Physics 4A
Lecture 8: Feb. 17, 2015

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UCSD Physics

What's In Common ?

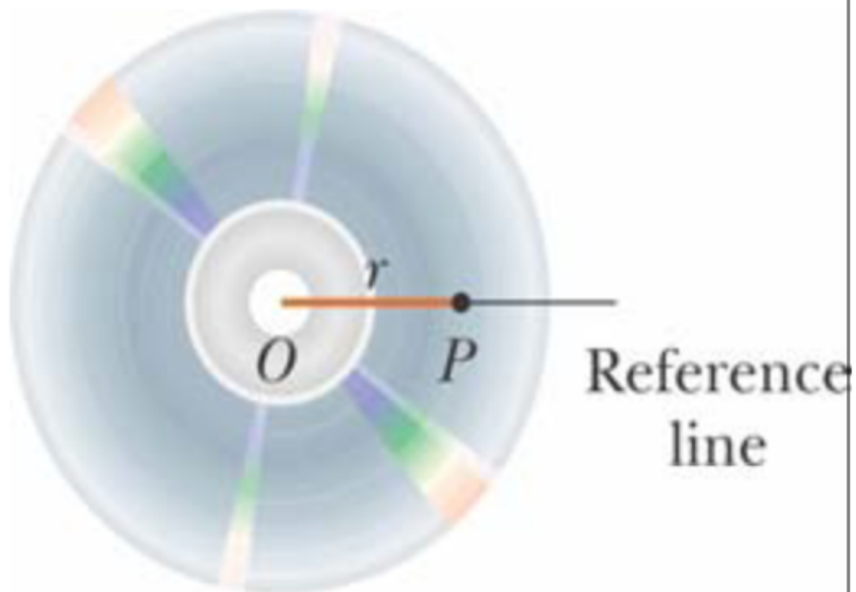
Each involves a body that rotates about a fixed axis

Rotation of Rigid Bodies

- Rigid body is one that is <u>non-deformable</u>
  - relative location of all particles making up the object remain constant
- All real objects are deformable so "rigid body" is an *idealized model* but a useful one
- This week:
  - kinematic language to describe rotation
    - Like chapter 2, derive equation of motion
  - Kinetic Energy in rotation
  - Dynamics of rotation

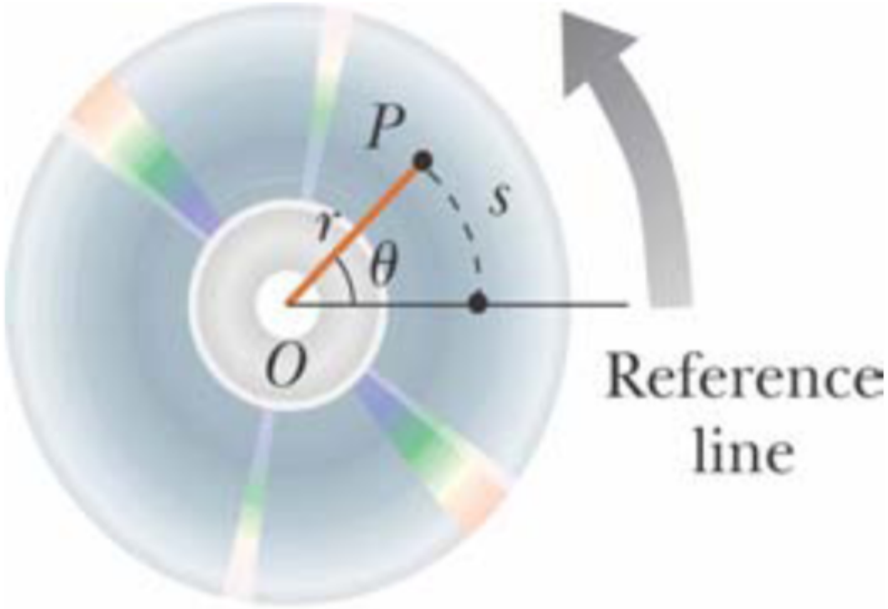
Angular Position In Rotation

- Axis of rotation passes thru center of disc & is ⊥ to plane of picture
- Choose a fixed reference line
- Point *P* is at a **fixed** distance *r* from the origin
- Point *P* will rotate about the origin in a circle of radius *r*



Change In Angular Position

- As the particle moves, the only coordinate that changes is θ
- As the particle moves through θ, it moves though an arc length s
- The arc length and r are related:
  - $s = r\,\theta$



Angular Displacement

Angular displacement $\theta = \dfrac{s}{r}$

θ is a pure number, but

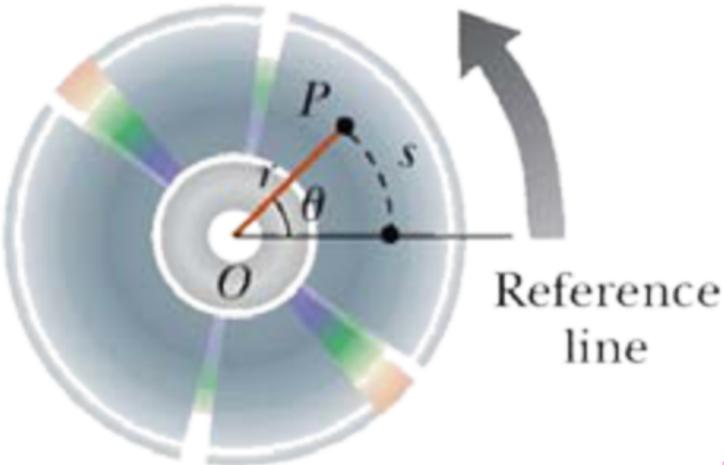
is given the artificial unit, radian

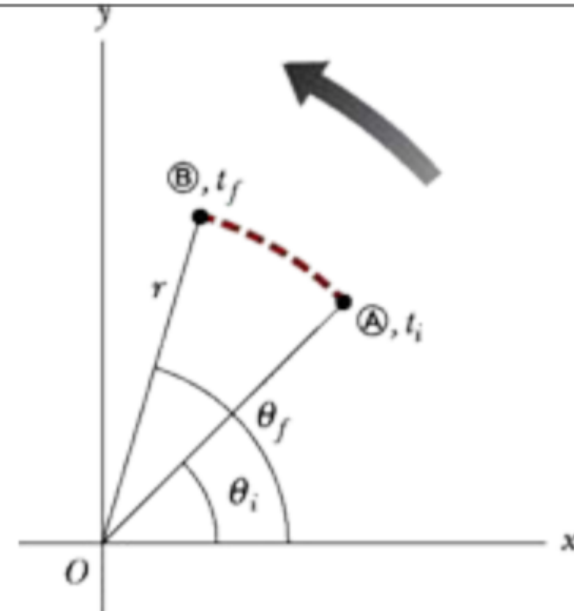
One radian is the angle subtended by an arc length equal to the radius of the arc

Comparing Degree and Radians:

$$1 \text{ rad} = \frac{360^{o}}{2\pi \text{ rad}} \simeq 57.3^{o}$$

Angular Velocity ω





$$\omega_{av-z} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

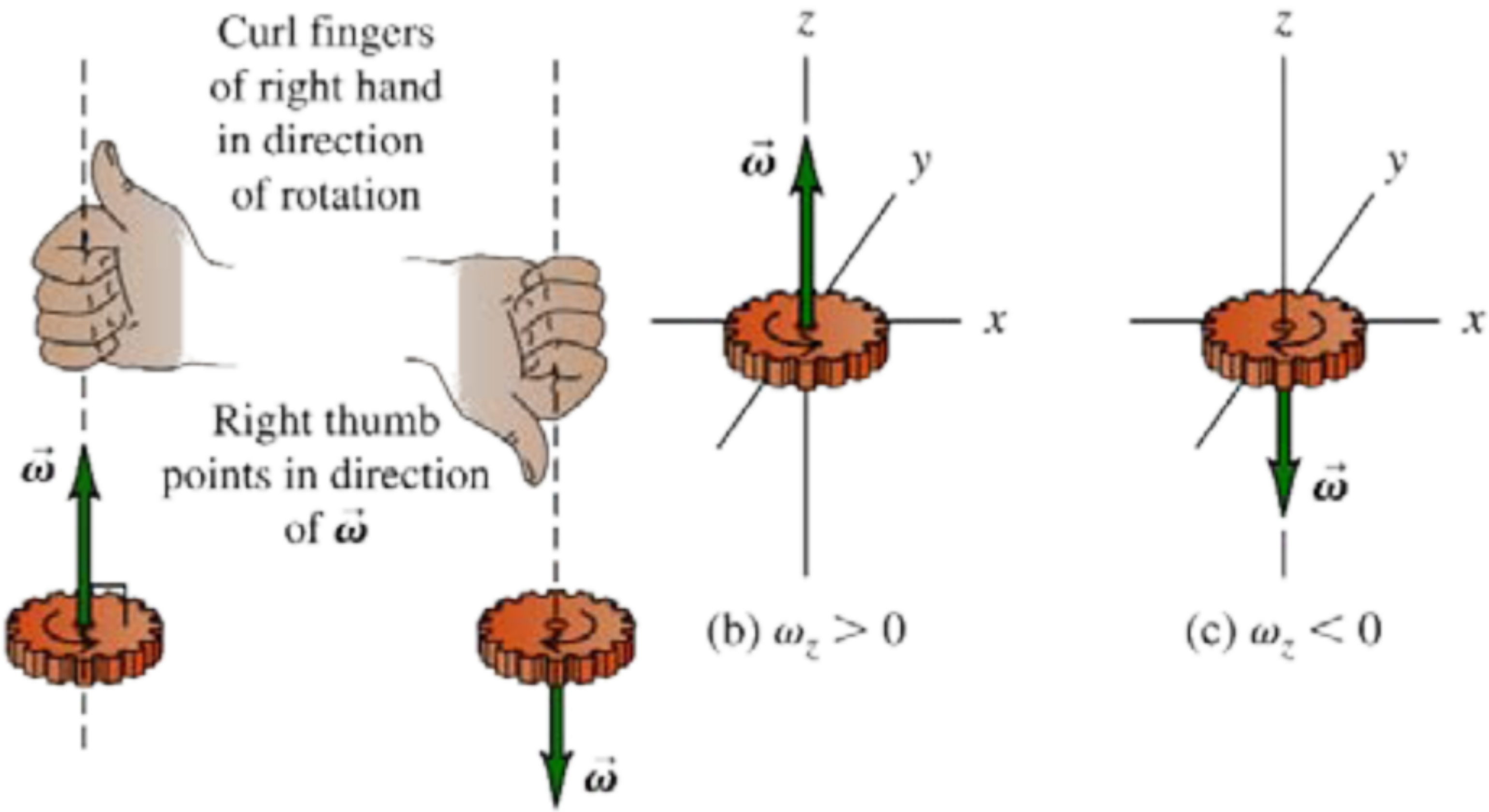
Unit of ω : rad/s

rpm : revs/second

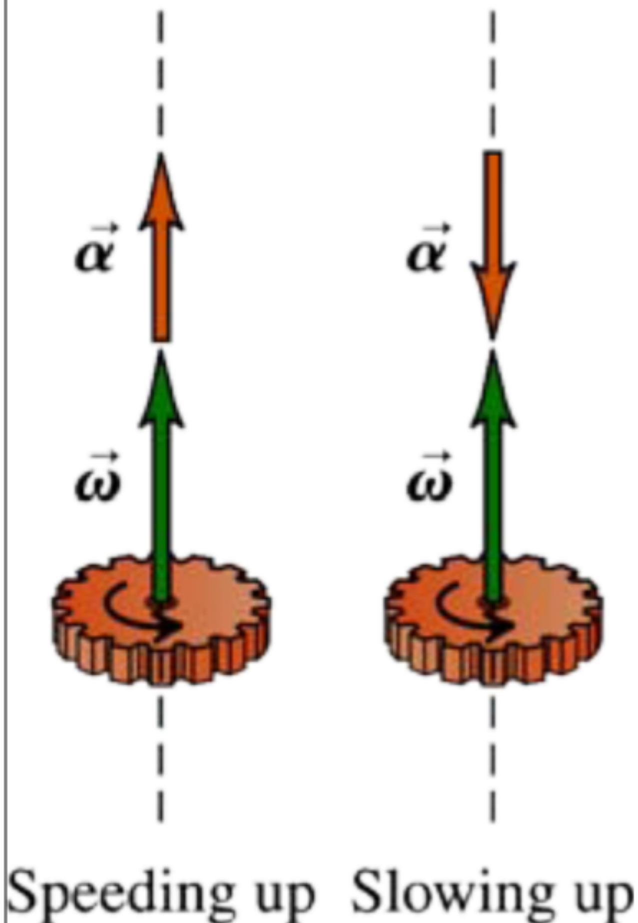
$$1\text{rpm} = \frac{2\pi}{60} rad / s$$

1 rad/s $\simeq$ 10 rpm

Angular Velocity $\vec{\omega}$ & Right Hand Rule



Angular Acceleration $\vec{\alpha}$



$$\alpha_{av-z} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega_z}{dt} = \frac{d^2\theta}{dt^2}$$

$$\boxed{\vec{\alpha}_z > 0 \Rightarrow \vec{\omega} \text{ increasing}}$$

Eqns. of Motion: Rotation with constant $\vec{\alpha}$

At t = 0, body at θ_0 moves with ang. vel.=$\vec{\omega}_{0z}$ & $\vec{\alpha}_z = const$

at time t=t, body at θ moves with ang. vel.=$\vec{\omega}_z$ & $\vec{\alpha}_z = const$

By def: $\alpha_z = \dfrac{\omega_z - \omega_{0z}}{t-0} \Rightarrow \omega_z = \omega_{0z} + \alpha_z t$

By def: $\omega_{av\text{-}z} = \dfrac{\omega_{0z} + \omega_z}{2} = \dfrac{\theta - \theta_0}{t-0} \Rightarrow \theta - \theta_0 = \dfrac{\omega_{0z} + \omega_z}{2} t$

substitute for $\omega_z \Rightarrow \theta - \theta_0 = (1/2)[\omega_{0z} + \omega_{0z} + \alpha_z t]t$

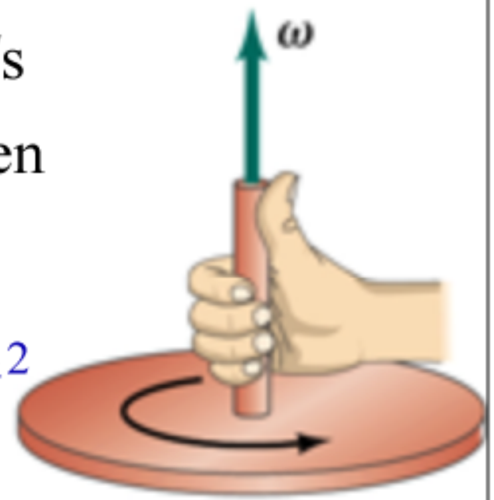
$\Rightarrow \theta - \theta_0 = \omega_{0z} t + (1/2)\alpha_z t^2$ ......for constant α_z

Substitute t =$(\omega_z - \omega_{0z})/\alpha_z \Rightarrow \omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$

Translational & Rotation Motion Compared

Straight-Line Motion with Constant Linear Acceleration	Fixed-Axis Rotation with Constant Angular Acceleratio
$a_x = \text{constant}$	$\alpha_z = \text{constant}$
$v_x = v_{0x} + a_x t$	$\omega_z = \omega_{0z} + \alpha_z t$
$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$	$\theta = \theta_0 + \omega_{0z} t + \frac{1}{2} \alpha_z t^2$
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_x + v_{0x})t$	$\theta - \theta_0 = \frac{1}{2}(\omega_z + \omega_{0z})t$

A wheel rotates with constant angular acceleration of $3.50\ \text{rad/s}^2$. If the angular speed of wheel is 2.00 rad/s at t = 0, thru what angle does the wheel rotate between t = 0 and t = 2s ? what's wheel's ang. speed after 2 s?



$$\theta = \theta_0 + \omega_{0z} t + \frac{1}{2}\alpha t^2 \Rightarrow \theta - \theta_0 = \omega_{0z} t + \frac{1}{2}\alpha t^2$$

$$\Delta\theta = (2.00\ \text{rad/s})(2.00\text{s}) + \frac{1}{2}(3.50\ \text{rad/s}^2)(2.00\text{s})^2$$

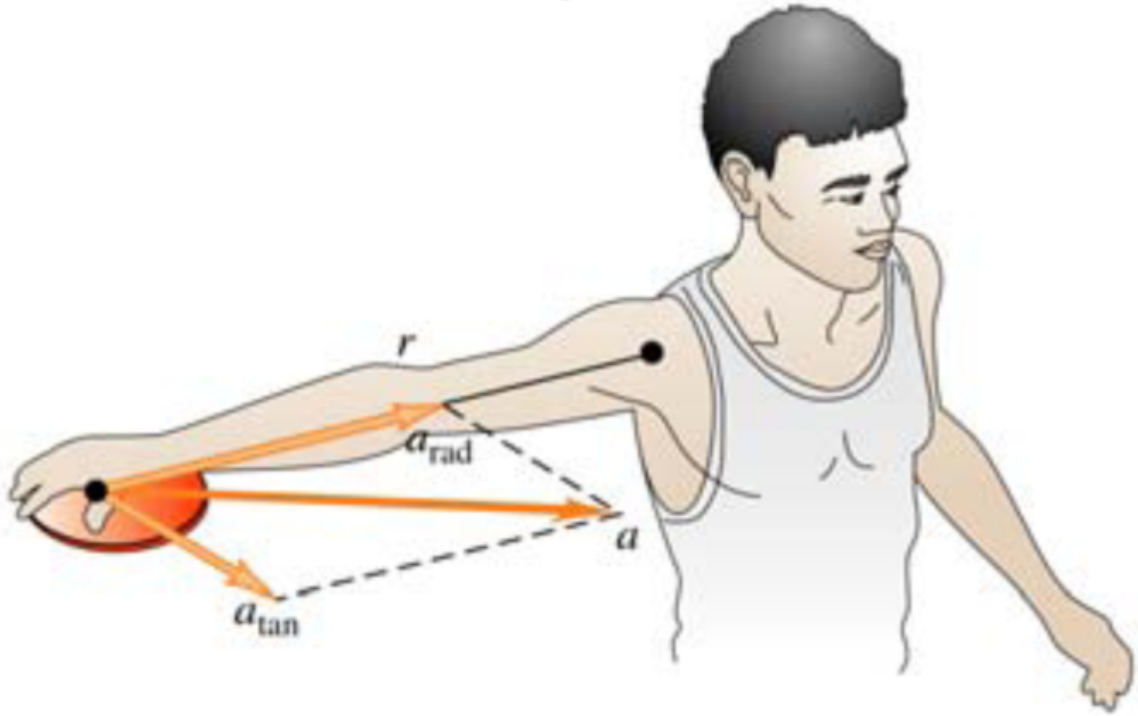
$$= 11.0\ \text{rad} \equiv \frac{11\ \text{rad}}{2\pi\ \text{rad/rev}} = 1.75\ \text{rev}$$

After 2 sec, ang. speed $\omega_z = \omega_{0z} + \alpha t$

$= 2.00\ \text{rad/s} + (3.50\ \text{rad/s}^2)(2.00s) = 9.00\ \text{rad/s}$

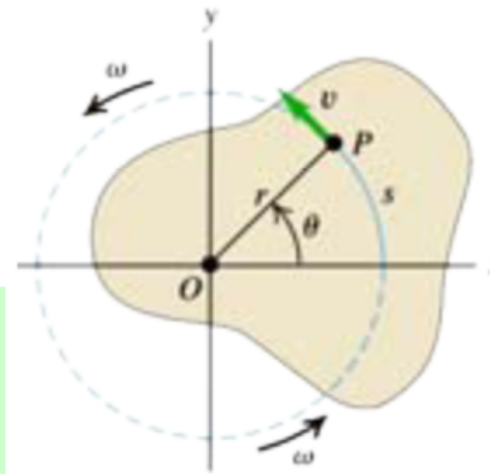
Relating Rotational & Linear Motion

Man moves discuss in circle of radius 80.0cm. At some instant when he is spinning at angular speed of 10.0rad/s, with the speed increasing at 50.0rad/s^2. What is tangential & radial component of acceleration of discuss & magnitude of acceleration



Relating Linear & Angular Kinetic Variables

consider a point P on rotating body



since s = rè $\Rightarrow \left|\frac{ds}{dt}\right| = r\left|\frac{d\theta}{dt}\right| \Rightarrow v = r\omega$

Tangential component of acceleration

$\vec{a}_{tan} \parallel \vec{v}$ changes magnitude of particle's

linear speed $|\vec{v}|$: $a_{tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$

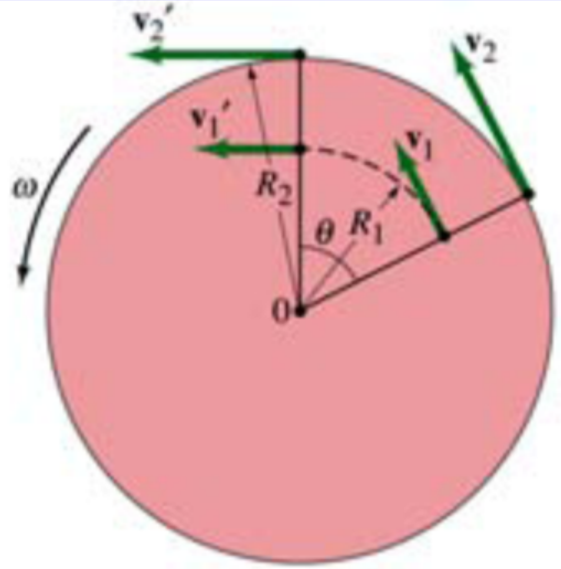
a_{rad} = centripetal component of body's acceleration

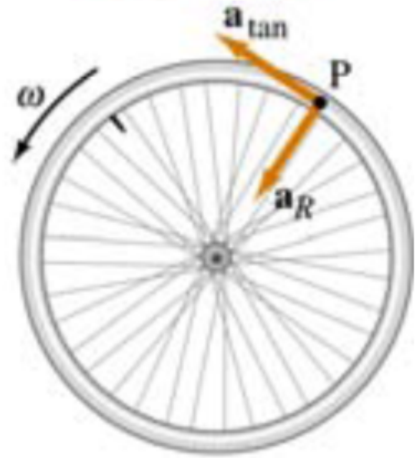
related to the change in direction of linear velocity

$a_{rad} = \frac{v^2}{r} = \frac{\omega^2 r^2}{r} = \omega^2 r$ (directed radially in)

Speed and Acceleration: Watch Out !

- All points on the rigid object will have the same *angular speed*, but **not** the same *tangential speed*
- All points on the rigid object will have the same *angular acceleration*, but **not** the same *tangential acceleration*
- The tangential quantities depend on ***r***, and *r* is not the same for all points on the object

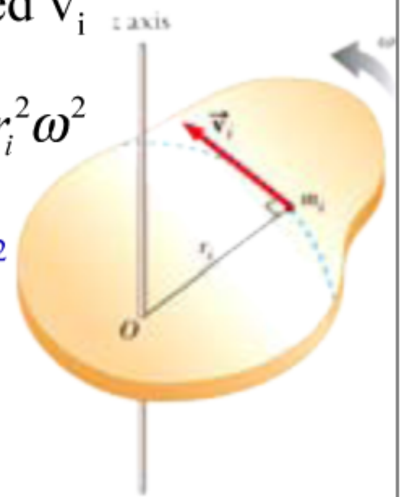




The Joy of Rotation !

Rotational Kinetic Energy

Rigid body=collection of particles of mass m_i, speed v_i



Kin. energy of ith particle $K_i = (1/2) m_i v_i^2 = (1/2) m_i r_i^2 \omega^2$

Total kin. energy $K = \sum_i K_i = \frac{1}{2}(\sum_i m_i r_i^2)\omega^2 = \frac{1}{2} I \omega^2$

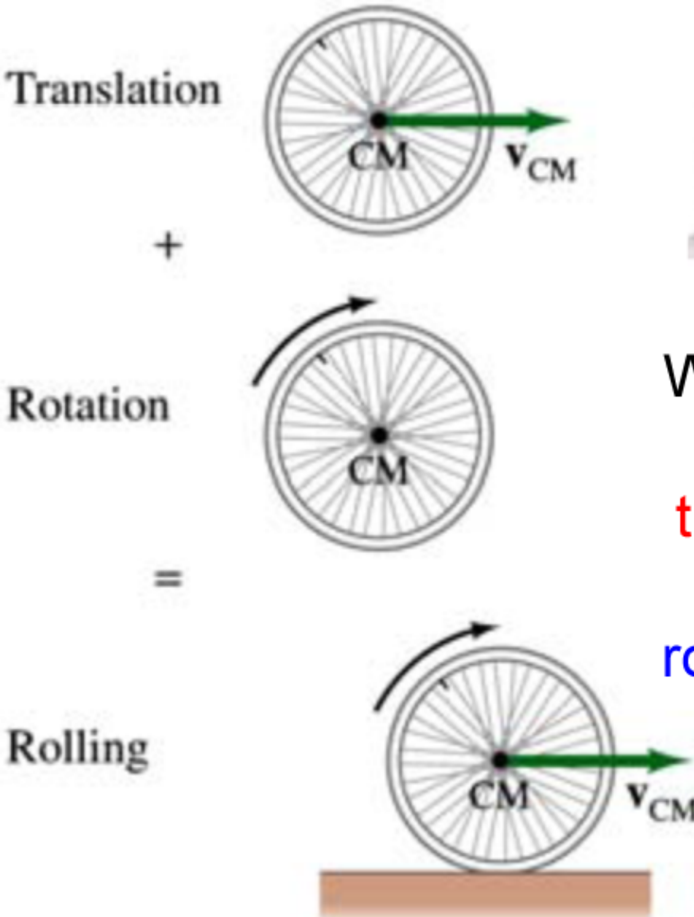
I = body's moment of Inertia for *this* rotation axis

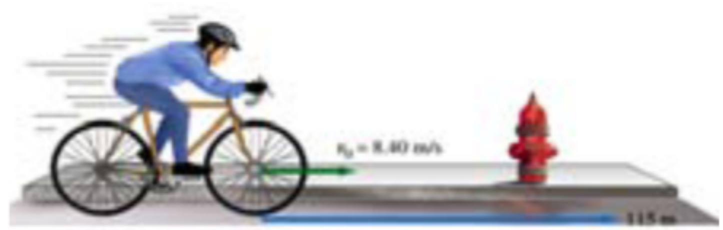
Moment of Inertia is measure of object's resistence to change in its angular speed. $\Rightarrow$ rotational inertia

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 .... = \sum_i m_i r_i^2$$

depends not just on the total mass but how it is arranged *around* the rotation axis

Rolling Motion Of A Ball or Wheel



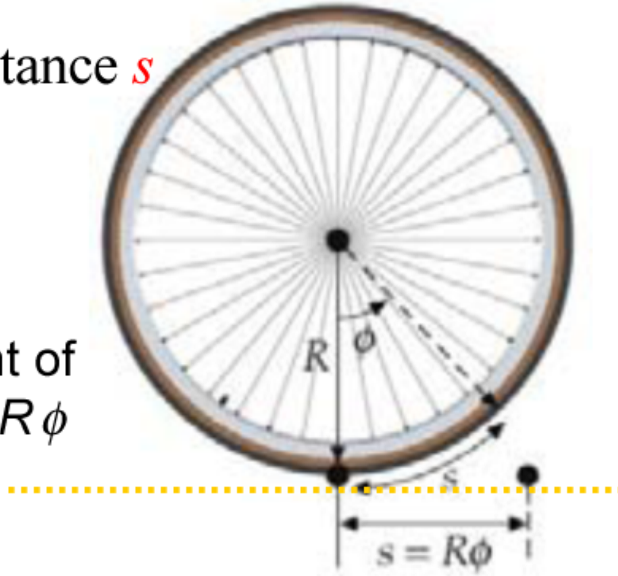


Wheel rolling without slipping can be considered as translation of the wheel as a whole with velocity v_{CM} + rotation about center of mass

Rolling Without Slipping

As a wheel rotates thru ϕ without slipping, point of contact between wheel and surface moves distance s

$$Then\ S = R\phi$$



If wheel rolling on flat surface then wheel's CM remains directly over point of contact, so it too moves a distance $s=R\phi$

$$\Rightarrow v_{CM} = R\omega$$

$$\text{and } a_{CM} = R\alpha$$

Pop Quiz

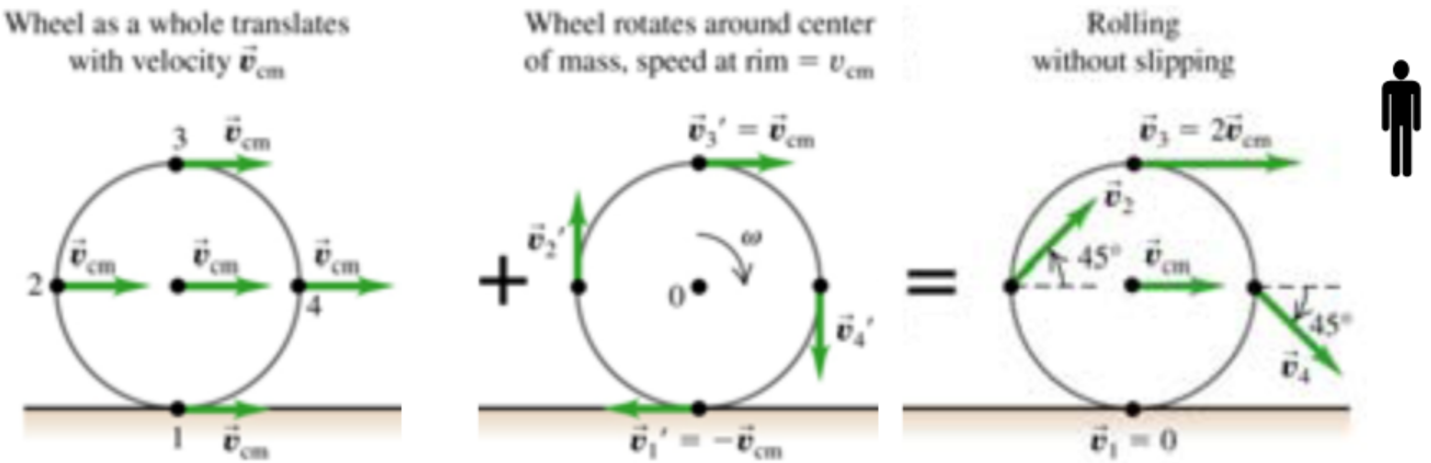
- A disc of radius 2 m is rolling on level ground without slipping. Its CM is moving horizontally at a velocity of 4 m/s. The instantaneous velocity of the top of the disc is
- (A) 8 m/s
- (B) 4 m/s
- (C) – 4 m/s
- (D) 0 m/s

Wheel (Radius R, Ang. speed ω) Rolls Without Slipping

Symmetric Wheel ⇒ CM= Geometric center

Imagine observer at rest w.r.t surface on which a wheel rolls

In Observer's frame: point on wheel touching surface must **instantaneously be at rest** so that it does not **slip**



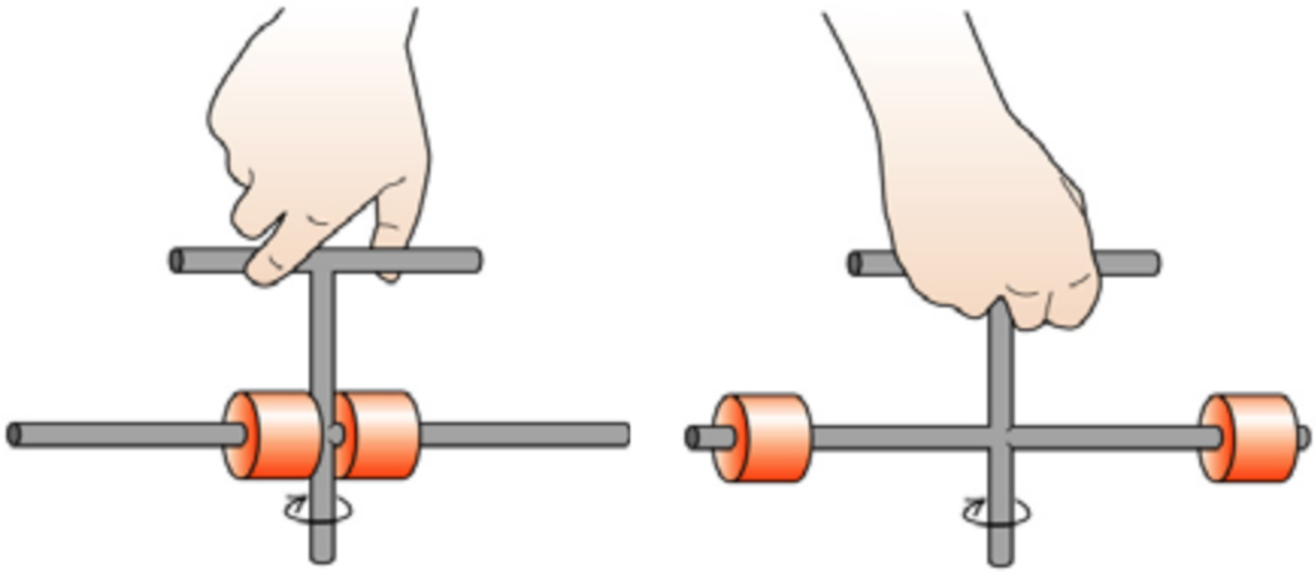
Velocity of point of contact $\vec{v}_i' = R\vec{\omega}$ (rel. to CM) must have same magnitude but opposite direction as $\vec{v}_{cm}$ ⇒ $\boxed{\vec{v}_{cm} = -R\vec{\omega}}$

Same Total Mass, Different Moment of Inertia

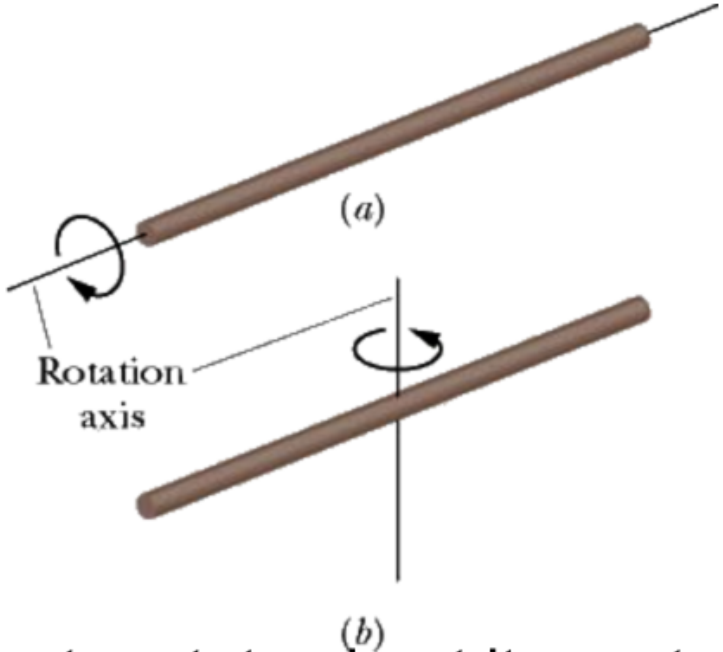
$$I = m_1 r_1^2 + m_2 r_2^2$$

- Mass close to axis
- Small moment of inertia
- Easy to start apparatus rotating

- Mass farther from axis
- Greater moment of inertia
- Harder to start apparatus rotating



Diff. Rotation Axis ⇒ Diff. MOI for Same Body



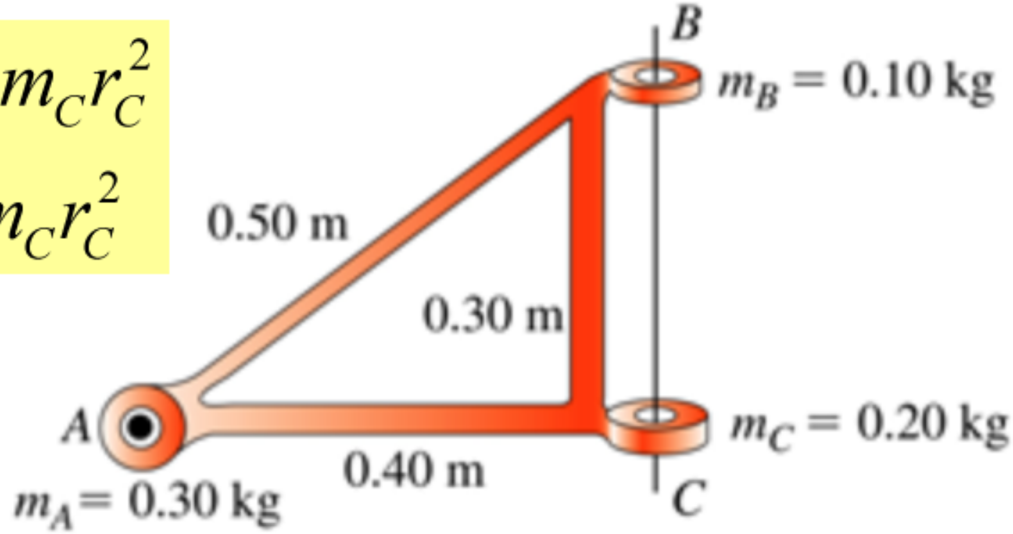
Long rod easier to rotate about its central axis (longitudinal) axis (a) that about an axis (b) through its center and ⊥ to its length

Because mass is distributed closer to longitudinal axis (a) than (b)

Diff. Rotation Axis $\Rightarrow$ Diff. MOI for Same Body

3 heavy connectors linked by "massless" plastic strut

$$I_A = m_A r_A^2 + m_B r_B^2 + m_C r_C^2$$
$$= 0 + m_B r_B^2 + m_C r_C^2$$

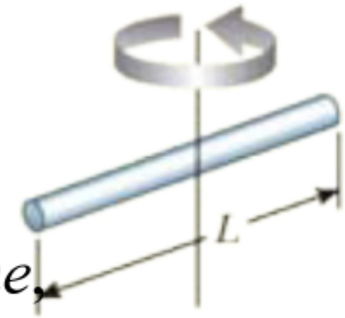


$$I_B = m_A r'^2_A + m_B r'^2_B + m_C r'^2_C$$
$$= m_A r'^2_A + 0 + 0$$

$$I_A \neq I_B. \text{ Since } K = \frac{1}{2} I\omega^2 \Rightarrow K_A \neq K_B$$

MOI For A Continuous Distribution Of Mass

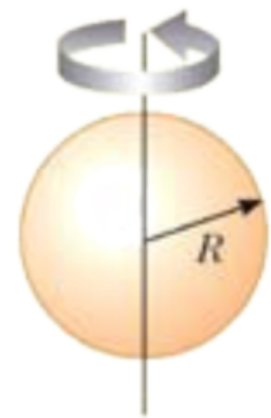
$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$$



For objects of uniform density $\rho = mass / volume$,

easiest to express dm = ρdV

$$I = \int r^2 dm = \int r^2 \rho dv = \rho \int r^2 dv$$



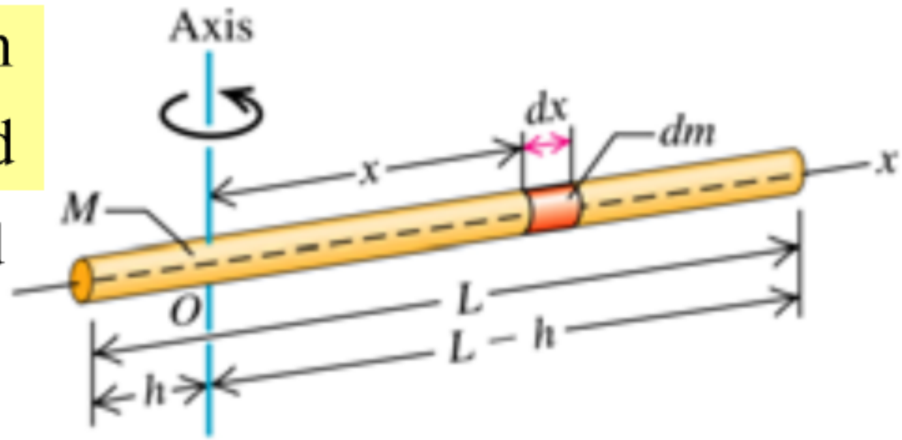
Volume element dv = dxdydz

Limits of integration defined by

shape & dimension of the object

Moment Of Inertia For Uniform Thin Rod

Calculate I about axis of rotation thru O, distance h from rod's end



Mass uniformly distributed on rod

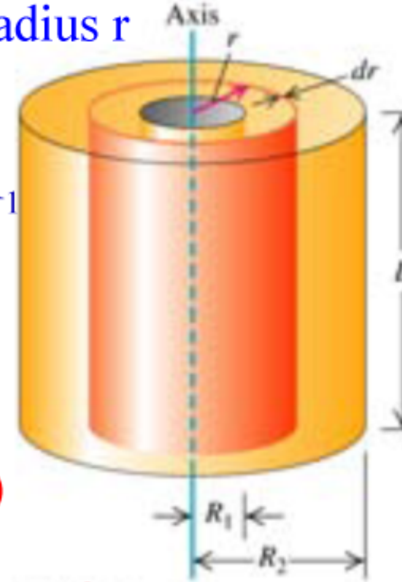
$$\frac{\mathrm{dm}}{\mathrm{M}} = \frac{dx}{L} \Rightarrow \mathrm{dm} = \frac{M}{L} dx$$

$$\mathrm{I_h} = \int x^2 dm = \frac{M}{L} \int_{-h}^{L-h} x^2 dx = \left[\frac{M}{L} \left(\frac{x^3}{3} \right)_{-h}^{L-h} \right] = \frac{1}{3} M (L^2 - 3Lh - 3h^2)$$

MOI if rod rotating thru left (h=0) or right (h=L) end:

$$\mathrm{I_{0,L}} = \frac{1}{3} ML^2$$

Moment Of Inertia For Thick Hollow Cylinder



Pick as volume element a thin cylindrical shell of radius r thickness dr and length L $\Rightarrow dV=(2\pi r)Ldr$

$\Rightarrow dm=\rho(2\pi r)Ldr$. Cylinder thickness $= R_2 - R_1$

$$I = \int_{R_1}^{R_2} r^2 \rho(2\pi r)Ldr = 2\pi\rho L \int_{R_1}^{R_2} r^3 dr$$

$$= \frac{2\pi\rho L}{4}(R_2^4 - R_1^4) = \frac{2\pi\rho L}{4}(R_2^2 - R_1^2)(R_2^2 + R_1^2)$$

Cylinder volume $V=\pi L(R_2^2 - R_1^2) \Rightarrow M = \rho\pi L(R_2^2 - R_1^2)$

$$\Rightarrow \boxed{I=\frac{1}{2}M(R_2^2 + R_1^2)}$$

For a solid cylinder $R_1=0 \Rightarrow I=\frac{1}{2}MR^2$

Pop Quiz

- A rope is being wound without slipping onto a cylinder of radius 2 m which is rotating at 1 revolution/minute. The speed with which a point on the rope is moving is
- (A) 2/60 m/s
- (B) $4\pi/60$ m/s
- (C) $60/2\pi$ m/s
- (D) 4π m/s

Cable Unwinding Off A Cylinder

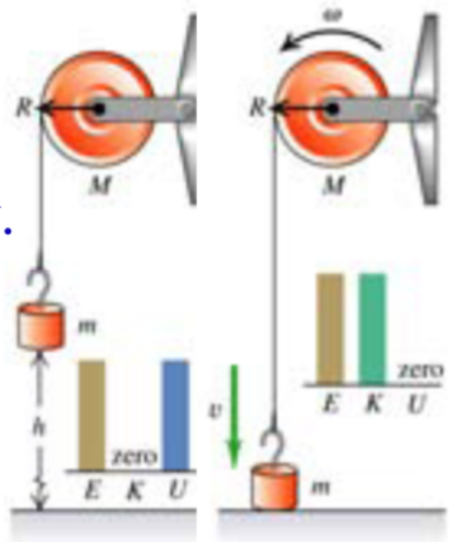
Cable wrapped around cylinder (mass M) is attached to object mass m. As cable unwinds, U_{grav} converted to kinetic energy. Find speed of object as it hits floor

$$K_1 + U_1 = K_2 + U_2$$

$$0 + mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + 0$$

$$\Rightarrow mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2$$

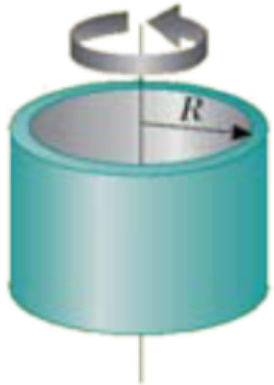
$$\Rightarrow v = \sqrt{\frac{2gh}{1 + M/2m}}$$



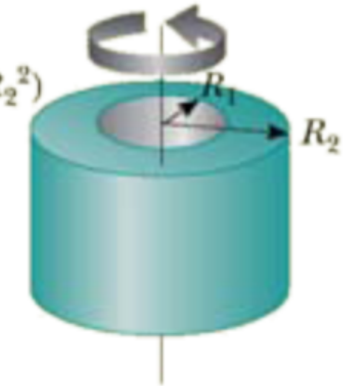
Cylinder is Solid

Moments of Inertia of Homogeneous Rigid Objects With Different Geometries

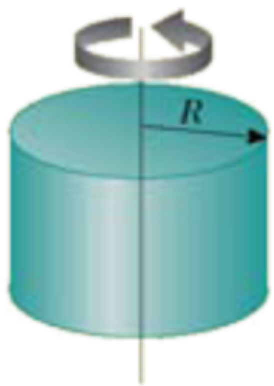
Hoop or thin cylindrical shell
$I_{CM} = MR^2$



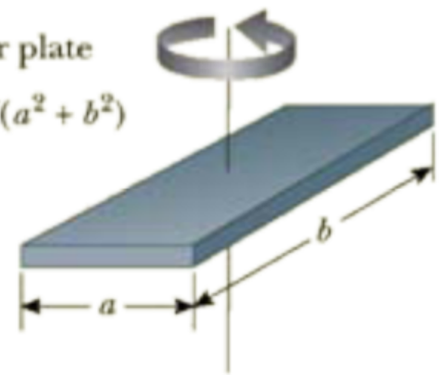
Hollow cylinder
$I_{CM} = \frac{1}{2}M(R_1^2 + R_2^2)$



Solid cylinder or disk
$I_{CM} = \frac{1}{2}MR^2$



Rectangular plate
$I_{CM} = \frac{1}{12}M(a^2 + b^2)$



Moment of Inertia of rectangle about normal axis through center

$$I_{strip} = \frac{1}{12} b^2 dm$$

$$dm = \sigma b dy$$

$$I_{strip} = \frac{1}{12} b^2 \sigma b dy$$

$$dI_{cm} = I_{strip} + y^2 dm = \frac{1}{12} b^2 \sigma b dy + y^2 \sigma b dy$$

$$I_{cm} = \sigma b \int_{-a/2}^{a/2} [\frac{1}{12} b^2 + y^2] dy = \sigma b [\frac{1}{12} b^2 a + \frac{1}{12} a^3]$$

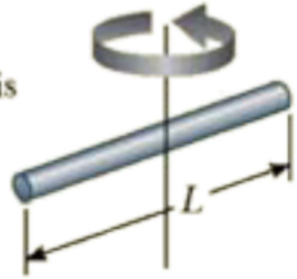
$$= \frac{1}{12} \sigma ab [a^2 + b^2] = \frac{1}{12} M [a^2 + b^2]$$

$$(M = \sigma ab)$$

Moment of Inertia of Objects About Their Axis Of Rotation

Long thin rod with rotation axis through center

$I_{CM} = \frac{1}{12} ML^2$

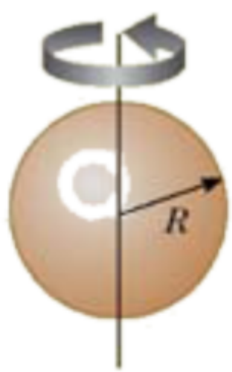


Long thin rod with rotation axis through end

$I = \frac{1}{3} ML^2$

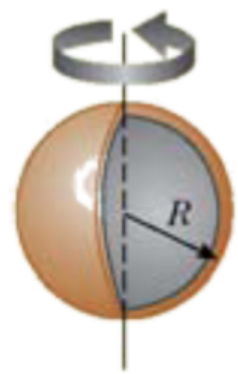
Solid sphere

$I_{CM} = \frac{2}{5} MR^2$

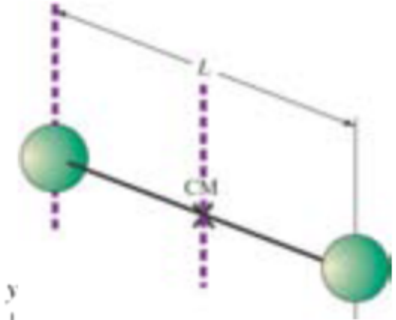


Thin spherical shell

$I_{CM} = \frac{2}{3} MR^2$



Parallel Axes Theorem



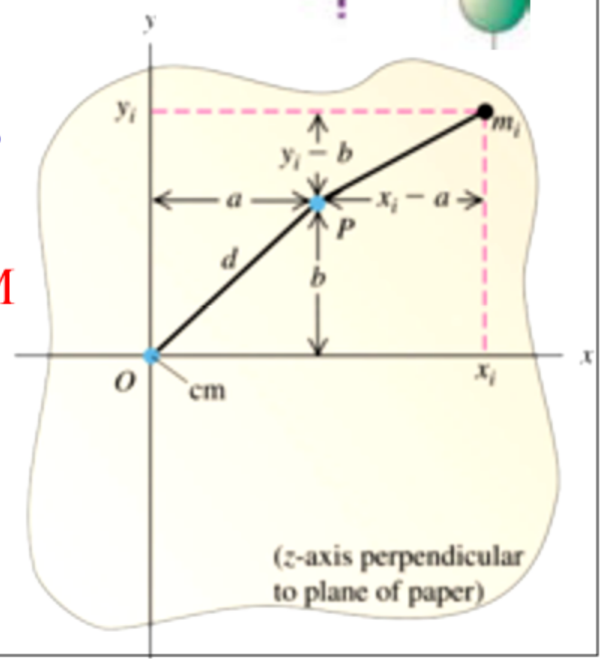
A body does not have only one moment of Inertia, its has many!

As many as the *many* axes about which it can rotate

Relation between MOI about an axis of rotation thru its center of mass &

MOI about any other axis || to C.O.M *axis* but displaced by distance d

$$I_P = I_{cm} + Md^2$$



Moment of inertia of rectangle about in-plane axis going through center of side of length a = (1/12) Ma^2

Moment of inertia of rectangle about in-plane axis going through center of side of length b = (1/12) Mb^2

Torque $\vec{\tau}$: Rotational Analog Of Force

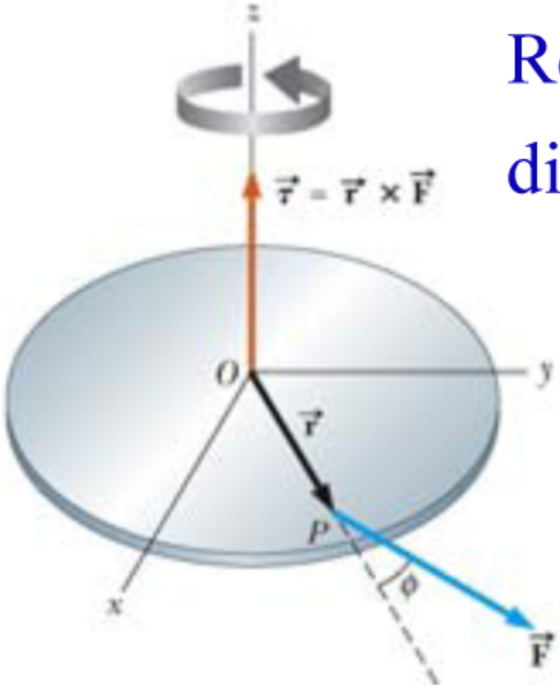
Torque at a point, due to an applied force causes angular acceleration

$\vec{\tau}$ lies in direction

$\perp$ to plane formed by

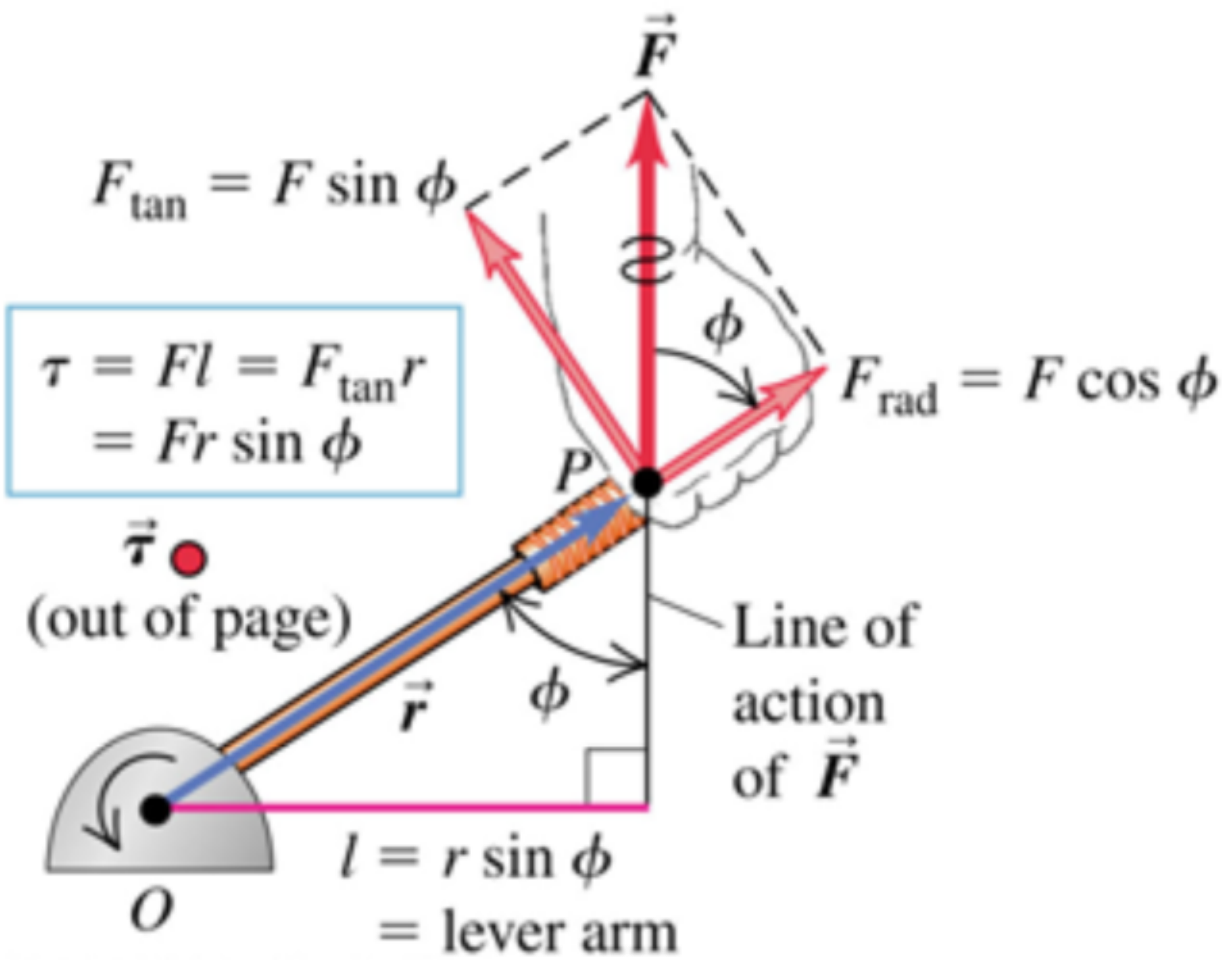
position vector $\vec{r}$ &

applied force $\vec{F}$



Rotation in

direction of $\vec{\tau}$

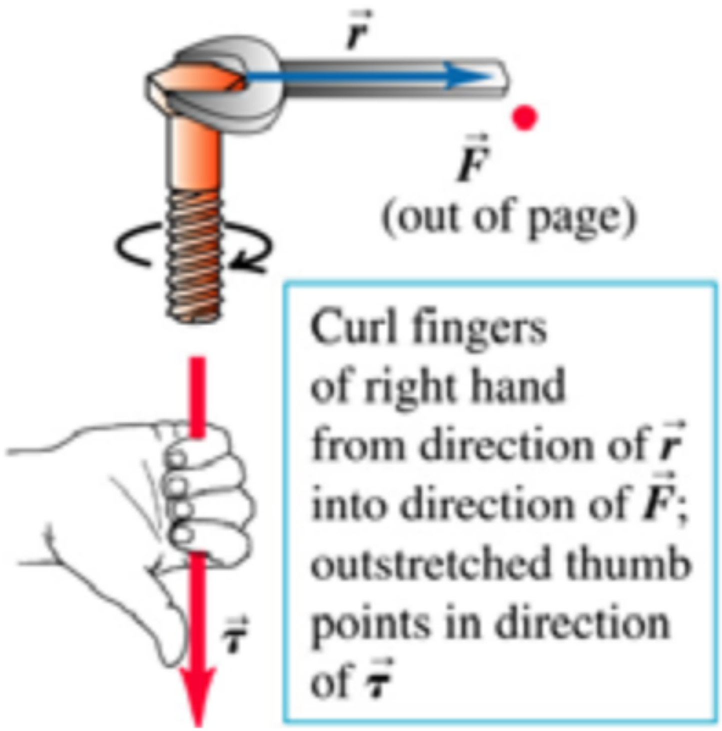


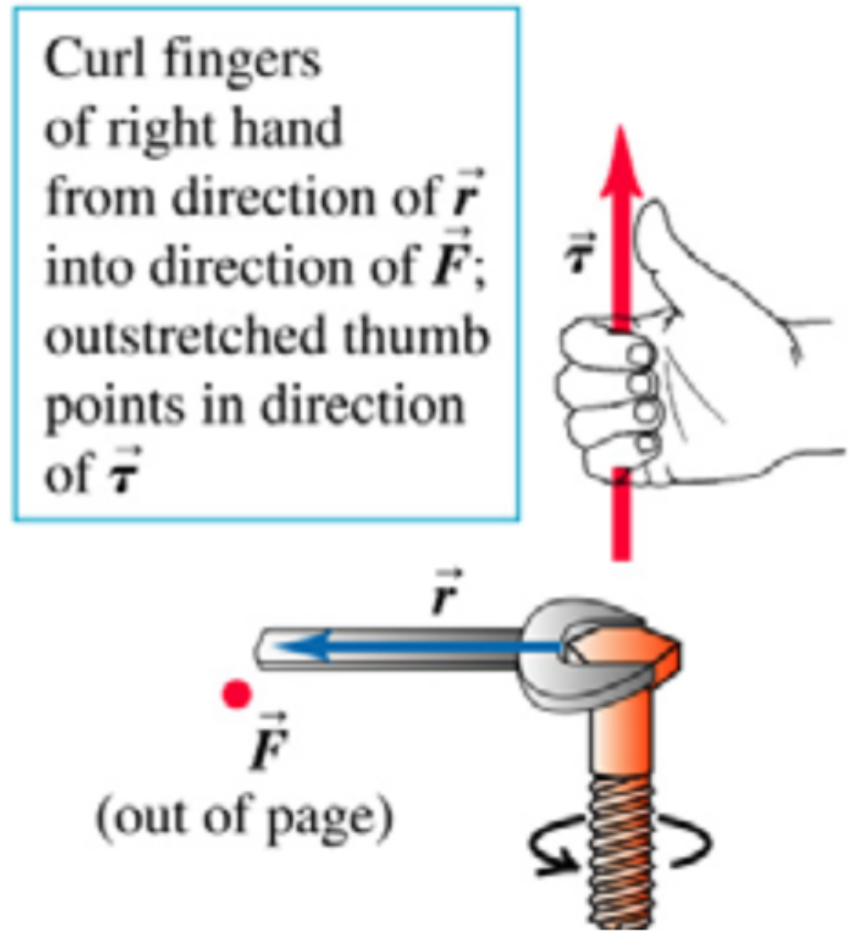
Lever arm = ⊥ distance between axis of rotation & line of action of force

Direction of The Torque Vector

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$|\vec{\tau}| = rF\sin\theta$$





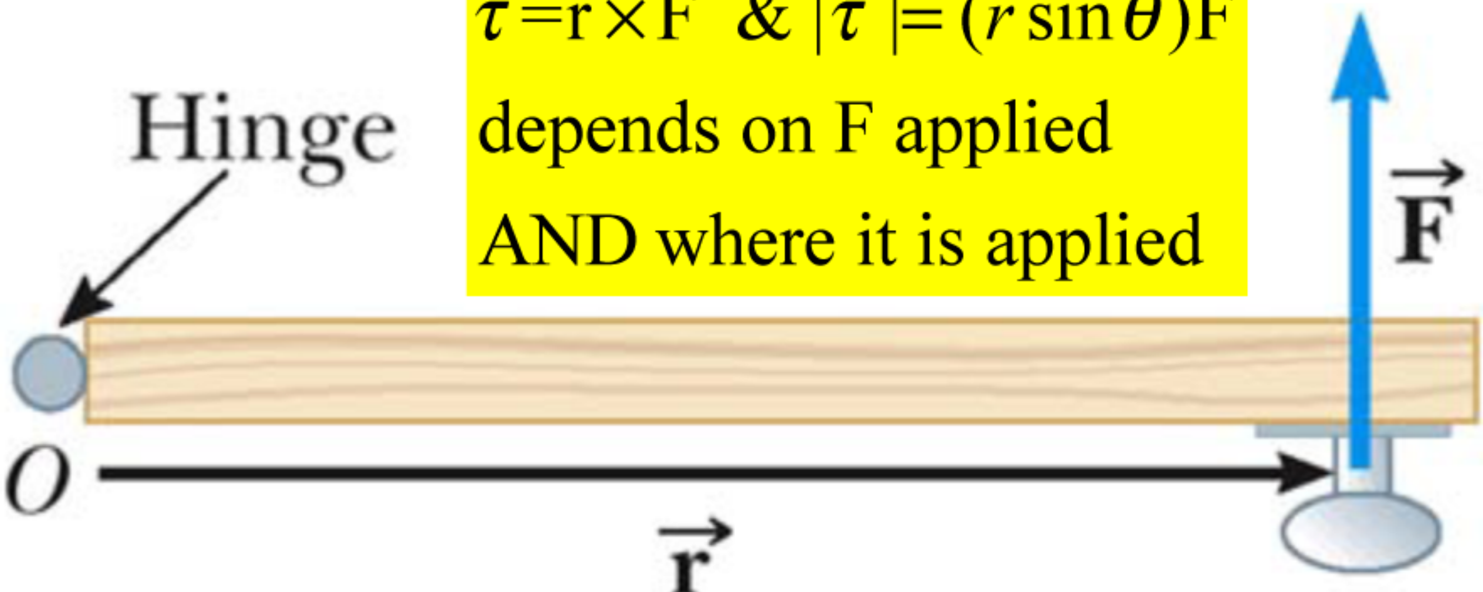
Door Knob & The Hinge

Why are ALL doorknobs always located the furthest from the hinge around which they rotate ?

$\vec{\tau} = \vec{r} \times \vec{F}$ & $|\vec{\tau}| = (r \sin\theta)F$

depends on F applied

AND where it is applied



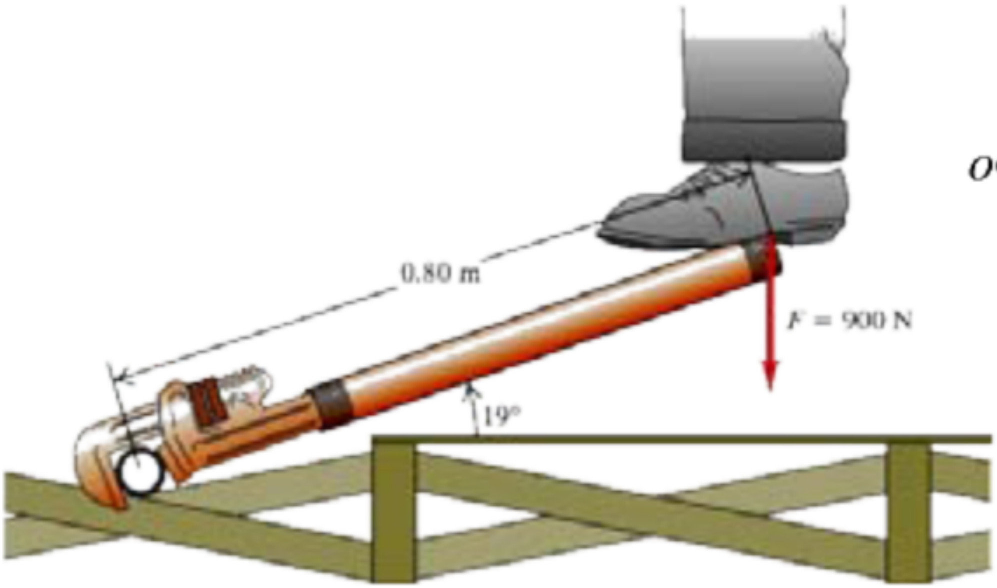
Lever arm = $\perp$ distance between axis of rotation & line of action of force

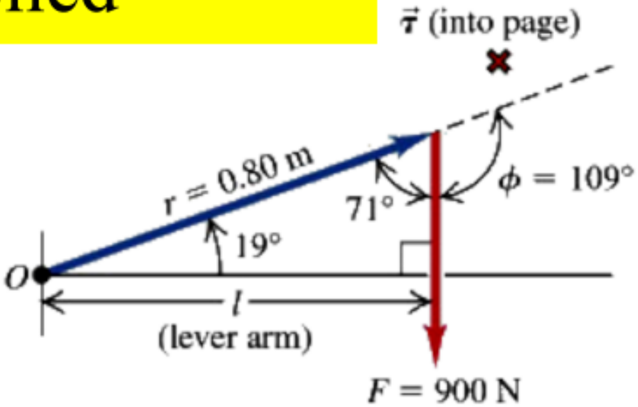
Larger the r $\Rightarrow$ less effort (F) needed

How to Get *More* Out Of Same

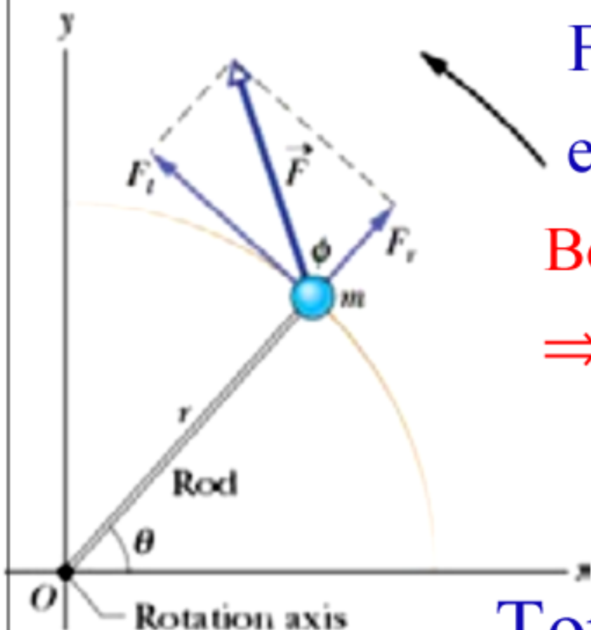
$\vec{\tau} = \vec{r} \times \vec{F}$ & $|\vec{\tau}| = (r \sin\theta) F$

Torque depends on force applied AND where it is applied





Torque & Angular Acceleration



Force $\vec{F}$ acts on body of mass m on one end of a massless rod of length r

Body rotates around an axis ⊥ to x-y plane

⇒ circular motion in x-y plane

$$F_{tangential} = ma_{tangential}$$

$$\text{Torque } \tau = F_{tangential} r = ma_{tangential} r$$

$$\Rightarrow \tau = m(r\alpha)r = (mr^2)\alpha \quad (\alpha \text{ in radians!})$$

$$\text{since } I = mr^2 \Rightarrow \boxed{\vec{\tau} = I\vec{\alpha}}$$

Newton's 2nd Law For Rotational Motion

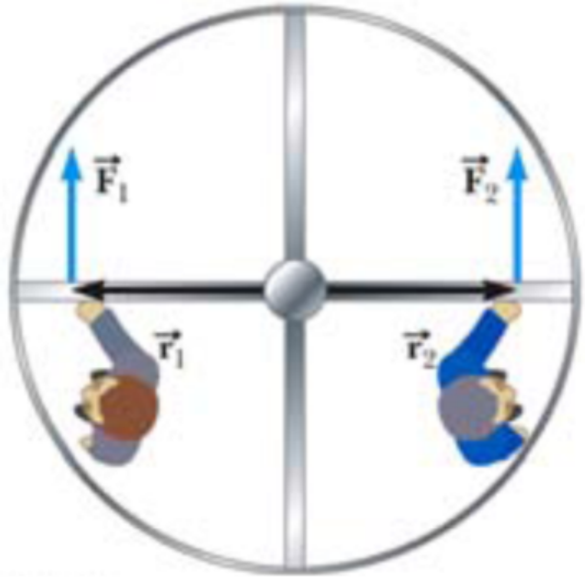
Battle Of The Revolving Door !

Two fired Trump apprentices are trying to use a revolving door. If $|r_1| > |r_2|$ & $|F_1| = |F_2|$, which way will the door turn?

If $\tau_1 = -r_1 F_1$ then $\tau_2 = +r_2 F_2$

$$\sum \vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 = -r_1 F + r_2 F < 0$$

$\Rightarrow$ Net torque is negative, will produce a clockwise rotation with downward angular acceleration $\vec{\alpha}$



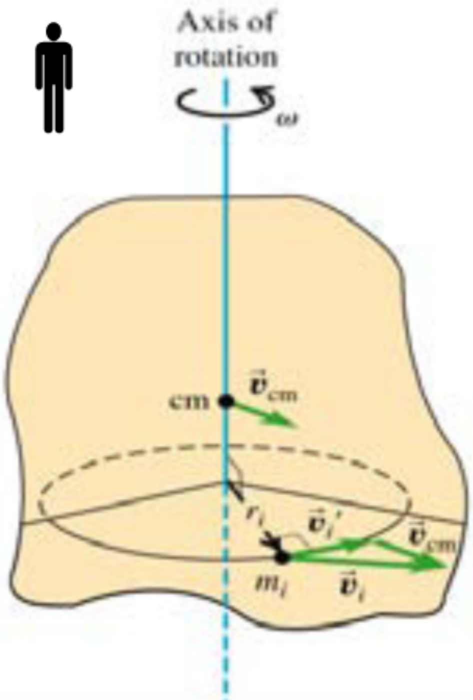
Which will no doubt upset the blue suit

Rigid Body Rotation About a Moving Axis

A rigid body's motion = sum of translation motion $\vec{v}_{CM}$ of CM

& rotation about an axis through the CM

Component particle m_i at $\vec{r}_i$ has $\vec{v}_i = \vec{v}_{cm} + \vec{v}_i'$ ⇐ vel. rel. to CM



$$K_i = \frac{1}{2} m_i (\vec{v}_{cm} + \vec{v}_i').(\vec{v}_{cm} + \vec{v}_i')$$

$$\Rightarrow K_i = \frac{1}{2} m_i (v_{cm}^2 + 2\vec{v}_{cm}.\vec{v}_i' + v_i'^2)$$

$$\Rightarrow K = \frac{1}{2}(\sum_i m_i) v_{cm}^2 + \vec{v}_{cm}.(\sum_i m_i \vec{v}_i') + \sum_i \left(\frac{1}{2} m_i v_i'^2 \right)$$

Since $\sum_i m_i \vec{v}_i' = \mathrm{M} \times$ Vel. of CM relative to CM = 0

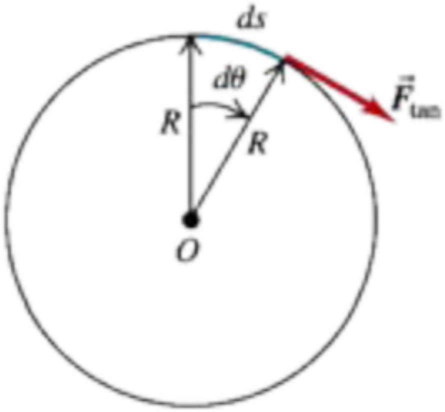
$$\Rightarrow \quad K = \frac{1}{2} M v_{cm}^2 + 0 + \frac{1}{2} I_{cm} \omega^2$$

Work Done By Torque In Rotational Motion

Tangential force $\vec{F}$ over time dt applied at rim of a disk causes torque $\vec{\tau}$, leads to ang. displacement dè

Work done $dW = F_{tan} ds = F_{tan} R d\theta$

$$\Rightarrow dW = \tau_z d\theta \Rightarrow W = \int_{\theta_1}^{\theta_2} \tau_z d\theta$$



Overhead view
of merry-go-round

If applied torque is constant

$$\Rightarrow W = \int_{\theta_1}^{\theta_2} \tau_z d\theta = \tau_z (\theta_2 - \theta_1)$$

Work & Power In Rotational Motion

As result of work done by $\vec{\tau}$, kinetic energy changes

Since $\vec{\tau} = I\vec{\alpha}_z \Rightarrow \tau_z d\theta = I\alpha_z d\theta$

$$\tau_z d\theta = I\frac{d\omega_z}{dt}d\theta = I\frac{d\theta}{dt}d\omega = I\omega_z d\omega_z$$

$$\Rightarrow W = \int_{\theta_1}^{\theta_2} \tau_z d\theta = \int_{\omega_1}^{\omega_2} I\omega_z d\omega_z = \frac{1}{2}I(\omega_2^2 - \omega_1^2) = \Delta K$$

work-energy theorem for rotating rigid bodies

Power associated with applied external torque:

$$\text{P}=\frac{\text{dW}}{\text{dt}} = \tau_z \frac{d\theta}{dt} = \tau_z\omega_z$$

Cable Unwinding Off A Cylinder

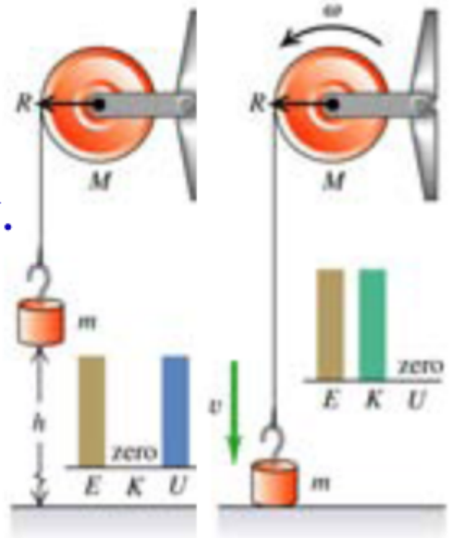
Cable wrapped around cylinder (mass M) is attached to object mass m. As cable unwinds, U_{grav} converted to kinetic energy. Find speed of object as it hits floor

$$K_1 + U_1 = K_2 + U_2$$

$$0 + mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + 0$$

$$\Rightarrow mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2$$

$$\Rightarrow v = \sqrt{\frac{2gh}{1 + M/2m}}$$



Cylinder is Solid