial\tilde{\phi}}{\partial t} = c_s^2\tilde{\rho}$$
$$\Rightarrow \nabla^2\tilde{\phi} - \frac{1}{c_s^2}\frac{\partial^2\tilde{\phi}}{\partial t^2} = 0$$

wave eqn.

Note: - sound is longitudinal wave $\Leftrightarrow$ contrast string
- c_s is sound speed.
- $\underline{k} = \frac{\omega}{c_s}\underbrace{\hat{n}}_{\text{unit vector in direction propagation}}$

Also, recall energy / Poynting thm for acoustic waves, i.e.

$$\frac{\partial \mathcal{E}}{\partial t} + \underline{\nabla}\cdot\underline{S} = 0 \quad , \quad \text{in absence} \begin{cases}\text{sources}\\ \text{dissipation}\end{cases}$$

$\mathcal{E} \equiv$ energy density

$$\mathcal{E} = \frac{1}{2}\rho_0 \tilde{V}^2 + \frac{1}{2} c_s^2 \rho_0 \left(\tilde{\rho}/\rho_0\right)^2$$

Kinetic (mechanical) — potential (internal)

$\underline{S} \equiv$ wave energy density flux

$$\underline{S} = c_s^2 \tilde{\rho}\, \underline{\tilde{V}}$$

$$\rho_0 + i\omega \tilde{\phi} = c_s^2 \tilde{\rho}$$

$$\underline{\tilde{V}} = \tilde{v}\,\hat{n} = i\underline{k}\,\tilde{\phi}$$

$$\therefore\ \underline{S} = \cancel{c_s^2}\,\frac{-i\omega\tilde{\phi}}{\cancel{c_s^2}}\, i\underline{k}\,\tilde{\phi}\,\rho_0$$

$$= \rho_0 k^2 |\tilde{\phi}|^2 c_s\, \vec{n}$$

$$\boxed{\underline{S} = c_s\, \mathcal{E}\, \vec{n}}$$

n.b. $\underline{S} = \underline{v}_{gr}\, \mathcal{E}$, in general.

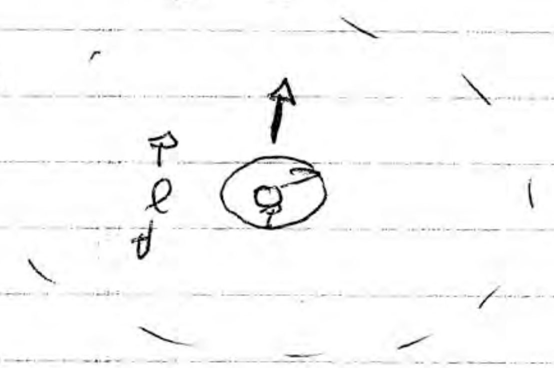
n.b. in eikonal theory, $\hat{n} = \dfrac{\underline{\nabla}\psi}{|\underline{\nabla}\psi|}$

$\phi = \phi_0 e^{i\psi}$, etc.

b.) Acoustic Radiation

Radiation ⟶ How is sound generated?

i.e.



i.e. relate far-field
$r \gg \ell, \lambda$
intensity, etc. to
source structure
and dynamics.

so, first

i.) Spherical Waves.

- when sound field depends only on distance from some point ⇒ spherical symmetry
- sound wave ⟶ spherical wave

i.e. if $\phi = \phi(r, t)$, then

$$\frac{\partial^2 \phi}{\partial t^2} = c_s^2 \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r} \right)$$

suggests immediately $\phi = \dfrac{f(r,t)}{r}$

$$\frac{\partial^2\phi}{\partial t^2} = c_s^2 \, \frac{1}{r^2}\frac{\partial}{\partial r}\left\{ r^2\left(-\frac{f}{r^2}+\frac{f'}{r}\right)\right\}$$

$$= c_s^2 \, \frac{1}{r^2}\frac{\partial}{\partial r}\left\{-f + r f'\right\}$$

$$\frac{1}{r}\frac{\partial^2 f}{\partial t^2} = c_s^2 \, \frac{1}{r^2}\left\{\cancel{-f'}+\cancel{f'}+r f''\right\}$$

$$\frac{\partial^2 f}{\partial t^2} = c_s^2 \, \frac{\partial^2 f}{\partial r^2}$$

$$f = f_1(ct-r) + f_2(ct+r)$$

or

$$\boxed{\phi = \frac{f_1(ct-r)}{r} + \frac{f_2(ct+r)}{r}}$$

general spherical wave soln.

outgoing
(causal)
relevent to radiation

incoming
(almost always non-causal
→ exception: bounded
system)

For spherical wave with $kr >> 1$;
(far field radiation)

$$\underline{V} = V\hat{r} = -\frac{f_1'}{r} \qquad \boxed{f' = -c_s \dot{f}}$$

$$\mathcal{E} = \rho_0 \frac{f_1'^2}{r^2}, \qquad \underline{S} = \hat{r}\, c_s\, \rho_0 \frac{f_1'^2}{r^2}$$

and $r^2 \underline{S} = \hat{r}\, c_s \rho_0 f_1'^2$

- intensity $\sim 1/r^2$ so flux thru surface at r ~ const.

- derivative terms $\} \rightarrow$ flux decreases in far field.

- if no source, ϕ must be finite at $r = 0$

$$\phi = \frac{f_1(ct-r)}{r} + \frac{f_2(ct+r)}{r}$$

$$= \frac{f(ct-r)}{r} - \frac{f(ct+r)}{r}$$

- outgoing, monochromatic wave

$$\phi = \frac{A}{r} e^{i(kr-\omega t)}$$

- standing monochromatic wave

$$\phi = A \frac{\sin kr}{r} e^{-i\omega t}$$

ii.) Sound Emission - I

- am concerned with acoustic emission from moving body
  → deformation → compression
  displacement

now, acoustics ↔ potential flow :

$$\nabla^2 \phi = +\frac{1}{c_s^2} \frac{\partial^2 \phi}{\partial t^2} \quad \rightarrow \text{eqn.}$$

$$u = \frac{\partial \phi}{\partial n} \quad \rightarrow \text{b.c.}$$

body velocity

→ fluid motion normal to body

- in all cases: $|\underline{V}| \ll c_s$

$$|\underline{V}| \sim \omega \underbrace{a}_{\text{displacement}} \Rightarrow \omega a \ll c_s$$

$$a \ll \underbrace{\lambda}_{\text{wavelength}}$$

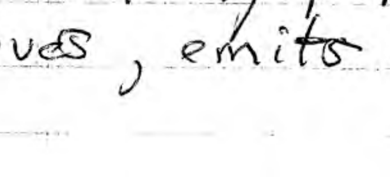
- can consider 2 cases:

1.) $\lambda \ll \ell \Leftrightarrow$ body size (high freq./short wavelength)

2.) $\lambda \gg \ell$ (low freq./long wavelength)

1a) $\lambda \ll \ell$ (easy)

- treat body as array of planar facets, each $\ell' > \lambda$



- body motion/pulsation → each facet plate moves, emits sound.

so, each facet emits <u>plane</u> wave. Thus, for facet

$$\underline{S} = c_s \rho V_n^2 \hat{n} = c_s \rho u_n^2 \hat{n} \qquad \boxed{I = P_{rad}}$$

$$\therefore \quad I = \int d\underline{a} \cdot \underline{S} \quad \rightarrow \text{ emitted intensity / radiated power (Watts)}$$

$$I = \int da \; \hat{n} \cdot \underline{S} = \int da \; c_s \rho \underbrace{V_n^2}_{u_n^2}$$

$$\therefore \quad \boxed{I = \int da \; c_s \rho \underbrace{V_n^2}_{u_n^2}} \qquad \text{(Radiated Power)}$$

n.b. for I;

$$\frac{\partial \mathcal{E}}{\partial t} + \underline{\nabla} \cdot \underline{S} = 0$$

$$\int d^3x \, \frac{\partial \mathcal{E}}{\partial t} + \int d^3x \, \underline{\nabla} \cdot \underline{S} = 0$$

$$\underbrace{\frac{\partial E}{\partial t}}_{\hookrightarrow I} = - \int d^3x \, \underline{\nabla} \cdot \underline{S} = + \int_{\text{within}} d\underline{a} \cdot \underline{S}$$

Now,

2) $\ell \ll \lambda$ (non-trivial)

for emission at frequency ω:

$$\underbrace{\nabla^2\phi}_{\sim 1/\ell^2 \text{ (near body)}} + \underbrace{\frac{\omega^2}{c_s^2}}_{1/\lambda^2}\phi = 0$$

$\Rightarrow$ so, can break pblm into 3 zones

i) ultra-near field

$r \sim \ell \ll \lambda$ $\qquad$ $\nabla^2 \sim 1/\ell^2 \gg \omega^2/c_s^2$

$\rightarrow \nabla^2\phi = 0$ (potential flow!) and

body configuration details matter.

ii) near field

$\lambda > r > \ell$ $\qquad$ $\nabla^2\phi = 0$

but, represent by multipole expansion!

(iii.) Far field → region sound radiated to...

$$\underbrace{r \gg \lambda}_{\text{outgoing spherical wave.}} \qquad \phi = -\frac{f(t - r/c_s)}{r}$$

$$V_r = \frac{1}{c}\,\frac{f'(t - r/c_s)}{r}$$

→ idea will be to match (ii.) to (iii.)!

Now, in general: $\nabla^2 \phi = 0$

$$\Rightarrow \phi = -\frac{\overset{\text{monopole moment}}{q}}{r} + \underbrace{\underline{A}}_{\text{dipole moment}} \cdot \nabla(1/r) + \cdots$$

consider term-by-term, ...

(i.) monopole $\quad$ i.e. $\circ \to \bigcirc \to \circ$

monopole → body pulsates
⟺ volume varies

Aside: Multipole Expansions

$\nabla^2\phi = -4\pi\rho$ $\rho \equiv \left\{\begin{array}{l}\text{charge or} \\ \text{arbitrary source}\end{array}\right.$

i.e. Poisson Eqn.

or

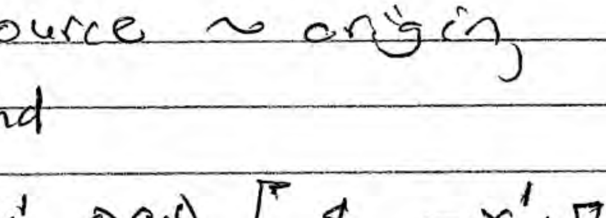
$\underline{\nabla}\cdot\underline{v} = \rho$, $\underline{v} = \underline{\nabla}\phi$

$\Rightarrow$

$$\phi(x) = \int \frac{d^3x'\,\rho(\underline{x}')}{|\underline{x}-\underline{x}'|} \qquad G(x,x') = \frac{1}{4\pi|\underline{x}-\underline{x}'|}$$

Consider localized source/charge distribution:

i.e.



$\underline{x}$

$\rho(\underline{x}')$

taking source ~ origin, $|x'| \ll |x|$
so expand

$$\phi = \int d^3x'\,\rho(x')\left[\underset{①}{\frac{1}{|x|}} - \underset{②}{\underline{x}'\cdot\underline{\nabla}\frac{1}{|x|}} + \underset{③}{\frac{1}{2}x'_\alpha x'_\beta \frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}\frac{1}{|x|}}\right]$$

q   es   → es.

$$= \frac{q}{|x|} - \begin{pmatrix} d \\ \underline{A} \end{pmatrix}\cdot\underline{\nabla}\frac{1}{|x|} + Q_{\alpha\beta}\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta}\frac{1}{|x|}$$

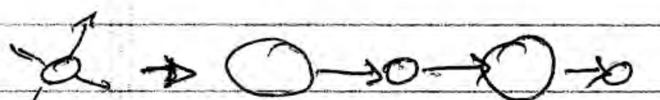
$+ \ldots$ acoustics

① → monopole → scalar •

$\sim 1/|x|$

$q, Q = \int d^3x' \rho(x')$

Physics
- e.s. total net charge
- acoustics : total $\dot{V}$ → compression
  - ↳ net, bulk property of source.



etc.

② → dipole → vector $\underline{p}$

$\sim 1/|x|^2$

$$\begin{pmatrix} \underline{d} \\ \underline{A} \end{pmatrix} = \int d^3x' \rho(\underline{x}') \, \underline{x}'$$

Physics
- e.s. net charge polarization (relative displacement)
- acoustics : net displacement of body.

recall : sphere $\underline{A}(t) = \frac{R^3}{2} \underline{U}(t)$

vector defining dipole moment

(3) → quadrupole → tensor ↔ dyad

$$\sim 1/|x|^3$$

N.B. – for e.g. mass,

$$Q_{\alpha\beta} = \int d^3x' \rho(x') \frac{x'_\alpha x'_\beta}{2}$$

but as $\nabla^2 \frac{1}{|x|} = \delta_{\alpha\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(\frac{1}{|x|}\right) = 0$

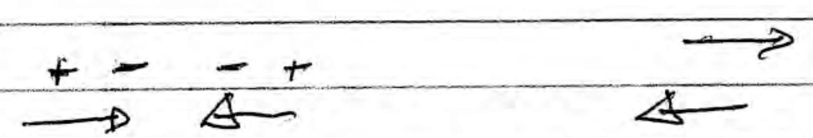
$$|x| \neq 0$$

$$Q_{\alpha\beta} = \int d^3x' \frac{\rho(x')}{2}\left[x'_\alpha x'_\beta - \frac{r^2}{3}\delta_{\alpha\beta}\right]$$

∴ quadrupole moment of mass distribution is "moment of inertia" tensor.

– can view quadrupole as pair of opposite dipoles (just as dipole as pair of opposite charges)

i.e.



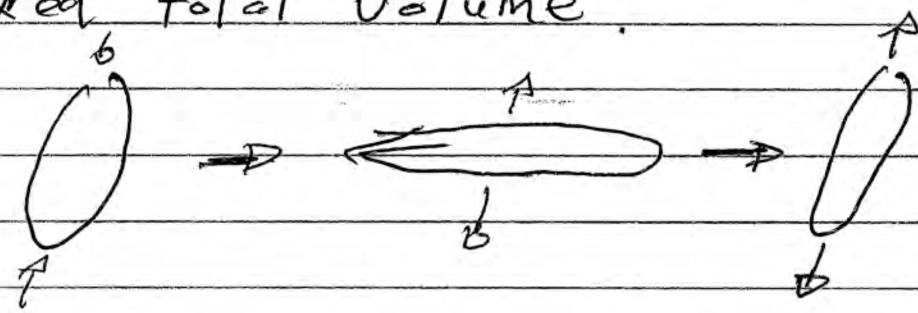
longitudinal quadrupole   lateral quadrupole.

Physics:

(moments)

→ oscillation in distribution / (shape) of
e.s. charge at fixed total charge.

→ oscillation in (distribution) shape of
acoustics.
body (which sets b.c. for fluid) at fixed total volume.

i.e. 

(but volume fixed!).

$\phi = -a/r$ $\qquad$ $a \sim L^3/T$ (dimensionally)

$V_r = a/r^2$

so for emitted mass flux due pulsations
at some enclosing surface

$$\int 4\pi r^2 \, d\Omega \, \rho V_r = 4\pi \rho a$$

surface

emitted flux of mass

mass flux

but then $\dfrac{4\pi\rho a}{\rho} = \dot{V}$

rate of change in volume of pulsating object.

(N.B. $\nabla^2 \phi = 0$ in near field $\rightarrow$ ⑥ $\nabla \cdot \underline{V} = 0$)

$$\therefore \quad \phi = -\frac{\dot{V}}{4\pi r} = -\frac{\dot{V}(t - r/c_s)}{4\pi r}$$

so $V_r = \frac{\partial}{\partial r}\phi$

$= -\frac{1}{c_s}\frac{\partial}{\partial t}\phi$ radial velocity at r set by volume change at retarded time $t - r/c_s$.

$= \frac{1}{c_s}\frac{\ddot{V}}{4\pi r} = \frac{1}{c_s}\frac{\ddot{V}(t - r/c_s)}{4\pi r}$

$$I = \int \hat{r}\cdot d\underline{a}\ \rho V_r^2 c_s$$

$$= \rho c_s \int da\ \frac{1}{c_s^2}\frac{|\ddot{V}|^2}{(4\pi)^2 r^2}$$

$$\boxed{I = \frac{\rho}{(4\pi)^2 c_s}\int da\ \frac{\overline{|\ddot{V}|^2}}{r^2}}$$

time avg'd. → radiated power

see 76a.

N.B.

- harmonic pulsations at ω

$\ddot{V} \sim \omega\tilde{V} \Rightarrow I \sim \omega^2$, for given surface velocity amplitude

- harmonic pulsations at fixed spatial amplitude given

For spherical surface of integration:

$$A = 4\pi r^2$$

$$\Rightarrow I = P_{rad} = \frac{\rho}{4\pi c_s} \overline{|\ddot{V}|^2}$$ → total intensity of emitted sound.

Example: L+L, Pg. 286 #3.

Sound from sphere executing small harmonic pulsations with

$$\underline{u}_s = u_0 e^{-i\omega t} \vec{r}$$

Assume sphere has radius R, with $\delta R/R \ll 1$.

P_{rad}?

Consider:

1) Low Frequency limit
2) high Frequency limit
3) unified

1) Can plug in, with

$$V = \tfrac{4}{3}\pi R^3 \qquad \ddot{V} = 4\pi R^2 \ddot{R} + \cancel{8\pi R \dot{R}^2} \quad \text{NL}$$

$$\dot{V} = 4\pi R^2 \dot{R}$$

$$\dot{R} = U_0 e^{-i\omega t}$$

$$\ddot{R} = -i\omega U_0 e^{-i\omega t}$$

so $$\ddot{V} = -4\pi i\omega R^2 U_0 e^{-i\omega t_{ret}}$$
$$= -4\pi i\omega R^2 U_0 e^{-i\omega(t-\frac{r}{c})}$$
$$\sim e^{i(kr-\omega t)}$$

and

$$P_{rad} = \frac{\rho}{4\pi C_s} \overline{\ddot{V}^2}$$

$$= \frac{\rho}{4\pi C_s} (4\pi)^2 \omega^2 R^4 \frac{U_0^2}{2} \rightarrow \text{time avg}$$

$$\therefore \boxed{P_{rad} = 2\pi \rho \omega^2 \frac{R^4 U_0^2}{C_s}} \sim \omega^2 U_0^2$$

2) Can plug in, with

$$P_{rad} = C_s \rho \int ds \, \overline{U_n^2}$$

$$U_n = U_0 e^{-i\omega t}$$

$$\overline{U_n^2} = U_0^2/2$$

$$A = 4\pi R^2$$

$$\Rightarrow P_{rad} = c_0 \rho\, 4\pi R^2 \frac{U_0^2}{2}$$

$$= 2\pi c_0 \rho\, R^2 U_0^2$$

3) Unified.

Now, - near field

$$\phi \cancel{=\sigma a/r} = a U_0/r$$

- far field ⟶ shift due finite radius

$$\phi \sim \frac{e^{ik(r-R)}}{r} a U$$

$$\therefore \phi = \frac{a U_0}{r} e^{ik(r-R)}$$

a?

$$\left.\frac{\partial \phi}{\partial r}\right|_R = U_0$$

$$\Rightarrow -\frac{a U_0}{R^2} + ik\frac{a U_0}{R} = U_0$$

$\Rightarrow \quad a = R^2 / (ikR - 1)$

$\therefore \quad I = 2\pi \rho c_0 |u_0|^2 k^2 R^4 / (1 + k^2 R^2)$

limits check

∴ $I \sim \omega^4$

ii.) dipole $\Rightarrow$ body moves

i.e. 

emission from oscillating sphere,

for pot. flow,

$$\phi = \underline{\nabla} \cdot \left[\frac{\underline{A}(t)}{r} \right] = \underline{A}(t) \cdot \nabla (1/r)$$ $\Rightarrow$ dipole potential

∴ for $r > \ell$

$$\phi = \underline{\nabla} \cdot \left[\underline{A}\left(t - \frac{r}{c_s}\right) / r \right]$$

and for $r > \lambda$, thus:

$$\phi = -\frac{\underline{\dot{A}}(t - r/c_s) \cdot \hat{n}}{c_s r}$$ $\hat{n} \equiv$ direction prop

but $\underline{V} = \underline{\nabla} \phi$

$$\Rightarrow \quad \underline{V} = -\frac{\underline{\ddot{A}}}{c_s^2 r} \cdot \hat{n} \, \underline{\nabla} \left(t - \frac{r}{c_s} \right)$$

so $$\nabla\left(t-\frac{r}{c_s}\right) = -\frac{\hat{n}}{c_s} = \frac{-1}{c_s}\nabla r$$

$$\underline{V} = \hat{n}\,\left(\hat{n}\cdot\ddot{\underline{A}}(t-r/c_s)\right)/c_s^2 r$$

so

$$\boxed{I = \frac{\rho}{c_s^3}\int da\,\overline{\frac{(\hat{n}\cdot\ddot{\underline{A}})^2}{r^2}}}$$

time-avg'd radiated power

– for given velocity amplitude ($A \sim u$)

$I \sim \omega^4$

– for given position amplitude

$I \sim \omega^6$

Integrating: $$\boxed{I = \frac{4\pi\rho}{3c_s^3}|\ddot{\underline{A}}|^2}$$

Note: For (electric) dipole EM radiation, have

$$\boxed{P_{rad} = \frac{2}{3c^3}|\ddot{\underline{d}}|^2}$$

→ Larmor Formula.

Analogy obvious.

Ex. P_{rad} from sphere oscillating at ω

with $\delta x < R$

$$\underline{u} = u_0 e^{-i\omega t}$$

a) Low Frequency

Flow is dipolar! Radiation is dipolar!

Acoustic 'Larmor Formula' $\Rightarrow$

$$P_{rad} = \frac{4\pi\rho}{3c_s^3}\overline{\ddot{\underline{A}}^2}$$

For $\underline{A}$;
$$\nabla^2\phi = 0$$

$$\phi = \frac{-q}{r} + \underline{A}\cdot\underline{\nabla}(1/r) + \cdots$$

$\underline{u}$
$$\doteq \underline{A}\cdot\underline{\nabla}(1/r)$$

$$= \alpha\,\underline{u}\cdot\underline{\nabla}(1/r) = -\alpha\frac{u\cos\theta}{r^2}$$

$$V_r = \frac{2\alpha u\cos\theta}{r^3}$$

$$V_r\Big|_R = u\cos\theta \quad\Rightarrow\quad \alpha = \frac{1}{2}R^3$$

$$\underline{A} = (R^3/2)\,\underline{u}$$

$$\Rightarrow P_{rad} = \frac{4\pi\rho}{3c_s^3}\frac{R^6}{4}\overline{\ddot{u}^2}$$

$$\underline{u} = \underline{u}_0 e^{-i\omega(t - r/c_s))}$$

$$\Rightarrow P_{rad} = \frac{\pi\rho}{6c_s^3} R^6 \omega^4 |u_0|^2$$

b.) high frequency

$$P_{rad} = c_s\rho \int \overline{u_n^2}\, dS$$

$$= c_s\rho \left(\frac{4\pi R^2}{3}\right)\frac{|u_0|^2}{2}$$

$$P_{rad} = \frac{2\pi}{3}\rho c_s |u_0|^2$$

c.) General:

$$\nabla^2\phi + \frac{\omega^2}{c_s^2}\phi = \nabla^2\phi + k^2\phi = 0$$

$$\phi = -\frac{a}{r} + \underline{A}\cdot\underline{\nabla}(1/r) \quad , \quad \underline{A} \sim \alpha\,\underline{u}$$

in near field.

$\therefore$ can write : $\phi = \underline{u} \cdot \underline{\nabla} f(r)$

$\Rightarrow$ $\underline{u} \cdot \underline{\nabla} \left(\nabla^2 f + k^2 f \right) = 0$

$\therefore$ $\nabla^2 f + k^2 f = \text{const.}$

$\Rightarrow$ $f = \frac{A}{r} e^{ik(r-R)} + \cancel{\text{const}}$

Matching $\Rightarrow$

$$\underline{A} = \frac{+\underline{u} R^3}{2 - 2ikR - k^2R^2}$$ ✓

→ General Theory of Acoustic Radiation

i.) Green's Function for Wave Equation

Now,

- Radiation Theory :

- Green's Function : $\phi(\underline{r}, t) = \int d^3x' \int dt' \, G(\underline{x}-\underline{x}', t-t') \, S(\underline{x}', t')$

response function

∴ Wave equation Green's Function central to Radiation theory.

Obtain via :

a.) substitution

b.) contour integration

a.) Substitution $\qquad c = c_s$

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right) G(\underline{x}-\underline{x}', t-t') = -\delta(x-x')\,\delta(t-t')$$

$$G(\underline{x}-\underline{x}', \omega) = \int dt \, e^{i\omega(t-t')} \, G(\underline{x}-\underline{x}', t-t')$$

so

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) G(\underline{x}-\underline{x}'; \omega) = -\delta(\underline{x}-\underline{x}')$$

Now, no loss of generality to assume:

- spherical symmetry $\Rightarrow$ propagation isotropic
- $\underline{x}' = 0$.

$$\left(\nabla_r^2 + \frac{\omega^2}{c^2}\right) G = -\delta(r)$$

now, $\nabla_r^2 G = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial G}{\partial r}\right)$

$$= G'' + \frac{2}{r}\frac{\partial G}{\partial r}$$

but $\frac{d^2}{dr^2}(rG) = \frac{d}{dr}(rG' + G)$

$$= rG'' + 2G'$$

$$\frac{1}{r}\left(\frac{d^2}{dr^2}(rG)\right)+\left(\frac{(rG)}{r}\frac{\omega^2}{c^2}\right) = -\delta(r)$$

$r\delta(r) = 0$, except at $r=0$

$$\frac{d^2}{dr^2}(rG)+\frac{\omega^2}{c^2}(rG) = 0$$

$$\Rightarrow \quad G = \frac{Ae^{i\frac{\omega}{c}r}}{r} + \frac{Be^{-i\frac{\omega}{c}r}}{r}$$

$Kr \ll 1$ reduction $\Rightarrow A = B = 1/4\pi$

$$G(|\underline{x}-\underline{x}'|,\omega) = \left(\frac{1}{4\pi|\underline{x}-\underline{x}'|}\right)\left[e^{i\frac{\omega}{c}|\underline{x}-\underline{x}'|} + e^{-i\frac{\omega}{c}|\underline{x}-\underline{x}'|}\right]$$

$$G(|\underline{x}-\underline{x}'|, t-t') = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, e^{-i\omega(t-t')}\, G(\omega, |\underline{x}-\underline{x}'|)$$

$$G = \frac{1}{4\pi|\underline{x}-\underline{x}'|}\left\{\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, e^{i\left(\omega\left(\frac{|\underline{x}-\underline{x}'|}{c}-(t-t')\right)\right)} + \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, e^{-i\omega\left(\frac{|\underline{x}-\underline{x}'|}{c}+(t-t')\right)}\right\}$$

①

$$= \frac{1}{4\pi|\underline{x}-\underline{x}'|}\left\{\delta\left(\frac{|\underline{x}-\underline{x}'|}{c}-(t-t')\right) + \delta\left(\frac{|\underline{x}-\underline{x}'|}{c}+(t-t')\right)\right\}$$

②

Note G will sample source field at:
→ time of flight from $\underline{x}' \to \underline{x}$

$t' = t - \frac{|\underline{x}-\underline{x}'|}{c}$ → retarded time (physical)

⟺ outgoing wave

$t' = t + \frac{|\underline{x}-\underline{x}'|}{c}$ → advanced time (future strikes present)

unphysical!

⇓

reject term ② ⟺ incoming wave.

∴ G.F. for Wave Equation is:

$$\boxed{G = \frac{\delta\left(\frac{|\underline{x}-\underline{x}'|}{c_s} - (t-t')\right)}{4\pi|\underline{x}-\underline{x}'|}}$$

b.) Contour Integration

→ Causality

i.e. consider:
→ localized disturbance at $\underline{x}=0$, $t=0$

∴ → causality ⇒ $G(\underline{x},t) = 0$, $t<0$.

Now,

$$\tilde{G} \equiv \int_{-\infty}^{+\infty} dt\, e^{i\omega t} G(\underline{x},t) = \int_{0}^{\infty} dt\, e^{i\omega t} G(\underline{x},t)$$

Further assume:

$$\int_{-\infty}^{\infty} G(\underline{x},t)\, dt < \infty$$

$\therefore$ $\tilde{G}(\underline{x}, \omega = 0)$ exists and is bounded.

Now,

For <u>general</u>, <u>complex</u> values of $\omega = \omega_1 + i\omega_2$

we have

$$\tilde{G}(\underline{x}, \omega_1 + i\omega_2) = \int_0^\infty dt\, e^{i\omega_1 t} e^{-\omega_2 t} G(\underline{x}, t)$$

Now

$$|\tilde{G}(\underline{x}, \omega)| \le \int_0^\infty \underbrace{dt\, e^{-\omega_2 t}}_{\text{phase} \Rightarrow \text{oscillations}} |G(\underline{x}, t)| \le \underbrace{\int_0^\infty dt\, |G(\underline{x}, t)| < \infty}_{G(\underline{x}, t) \text{ absolutely integrable}}$$

$\therefore$ $\boxed{\tilde{G}(\underline{x}, \omega) \text{ bounded in upper half plane } (\omega_2 > 0)}$

also

all $d^n \tilde{G} / d\omega^n$ exist

$\Rightarrow$ $\boxed{\tilde{G}(\underline{x}, \omega) \text{ is analytic in UHP}}$

→ How is this used? ⇒ Pole Prescription!

$$\nabla^2 G - \frac{1}{c^2}\partial_t^2 G = -\delta(\underline{x})\,\delta(t)$$

$$\left(\frac{\omega^2}{c^2} - k^2\right)\tilde{\tilde{G}}(k,\omega) = -1$$

$$\Rightarrow \tilde{\tilde{G}}(k,\omega) = 1\Big/ k^2 - \omega^2/c^2$$

and understood $\omega \to \omega + i\eta$
$\lim \eta \to 0$

$\therefore$

$$\tilde{G}(\underline{x},\omega) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\underline{k}\cdot\underline{r}}\, \tilde{\tilde{G}}(\underline{k},\omega)$$

$$= \frac{1}{(2\pi)^3}\int_0^\infty k^2 dk \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\, \frac{e^{ikr\cos\theta}}{k^2 - \omega^2/c^2}$$

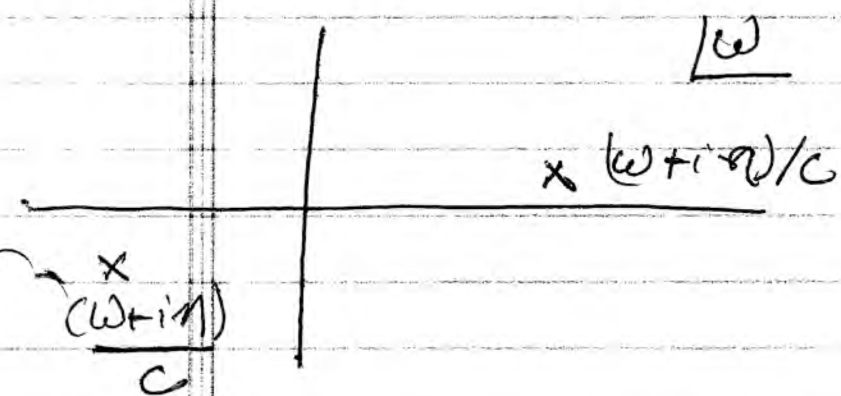
$r = |x|$

$$r = |x|$$

Angular integrals $\Rightarrow$

$$\tilde{G}(r, \omega) = \frac{1}{8\pi^2 i r} \int_{-\infty}^{+\infty} \frac{k\,dk\,(\overset{①}{e^{ikr}} - \overset{②}{e^{-ikr}})}{(k^2 - \omega^2/c^2)}$$

Now, for integral, $\omega \to \omega + i\eta$



Now, – 2 terms $\Rightarrow$ 2 integrals + contour prescriptions

– 2 poles $\quad k = \pm\left(\frac{\omega + i\eta}{c}\right)$

① $\Rightarrow r \geq 0$

$\Rightarrow \tilde{G}$ analytic

$\therefore \quad k_{pole} = \frac{\omega + i\eta}{c}$; i.e. $e^{i\left(\frac{\omega + i\eta}{c}\right) r} \sim$

$e^{-\eta r/c} \to 0$
converges.

(2) → $r \gtrsim 0$

→ $\tilde{G}$ analytic

$$K_{pole} = -\frac{(\omega + i\eta)}{c}, \quad e^{+i\frac{(\omega+i\eta)}{c} r} \sim e^{-\frac{\eta r}{c}}$$

$$\tilde{G}(r,\omega) = \frac{1}{8\pi^{\cancel{2}} \cancel{i} r}\left[\frac{\cancel{k}\, \cancel{2\pi i}}{\cancel{2\omega/c}} e^{i\frac{\omega}{c}r} + \frac{\cancel{k}(+\cancel{2\pi i})}{+\cancel{2\omega/c}} e^{i\frac{\omega}{c}r}\right]$$

$$= \frac{1}{4\pi r} e^{i\omega r/c}$$

so finally obtain:

$$\boxed{\tilde{G}(r,\omega) = \frac{1}{4\pi r} e^{i\frac{\omega}{c} r}}$$

ii) Greens Functions + Boundary

→ major challenge of radiation theory:

reconcile { Green's function ; boundary conditions }

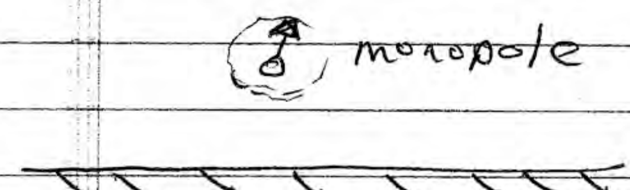
n.b.: $G(\underline{x}-\underline{x}', t-t') = \dfrac{\delta\left(\frac{|\underline{x}-\underline{x}'|}{c} - (t-t')\right)}{4\pi|\underline{x}-\underline{x}'|}$

is free space G.F. (retarded)

∴ can consider cases:

① homogeneous boundary condition
+
inhomogeneous acoustic wave equation (source present)

i.e.



hard bndry or soft bndry (pressure release)

$V_n = 0$ $p\big|_{\text{bndry}} = 0$

⇒ method of images!

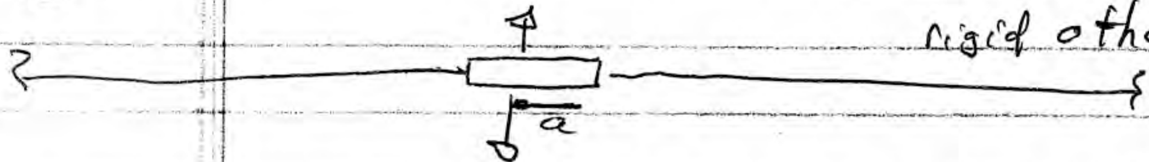
[cf: 201, 203A, B]

② homogeneous equation (no source f)

+

inhomogeneous boundary

i.e.



$$\text{circular piston: } \delta h = \underbrace{\epsilon}_{\ll 1} a e^{-i\omega t}$$

i.e. piece of boundary pulsates

⇒ inhomogeneous boundary condition

$$\left.\frac{\partial \phi}{\partial z}\right|_0 = \begin{cases} 0, & r > a \quad \text{(rigid)} \\ V_n = -i\omega\epsilon a, & r < a \quad \text{piston motion} \end{cases}$$

∴ Green's integral

Physical Insight: Represent piston emission as emission of spherical waves associated with motion of points on piston head

etc.

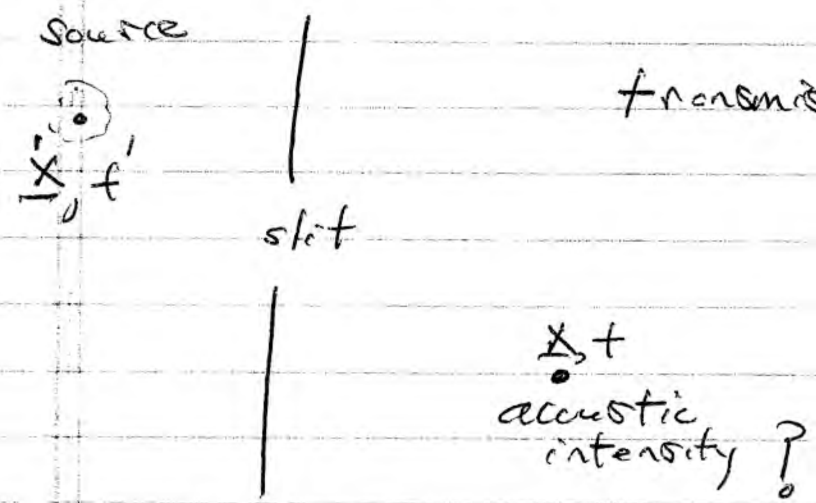
ⓒ Inhomogeneous boundary

$+$

Inhomogeneous Acoustic Eqn. (sources)

$= \boxed{\text{Diffraction}}$

i.e.



transmission $\Leftrightarrow G(\underline{x}, \underline{x}'; t, t')$

determines sound field on far side of slit.

Cases: – Fresnel
– Fraunhofer

so

$\Rightarrow$ Case ⓑ : Piston

First: Generalities

homogeneous boundary $\hat{z}$

b.c.

$$\alpha \frac{\partial \phi}{\partial z} - \beta \phi = 0 \Big|_{z=0}$$

$\beta = 0 \rightarrow$ hard

$\alpha = 0 \rightarrow$ soft

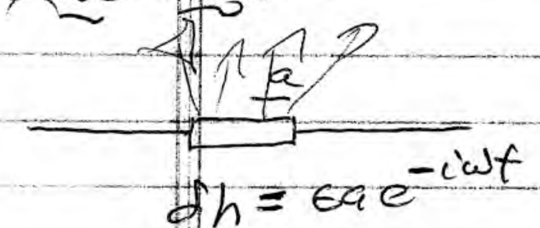
then, for G.F. for $z > 0$, apply 1D techniques to:

$$\frac{\partial^2 G}{\partial z^2} + \left(\frac{\omega^2}{c_s^2} - k_\perp^2\right) G = -\delta(z - z')$$

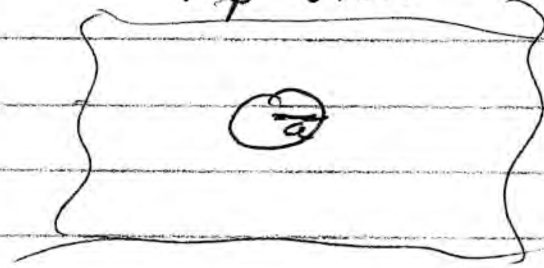
with b.c.

Now → the Piston

side view



top view



b.c.'s inhomogeneous:

$$\left.\frac{\partial \phi}{\partial z}\right|_0 = \begin{cases} 0, & r > a \\ -i\omega\epsilon a, & r < a \end{cases}$$

How deal with this? – Green's Identity?

Consider 2 fields: ϕ, G

1 → ϕ : sound field

$$\nabla^2\phi + \frac{\omega^2}{c^2}\phi = 0 \quad (1)$$ ⇝ homog. field

$$\underbrace{\hat{n}}_{\text{normal to } z=0 \text{ plane}}\cdot\nabla\phi = F(\underline{x})$$ ⇝ inhomogeneous b.c.

and

2 → $G(\underline{x}, \underline{x}')$: Green's function

$$\nabla^2 G + \frac{\omega^2}{c^2} G = +\delta(\underline{x}-\underline{x}') \quad (2)$$

with

$$\hat{n}\cdot\nabla G = 0$$ [key point: G is G.F. for rigid, homog. boundary case!]

Now, $\underbrace{\phi \otimes (2) - G \otimes (1)}_{\Delta} \Rightarrow$

$$\Delta = \phi\nabla^2 G + \cancel{\frac{\omega^2}{c^2}\phi G} + \phi\delta(\underline{x}-\underline{x}') - G\nabla^2\phi - \cancel{\frac{\omega^2}{c^2}\phi G}$$

So, define $I_G \equiv$ Green's Integral

$$I_G = \int d^3x' \left[\phi \nabla^2 G - G \nabla^2 \phi\right] \equiv \int \phi(x') \delta(x-x')\, d^3x$$
$$= \phi(\underline{x})$$

(gives expression for sound field)

Further

$$I_G = \int d^3x' \left[\phi\, \nabla\cdot\underline{\nabla} G - G\underline{\nabla}\cdot\underline{\nabla}\phi\right]$$
$$= \int d^3x' \left(\nabla\phi\cdot\nabla G - \underline{\nabla} G\cdot\underline{\nabla}\phi\right) + \int d\underline{a}\cdot\left(\phi\,\underline{\nabla} G - G\,\underline{\nabla}\phi\right)$$

$\therefore$ have:

$$\int d\underline{a}\cdot\left(\phi\,\underline{\nabla} G - G\,\nabla\phi\right) = \phi(\underline{x})$$

$\int d\underline{a}$ is over $z=0$ surface

Now, $\hat{n}\cdot\underline{\nabla} G = 0 \quad \therefore \quad da\cdot\nabla G = 0$

$$d\underline{a}\cdot\underline{\nabla}\phi \equiv da\,\hat{n}\cdot\nabla\phi = F$$

$$\therefore \quad \hat{n}\cdot\nabla\phi\big|_{sur} = F$$

$$\boxed{+\int da\, G(\underline{x},\underline{x}')\, F(\underline{x}') = \phi(\underline{x})}$$

expression for sound field!

$\rightarrow$ the Piston: G.F at fixed ω

$$\phi(\underline{x}) = +\int da \; G_\omega(\underline{x},\underline{x}') \, F(\underline{x}')$$

$$= +\int_{r<a} dx'dy' \; G_\omega(\underline{x},\underline{x}') \, (-i\omega\epsilon a)$$

$$G_\omega(\underline{x},\underline{x}') = \frac{e^{i\frac{\omega}{c}|\underline{x}-\underline{x}'|}}{4\pi|\underline{x}-\underline{x}'|}$$

$$\phi(x) = -\frac{i\omega\epsilon a}{4\pi} \int_{r<a} dx' \int dy' \frac{e^{ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|}$$

now, $|x'| \sim a \ll |x|$, in far field

$$|\underline{x}-\underline{x}'| = \left[(|\underline{x}-\underline{x}'|)^2\right]^{1/2}$$

standard approx $\rightarrow$ appears frequently

$$= |\underline{x}| \left(1 - \frac{2\underline{x}\cdot\underline{x}'}{x^2} + \cancel{\frac{x'^2}{x^2}}\right)^{1/2}$$

$$\cong |\underline{x}| \left(1 - \frac{\underline{x}\cdot\underline{x}'}{x^2}\right) = |\underline{x}| - \hat{\underline{x}}\cdot\underline{x}'$$

$k = \omega/c$

$\therefore$

$$\phi(\underline{x}) = \frac{-i\omega\epsilon a}{4\pi}\int dx'\int dy' \frac{e^{ik|\underline{x}|}}{|\underline{x}|} e^{-ik\hat{\underline{x}}\cdot\underline{x}'}$$

$$\boxed{\phi = -\frac{i\omega\epsilon a}{4\pi}\frac{e^{ikr}}{r}\int dx'\int dy'\, e^{-ik\hat{\underline{x}}\cdot\underline{x}'}}$$

Now, need perform I

$$I = \int dx'\int dy'\, e^{-ik\hat{\underline{x}}\cdot\underline{x}'}$$

in e.m. $\Rightarrow$ expansion in $kr' \ll 1 \rightarrow$ magnetic dipole, quadrupole terms

Now, $\int dx'\int dy' = \int_0^a dr'\, r' \int_0^{2\pi} d\phi'$

$$\underline{x}' = (r'\cos\phi',\ r'\sin\phi', 0)$$

$$\hat{\underline{x}} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$$\hat{x}\cdot\underline{x}' = r'\sin\theta\left(\cos\phi'\cos\phi + \sin\phi'\sin\phi\right)$$

$$= r'\sin\theta\cos(\phi'-\phi)$$

$\therefore$

$$I = \int_0^a dr'\, r' \int_0^{2\pi} d\phi'\, e^{-ikr'\sin\theta\cos(\phi'-\phi)}$$

$$\cos(\phi'-\phi) = \sin(\phi'-\phi+\pi/2)$$

$$I = \int_0^a dr'\, r' \int_0^{2\pi} d\phi'\, e^{-ikr'\sin\theta\sin(\phi'-\phi+\pi/2)}$$

$$e^{ix\sin\phi} = \sum_n J_n(x)\, e^{in\phi} \quad \left(\text{standard identity}\right)$$

$$\therefore\ I = \int_0^a dr'\, r' \int_0^{2\pi} d\phi' \sum_n J_n(-kr'\sin\theta)\, e^{in(\phi'-\phi-\pi/2)}$$

Kills all but $n=0$

$$I = 2\pi \int_0^a dr'\, r'\, J_0(+kr'\sin\theta)$$

as $J_0(x) = J_0(-x)$

Now, $I = 2\pi \int_0^a dr'\, r'\, J_0(kr\sin\theta)$

Now, $\frac{d}{dz}\left(z^n J_n(z)\right) = z^n J_{n-1}(z)$

$\therefore \int_0^z J_0(\rho)\rho\, d\rho = z J_1(z)$

$$\Rightarrow \phi(\underline{x}) = -\frac{i\omega\epsilon a}{4\pi}\frac{e^{ikr}}{r}(2\pi a^2)\frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

$$= \underbrace{\pi a^2}_{\text{area piston}} \underbrace{V_p}_{\text{velocity}} \underbrace{\frac{e^{ikr}}{r}}_{\text{pt. response.}} \underbrace{\left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right]}_{\text{angular directivity factor}}$$

Note: $\to ka \ll 1$ — low frequency limit

$$\lim_{x\to 0} \frac{2J_1(x)}{x} \cong 1 + \cancel{O(x^2)}$$

$$\phi(\underline{x}) \cong \underbrace{\pi a^2}_{\text{area}} \underbrace{\tilde{V}_p}_{\text{velocity}} \underbrace{\frac{e^{ikr}}{r}}_{\text{pt. response.}}$$

dims ✓

→ for radiated power

$$\underline{S} = \overline{c_s^2 \tilde{\rho}\, \tilde{\underline{v}}}$$

$$= \frac{1}{2} \text{re}\, \tilde{\underline{v}}\, \tilde{p}$$

$$\tilde{\underline{v}} = \underline{\nabla} \tilde{\phi} \ , \quad \tilde{p} = -\frac{\partial}{\partial t} \tilde{\phi}$$

so

$$\boxed{\underline{S} = \frac{\rho_0 \epsilon^2 c_s^3 k^4 a^6}{8 r^2} \left[\frac{2 J_1 (ka \sin\theta)}{ka \sin\theta} \right]^2 \hat{r}}$$

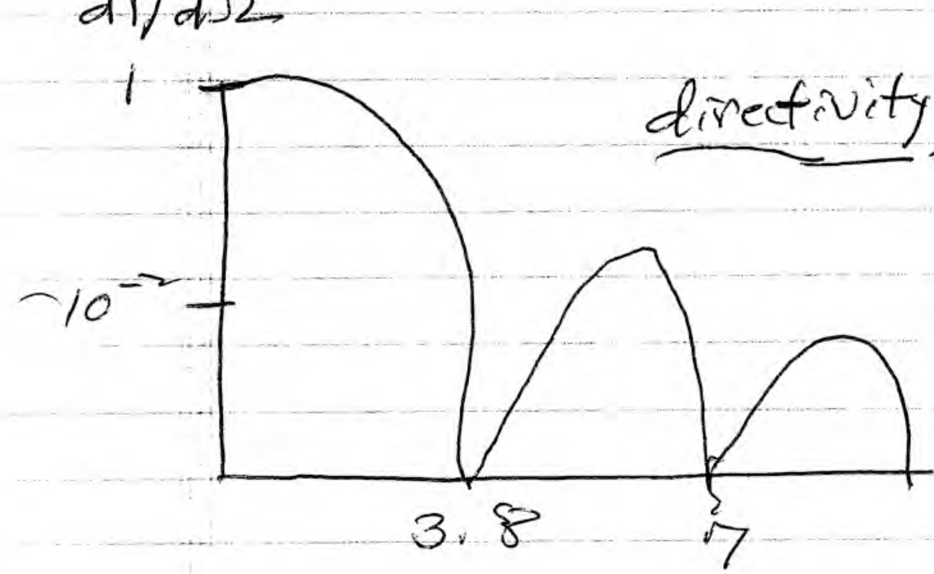
radiated wave energy density flux

→ for $dP/d\Omega$ → radiated power thru angular element $d\Omega$

$$dP = \underline{S} \cdot \hat{r}\, dA$$

$$= \underline{S} \cdot \hat{r}\, r^2 d\Omega$$

$$\therefore \quad \frac{dP}{d\Omega} = \underbrace{\frac{1}{8}\,\rho\,\epsilon^2 c_s^3 k^4 a^6}_{\epsilon^2 a^2 k^4 a^4} \underbrace{\left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right]^2}_{\text{directivity factor}}$$



directivity pattern (a.k.a. antenna)

→ radiated power maximal forward (obvious)

→ $ka \ll 1$ ⇒ isotropic radiation

→ for zero in radiated power

$ka\sin\theta = \alpha_{1,n}$ → n^{th} zero of J_1

$n = 1$ → bndry of first lobe
$n = 2$ → bndry of 2^{nd} lobe

→ compare pulsating sphere:

$$\begin{cases} k\delta R \ll 1 \\ kR_0 \ll 1 \end{cases}$$

$$P_{rad} = 2\pi\rho\,\omega^2 \frac{R^4 u_0^2}{c_s}$$

here

$$\frac{dP}{d\Omega} = \frac{1}{8}\rho\,\epsilon^2 c_s^3 \frac{\omega^4}{c_s^4} a^6 \left[\quad\right]^2$$

$$= \left(\frac{1}{8}\right)\frac{\rho}{c_s}\left[\epsilon^2 a^2 \omega^2\right] \omega^2 a^4 \left[\quad\right]^2$$

same!, up to
- numerical factor
- directivity factor.

∴ not surprising, as piston radiation clearly monopole.

→ Example 2: Oscillating Sphere

Familiar problem of oscillating sphere

$a \bigcirc \quad \delta z = \epsilon a e^{-i\omega t} \vec{z}$

(dipole radiator) is also case of inhomogeneous b.c. + homogeneous sound field.

Here, inhomogeneity due motion of sphere's surface!

Now:

- could proceed via Green's integral

or

- undertake direct approach, using b.c.

Now, will go direct route:

$$\underline{z}_{sph} = \epsilon a e^{-i\omega t} \vec{z}$$

$$\underline{V}_{sph} = -i\omega \epsilon a e^{-i\omega t} \vec{z}$$

and have b.c.

$$V_r\Big|_a = \frac{\partial \phi}{\partial r}\Big|_a = -i\omega \epsilon a e^{-i\omega t} \vec{z} \cdot \vec{r}$$

for ϕ_ω; $\left(\vec{z} \cdot \hat{r} = \cos\theta \right)$

$$\boxed{\left.\frac{\partial \phi}{\partial r}\right|_a = -i\omega \sigma a \cos\theta}$$ b.c

Now, ϕ satisfies Helmholtz, so:

$$\nabla^2 \phi + \frac{\omega^2}{c^2}\phi = 0$$

in spherical geometry: Helmholtz Eqn.

$$\left\{ \begin{array}{l} \left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{\underline{L}^2}{r^2} + \frac{\omega^2}{c_s^2} \right)\phi = 0 \\ \underline{L}^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \end{array} \right.$$

now, write: spherical harmonics.

$$\phi(r,\theta,\phi) = R(r)\, Y_{\ell,m}(\theta,\phi)$$

$$\underline{L}^2 Y_{\ell m}(\theta,\phi) = \ell(\ell+1)\, Y_{\ell,m}(\theta,\phi)$$

so

have:

for $R(r)$;

$$\left[\frac{1}{r^2}\frac{d}{dr}\,r^2\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} + \frac{\omega^2}{c^2}\right] R = 0$$

spherical Bessel eqn.!

$$R = \Big(a_\ell \underbrace{j_\ell(kr)}_{\text{finite, } r\to 0} + b_\ell \overbrace{n_\ell(kr)}^{\text{divergent, } r\to 0}\Big)$$

(excluded in H-atom pblm.)

but, for radiation, seek <u>outgoing</u> <u>wave</u>:

$$\therefore \quad R = a_\ell \underbrace{h_\ell(kr)}_{\text{spherical Hankel fctn.}} \qquad \begin{cases} h_\ell \sim e^{ikr} \\ h_\ell = j_\ell + i n_\ell \end{cases}$$

Now, as $\left.\dfrac{\partial\phi}{\partial r}\right|_{r=a} = +i\omega\epsilon a\cos\theta$

$$\phi(\underline{x}) = C\,Y_{1,0}(\theta,\phi)\,f(kr) \qquad (\ell = 1)$$

$$= c\cos\theta\, f(kr)$$

$\left(r \neq 0 \rightarrow \underset{\text{allowed}}{n}\right)$

and, appling outgoing wave b.c.;

$$f(kr) = h_1(kr) = -(kr+i)(kr)^{-2} e^{ikr}$$

$\therefore \quad \phi(r,\theta) = C\cos\theta\, f(kr)$

$f(kr) = h_1(kr)$

finally, $C\cancel{\cos}\theta\, k f'(kr) = i\omega\epsilon a\, \cancel{\cos}\theta$

$\Rightarrow \quad C = \dfrac{i\omega\epsilon a}{k f'(ka)} = \dfrac{i c_s \epsilon a}{f'(ka)}$

$$\therefore \quad \boxed{\phi(r,\theta,t) = \mathrm{Re}\left(i c_s \epsilon a \cos\theta \frac{f(kr)}{f'(ka)} e^{-i\omega t}\right)}$$

→ complete solution.
→ clearly dipole ($\sim \cos\theta$)

Now, $kr >> 1 \leftrightarrow$ far field

$\phi(r,\theta,t) \cong -\mathrm{re}\left[C\cos\theta \dfrac{e^{i(kr-\omega t)}}{kr}\right]$

$\Rightarrow \quad \tilde{V}_r \cong -i C\cos\theta \dfrac{e^{i(kr-\omega t)}}{r}$

$\tilde{p} \cong -i c_s \rho_0 C\cos\theta \dfrac{e^{i(kr-\omega t)}}{r}$

$\therefore \quad \frac{dP_{rad}}{d\Omega} = \frac{r^3}{2} \, re(\tilde{p}^* \tilde{v}_r)$

$= \frac{1}{2} c_s^2 |C|^2 \cos^2\theta$

$kr \gg 1$

finally, plugging in:

ka arb.

$$\boxed{\frac{dP_{rad}}{d\Omega} = \frac{1}{2} c_s \rho_0 \frac{(ka)^6 (c_s \epsilon a)^2 \cos^2\theta}{4 + k^4 a^4}}$$

and

$$P_{rad} = \int d\Omega \frac{dP_{rad}}{d\Omega} = \frac{2\pi}{3}\left[\frac{c_s \, \rho_0 \, (c_s a \epsilon)^2 (ka)^6}{4 + k^4 a^4}\right]$$

$ka \ll 1$

$$P_{rad} \cong \frac{2\pi}{3} c_s^3 \rho_0 (a\epsilon)^2 \frac{\omega^6 a^6}{c_s^6}$$

$$= \frac{2\pi}{3} \frac{\rho_0}{c_s^3} (\omega a \epsilon)^2 \frac{\omega^4 a^6}{4}$$

$$= \frac{2\pi}{3} \frac{\rho_0}{c_s^3} |\tilde{u}_0|^2 \frac{\omega^4 a^6}{4} = \frac{\pi}{6} \frac{\rho_0}{c_s^3} |\tilde{u}_0|^2 \omega^4 a^6$$

recall, from pg. 81; $kr >> 1$
$ka << 1$

$$P_{rad} = \frac{\pi}{6} \frac{\rho_0}{c_s^3} R^6 \omega^4 |U_0|^2$$

checks.

Aside:

- note here motion prescribed

- if external force applied to move sphere, would need account for induced mass

i.e.  flow of H_2O about sphere.

$$F_{ext} = M_{sph} \frac{dy}{dt} + \frac{d}{dt} P_{H_2O}$$

Now, is previous calculation of $M_{ind.}$ valid here?

No! → Radiation occurs ⟷ outgoing wave b.c.
⇒ energy propagates → ∞. Not closed system, as before!

How calculate induced mass?

→ Interpret induced mass as reaction force

i.e. sphere radiates, loses energy ⟺ equivalent to reaction force acting on sphere,

i.e.

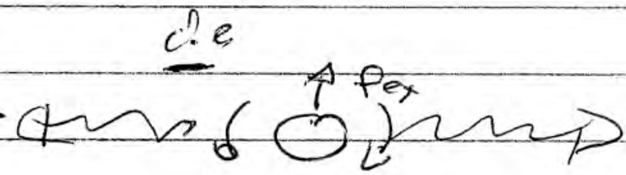
$$F_{ext} = m_{sph}\frac{du}{dt} + \frac{d}{dt} P_{H_2O}$$

$$\underline{F}_{ext} - \frac{d}{dt} P_{H_2O} = m_{sph}\frac{du}{dt}$$

$$\underline{P}_{H_2O_i} = \underbrace{M^{ind}_{ik}}_{\text{induced mass}} U_k$$

can re-write: $\underline{F}_{react} = -\frac{d}{dt}\underline{P}_{H_2O}$

$$\underline{F}_{ext} + \underline{F}_{react} = m_{sph}\frac{du}{dt}$$



$$\underline{F}_{react} = \int_{sphere} d\underline{A}\,\tilde{p}(\theta, \phi)\Big|_{r=a}$$

here, need $(\underline{F}_{react})_z \Rightarrow$

$$F_z = \int d\underline{A}\cdot\hat{z}\,\tilde{p}(\theta)\Big|_{r=a}$$

$$d\underline{A} = \hat{r}\, dA$$

$$\tilde{p} = -\rho_0 \frac{\partial \phi}{\partial t}$$

$$= -\rho_0 \omega \varsigma a \epsilon \cos\theta \frac{f(kr)}{f'(ka)}$$

from $\hat{z}\cdot\hat{n}$ and $\tilde{p}$

$$F_z^{react} = -a^2 \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \left(-\rho_0 \omega \varsigma a \epsilon \cos^2\theta \frac{f(kr)}{f'(kr)} \right)$$

$$= -\frac{4\pi}{3} \omega^2 a \epsilon \rho_0 a^3 \frac{f(ka)}{(ka) f'(ka)}$$

$$= \omega^2 a \epsilon (\rho_0 \bar{V}) \left[\frac{2 + k^2 a^2 + i k^3 a^3}{4 + k^4 a^4} \right]$$

$$\bar{V} = \frac{4\pi}{3} a^3$$

$$\rho_0 \bar{V} = M_{\substack{fluid \\ displ.}}$$

Now, $F_{ext_z}(\omega) + F_{react_z}(\omega) = M_{sph} \frac{d^2 z_{sph}}{dt^2}$

$$= -M_{sph} \omega^2 \epsilon a$$

so can re-write:

$$F_{ext_z}(\omega) = -\omega^2 \epsilon a \left[M_{sph} + \rho_0 V \underbrace{\left[\frac{2 + k^2a^2 + ik^3a^3}{4 + k^4a^4} \right]}_{\text{induced mass}} \right]$$

$$\underbrace{M^* = M_{eff}}_{\text{net effective mass}}$$

→ for $ka \to 0$ ⟺ must recover potential flow

$$ka \to 0 \quad M_{ind} = \rho_0 V \frac{2}{4} = \frac{\rho_0 V}{2} \quad ✓$$

→ finite ka ?! ⇒ M_{ind} is complex !!

observe if damping:

$$\underline{F}_{ext} - \eta \, \underline{y} = m \frac{dy}{dt}$$

$$F_{ext} = -\omega^2 m \tilde{z} + \underbrace{i\omega \eta \tilde{z}}_{\sim \text{ complex inertia!}}$$

$\therefore$ imaginary piece $M_{ind} \Leftrightarrow$ damping

Origin of damping $\Leftrightarrow$ radiation of energy to ∞

$\Leftrightarrow$ outgoing wave b.c.'s