What's In Common?

Each involves a body that rotates about a fixed axis.
Rotation of Rigid Bodies

- Rigid body is one that is non-deformable
  - relative location of all particles making up the object remain constant
- All real objects are deformable so “rigid body” is an idealized model but a useful one
- This week:
  - kinematic language to describe rotation
    - Like chapter 2, derive equation of motion
  - Kinetic Energy in rotation
  - Dynamics of rotation
Angular Position In Rotation

- Axis of rotation passes thru center of disc & is \( \perp \) to plane of picture
- Choose a fixed reference line
- Point \( P \) is at a fixed distance \( r \) from the origin
- Point \( P \) will rotate about the origin in a circle of radius \( r \)
Change In Angular Position

• As the particle moves, the only coordinate that changes is $\theta$
• As the particle moves through $\theta$, it moves though an arc length $s$
• The arc length and $r$ are related:
  $s = r \theta$
Angular Displacement

Angular displacement \( \theta = \frac{s}{r} \)

\( \theta \) is a pure number, but is given the artificial unit, radian

One radian is the angle subtended by an arc length equal to the radius of the arc

Comparing Degree and Radians:

\[ 1 \text{ rad} = \frac{360^\circ}{2\pi \text{ rad}} \approx 57.3^\circ \]
Angular Velocity $\omega$

\[
\omega_{av-z} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta \theta}{\Delta t}
\]

\[
\omega_z = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}
\]

Unit of $\omega$: rad/s

rpm: revs/second

\[
1 \text{ rpm} = \frac{2\pi}{60} \text{ rad/s}
\]

1 rad/s $\approx 10$ rpm
Angular Velocity \( \vec{\omega} \) & Right Hand Rule

Curl fingers of right hand in direction of rotation.

Right thumb points in direction of \( \vec{\omega} \).

(b) \( \omega_z > 0 \)

(c) \( \omega_z < 0 \)
Angular Acceleration $\ddot{\alpha}$

$\alpha_{av-z} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta \omega}{\Delta t}$

$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega_z}{dt} = \frac{d^2 \theta}{dt^2}$

$\ddot{\alpha}_z > 0 \Rightarrow \ddot{\omega}$ increasing

Speeding up  Slowing up
Eqns. of Motion: Rotation with constant $\ddot{\alpha}$

At $t = 0$, body at $\theta_0$ moves with ang. vel. $= \omega_{0z}$ & $\ddot{\alpha}_z = \text{const}$

At time $t = t$, body at $\theta$ moves with ang. vel. $= \dot{\omega}_z$ & $\dddot{\alpha}_z = \text{const}$

By def: $\alpha_z = \frac{\omega_z - \omega_{0z}}{t - 0} \Rightarrow \omega_z = \omega_{0z} + \alpha_z t$

By def: $\omega_{av-z} = \frac{\omega_{0z} + \omega_z}{2} = \frac{\theta - \theta_0}{t - 0} \Rightarrow \theta - \theta_0 = \frac{\omega_{0z} + \omega_z}{2} t$

substitute for $\omega_z \Rightarrow \theta - \theta_0 = (1/2)[\omega_{0z} + \omega_{0z} + \alpha_z t] t$

$\Rightarrow \theta - \theta_0 = \omega_{0z} t + (1/2)\alpha_z t^2$ ......for constant $\alpha_z$

Substitute $t = (\omega_z - \omega_{0z}) / \alpha_z \Rightarrow \omega_z^2 = \omega_{0z}^2 + 2\alpha_z (\theta - \theta_0)$
### Translational & Rotation Motion Compared

<table>
<thead>
<tr>
<th>Straight-Line Motion with Constant Linear Acceleration</th>
<th>Fixed-Axis Rotation with Constant Angular Acceleration</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ a_x = \text{constant} ]</td>
<td>[ \alpha_z = \text{constant} ]</td>
</tr>
<tr>
<td>[ v_x = v_{0x} + a_x t ]</td>
<td>[ \omega_z = \omega_{0z} + \alpha_z t ]</td>
</tr>
<tr>
<td>[ x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2 ]</td>
<td>[ \theta = \theta_0 + \omega_{0z} t + \frac{1}{2} \alpha_z t^2 ]</td>
</tr>
<tr>
<td>[ v_x^2 = v_{0x}^2 + 2a_x(x - x_0) ]</td>
<td>[ \omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0) ]</td>
</tr>
<tr>
<td>[ x - x_0 = \frac{1}{2}(v_x + v_{0x})t ]</td>
<td>[ \theta - \theta_0 = \frac{1}{2}(\omega_z + \omega_{0z})t ]</td>
</tr>
</tbody>
</table>
A wheel rotates with constant angular acceleration of 3.50 rad/s². If the angular speed of wheel is 2.00 rad/s at \( t = 0 \), thru what angle does the wheel rotate between \( t = 0 \) and \( t = 2 \) s? What's wheel's ang. speed after 2 s?

\[
\theta = \theta_0 + \omega_{0z} t + \frac{1}{2} \alpha t^2 \Rightarrow \theta - \theta_0 = \omega_{0z} t + \frac{1}{2} \alpha t^2
\]

\[
\Delta \theta = (2.00 \text{ rad/s})(2.00\text{s}) + \frac{1}{2}(3.50 \text{ rad/s}^2)(2.00\text{s})^2
\]

\[
= 11.0 \text{ rad} = \frac{11 \text{ rad}}{2\pi \text{ rad/rev}} = 1.75 \text{ rev}
\]

After 2 sec, ang. speed \( \omega_z = \omega_{0z} + \alpha t \)

\[
= 2.00 \text{ rad/s} + (3.50 \text{ rad/s}^2)(2.00\text{s}) = 9.00 \text{ rad/s}
\]
Relating Rotational & Linear Motion

Man moves discuss in circle of radius 80.0 cm. At some instant when he is spinning at angular speed of 10.0 rad/s, with the speed increasing at 50.0 rad/s². What is tangential & radial component of acceleration of discuss & magnitude of acceleration
Relating Linear & Angular Kinetic Variables

Consider a point P on rotating body

since \( s = r \dot{\theta} \Rightarrow \frac{ds}{dt} = r \frac{d\theta}{dt} \) \( \Rightarrow \) \( v = r \omega \)

Tangential component of acceleration

\( \vec{a}_{\text{tan}} \parallel \vec{v} \) changes magnitude of particle's

linear speed \( |\vec{v}| : \quad a_{\text{tan}} = \frac{dv}{dt} = r \frac{d\omega}{dt} = r \alpha \)

\( a_{\text{rad}} = \text{centripetal component of body's acceleration} \)
related to the change in direction of linear velocity

\( a_{\text{rad}} = \frac{v^2}{r} = \frac{\omega^2 r^2}{r} = \omega^2 r \quad \text{(directed radially in)} \)
Speed and Acceleration: Watch Out!

- All points on the rigid object will have the same angular speed, but not the same tangential speed.
- All points on the rigid object will have the same angular acceleration, but not the same tangential acceleration.
- The tangential quantities depend on $r$, and $r$ is not the same for all points on the object.
The Joy of Rotation!
Rotational Kinetic Energy

Rigid body = collection of particles of mass $m_i$, speed $v_i$

Kin. energy of ith particle $K_i = (1/2)m_i v_i^2 = (1/2)m_i r_i^2 \omega^2$

Total kin. energy $K = \sum_i K_i = \frac{1}{2} (\sum_i m_i r_i^2) \omega^2 = \frac{1}{2} I \omega^2$

$I = \text{body's moment of Inertia for } this \text{ rotation axis}$

Moment of Inertia is measure of object's resistance to change in its angular speed. $\Rightarrow$ rotational inertia

$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 ... = \sum_i m_i r_i^2$

depends not just on the total mass but how it is arranged around the rotation axis
Rolling Motion Of A Ball or Wheel

Wheel rolling without slipping can be considered as translation of the wheel as a whole with velocity $v_{CM}$ + rotation about center of mass.
Rolling Without Slipping

As a wheel rotates thru $\phi$ without slipping, point of contact between wheel and surface moves distance $s$

Then $S = R\phi$

If wheel rolling on flat surface then wheel’s CM remains directly over point of contact, so it too moves a distance $s=R\phi$

$$\Rightarrow v_{CM} = R\omega$$

and $a_{CM} = R\alpha$
Pop Quiz

• A disc of radius 2 m is rolling on level ground without slipping. Its CM is moving horizontally at a velocity of 4 m/s. The instantaneous velocity of the top of the disc is

  • (A) 8 m/s
  • (B) 4 m/s
  • (C) – 4 m/s
  • (D) 0 m/s
Wheel (Radius R, Ang. speed $\omega$) Rolls Without Slipping

Symmetric Wheel $\Rightarrow$ CM = Geometric center
Imagine observer at rest w.r.t. surface on which a wheel rolls

In Observer's frame: point on wheel touching surface must
**instantaneously be at rest** so that it does not **slip**

Wheel as a whole translates with velocity $\vec{v}_{cm}$

Wheel rotates around center of mass, speed at rim = $v_{cm}$

Rolling without slipping

Velocity of point of contact $\vec{v}_{i}' = R\vec{\omega}$ (rel. to CM) must have same magnitude but opposite direction as $\vec{v}_{cm}$ $\Rightarrow$ $\vec{v}_{cm} = -R\vec{\omega}$
Same Total Mass, Different Moment of Inertia

\[ I = m_1 r_1^2 + m_2 r_2^2 \]

- Mass close to axis
- Small moment of inertia
- Easy to start apparatus rotating

- Mass farther from axis
- Greater moment of inertia
- Harder to start apparatus rotating
Diff. Rotation Axis $\Rightarrow$ Diff. MOI for Same Body

Long rod easier to rotate about its central axis (longitudinal) axis (a) that about an axis (b) through its center and $\perp$ to its length

Because mass is distributed closer to longitudinal axis (a) than (b)
Diff. Rotation Axis $\Rightarrow$ Diff. MOI for Same Body

3 heavy connectors linked by "massless" plastic strut

$$I_A = m_A r_A^2 + m_B r_B^2 + m_C r_C^2$$

$$= 0 + m_B r_B^2 + m_C r_C^2$$

$$I_B = m_A r_A'^2 + m_B r_B'^2 + m_C r_C'^2$$

$$= m_A r_A'^2 + 0 + 0$$

$I_A \neq I_B$. Since $K = \frac{1}{2} I \omega^2 \Rightarrow K_A \neq K_B$
MOI For A Continuous Distribution Of Mass

\[ I = \lim_{\Delta m_i \to 0} \sum_{i} r_i^2 \Delta m_i = \int r^2 \, dm \]

For objects of uniform density \( \rho = \text{mass} / \text{volume} \), easiest to express \( dm = \rho \, dV \)

\[ I = \int r^2 \, dm = \int r^2 \rho \, dV = \rho \int r^2 \, dv \]

Volume element \( dv = dx \, dy \, dz \)

Limits of integration defined by shape & dimension of the object
Moment Of Inertia For Uniform Thin Rod

Calculate I about axis of rotation thru O, distance h from rod's end.

Mass uniformly distributed on rod

\[
\frac{dm}{M} = \frac{dx}{L} \Rightarrow dm = \frac{M}{L} dx
\]

\[
I_h = \int x^2 dm = \frac{M}{L} \int_{-h}^{L-h} x^2 dx = \left[ \frac{M}{L} \left( \frac{x^3}{3} \right) \right]_{-h}^{L-h} = \frac{1}{3} M (L^2 - 3Lh - 3h^2)
\]

MOI if rod rotating thru left (h=0) or right (h=L) end:

\[
I_{0,L} = \frac{1}{3} ML^2
\]
Moment Of Inertia For Thick Hollow Cylinder

Pick as volume element a thin cylindrical shell of radius \( r \)
thickness \( dr \) and length \( L \) \( \Rightarrow dV = (2\pi r)Ldr \)
\( \Rightarrow dm = \rho(2\pi r)Ldr \). Cylinder thickness = \( R_2 - R_1 \)

\[ I = \int_{R_1}^{R_2} r^2 \rho(2\pi r)Ldr = 2\pi\rho L \int_{R_1}^{R_2} r^3 dr \]

\[ = \frac{2\pi\rho L}{4} (R_2^4 - R_1^4) = \frac{2\pi\rho L}{4} (R_2^2 - R_1^2)(R_2^2 + R_1^2) \]

Cylinder volume \( V = \pi L(R_2^2 - R_1^2) \) \( \Rightarrow M = \rho\pi L(R_2^2 - R_1^2) \)

\[ \Rightarrow I = \frac{1}{2} M(R_2^2 + R_1^2) \]

For a solid cylinder \( R_1 = 0 \) \( \Rightarrow I = \frac{1}{2} MR^2 \)
Pop Quiz

• A rope is being wound without slipping onto a cylinder of radius 2 m which is rotating at 1 revolution/minute. The speed with which a point on the rope is moving is

• (A) $\frac{2}{60}$ m/s
• (B) $\frac{4\pi}{60}$ m/s
• (C) $\frac{60}{2\pi}$ m/s
• (D) $4\pi$ m/s
Cable Unwinding Off A Cylinder

Cable wrapped around cylinder (mass M) is attached to object mass m. As cable unwinds, $U_{grav}$ converted to kinetic energy.

Find speed of object as it hits floor

$$K_1 + U_1 = K_2 + U_2$$

$$0 + mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + 0$$

$$\Rightarrow mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2$$

$$\Rightarrow v = \sqrt{\frac{2gh}{1 + M/2m}}$$
Moments of Inertia of Homogeneous Rigid Objects With Different Geometries

- **Hoop or thin cylindrical shell**
  \[ I_{CM} = MR^2 \]

- **Hollow cylinder**
  \[ I_{CM} = \frac{1}{2} M (R_1^2 + R_2^2) \]

- **Solid cylinder or disk**
  \[ I_{CM} = \frac{1}{2} MR^2 \]

- **Rectangular plate**
  \[ I_{CM} = \frac{1}{12} M (a^2 + b^2) \]
Moment of Inertia of rectangle about normal axis through center

\[ I_{\text{strip}} = \frac{1}{12} b^2 dm \]

\[ dm = \sigma b dy \]

\[ I_{\text{strip}} = \frac{1}{12} b^2 \sigma b dy \]

\[ dI_{cm} = I_{\text{strip}} + y^2 dm = \frac{1}{12} b^2 \sigma b dy + y^2 \sigma b dy \]

\[ I_{cm} = \sigma b \int_{-a/2}^{a/2} \left[ \frac{1}{12} b^2 + y^2 \right] dy = \sigma b \left[ \frac{1}{12} b^2 a + \frac{1}{12} a^3 \right] \]

\[ = \frac{1}{12} \sigma ab [a^2 + b^2] = \frac{1}{12} M [a^2 + b^2] \]

\[ (M = \sigma ab) \]
Moment of Inertia of Objects About Their Axis Of Rotation

Long thin rod with rotation axis through center

\[ I_{CM} = \frac{1}{12} ML^2 \]

Long thin rod with rotation axis through end

\[ I = \frac{1}{3} ML^2 \]

Solid sphere

\[ I_{CM} = \frac{2}{5} MR^2 \]

Thin spherical shell

\[ I_{CM} = \frac{2}{3} MR^2 \]
Parallel Axes Theorem

A body does not have only one moment of Inertia, its has many! As many as the many axes about which it can rotate.

Relation between MOI about an axis of rotation thru its center of mass & MOI about any other axis || to C.O.M axis but displaced by distance d

\[
I_P = I_{cm} + M d^2
\]
Moment of inertia of rectangle about in-plane axis going through center of side of length $a = \frac{1}{12} Ma^2$

Moment of inertia of rectangle about in-plane axis going through center of side of length $b = \frac{1}{12} Mb^2$
Torque $\vec{\tau}$: Rotational Analog Of Force

Torque at a point, due to an applied force causes angular acceleration.

Rotation in direction of $\vec{\tau}$

$\vec{\tau}$ lies in direction perpendicular to plane formed by position vector $\vec{r}$ & applied force $\vec{F}$.
Lever arm = \perp distance between axis of rotation & line of action of force

\[ \tau = Fl = F_{\text{tan}}r = Fr \sin \phi \]

\[ F_{\text{rad}} = F \cos \phi \]

Line of action of \( \vec{F} \)

\( \vec{r} \) (out of page)
Direction of The Torque Vector

\[ \vec{\tau} = \vec{r} \times \vec{F} \]

\[ |\vec{\tau}| = rF\sin\theta \]

Curl fingers of right hand from direction of \( \vec{r} \) into direction of \( \vec{F} \);
outstretched thumb points in direction of \( \vec{\tau} \)

(out of page)
Door Knob & The Hinge

Why are ALL doorknobs always located the furthest from the hinge around which they rotate?

\[ \vec{\tau} = \vec{r} \times \vec{F} \quad \& \quad |\vec{\tau}| = (r \sin \theta)F \]

depends on F applied

AND where it is applied

Lever arm = \( \perp \) distance between axis of rotation \& line of action of force

Larger the \( r \) \( \Rightarrow \) less effort (F) needed
How to Get *More* Out Of Same

\[ \vec{\tau} = \vec{r} \times \vec{F} \quad \& \quad |\vec{\tau}| = (r \sin \theta)F \]

Torque depends on force applied AND where it is applied
Torque & Angular Acceleration

Force $\vec{F}$ acts on body of mass $m$ on one end of a massless rod of length $r$.  
Body rotates around an axis $\perp$ to x-y plane $\Rightarrow$ circular motion in x-y plane

$F_{\text{tangential}} = ma_{\text{tangential}}$

Torque $\tau = F_{\text{tangential}}r = ma_{\text{tangential}}r$

$\Rightarrow \tau = m(r\alpha)r = (mr^2)\alpha$ ($\alpha$ in radians!)

since $I = mr^2 \Rightarrow \vec{\tau} = I\vec{\alpha}$

Newton’s 2nd Law For Rotational Motion
Battle Of The Revolving Door!

Two fired Trump apprentices are trying to use a revolving door. If $|r_1| > |r_2| \AND |F_1| = |F_2|$, which way will the door turn?

If $\tau_1 = -r_1F_1$ then $\tau_2 = +r_2F_2$

$$\sum \tau = \tau_1 + \tau_2 = -r_1F + r_2F < 0$$

$\Rightarrow$ Net torque is negative, will produce a clockwise rotation with downward angular acceleration $\vec{\alpha}$

Which will no doubt upset the blue suit.
Rigid Body Rotation About a Moving Axis

A rigid body's motion = sum of translation motion \( \vec{v}_{cm} \) of CM & rotation about an axis through the CM.

Component particle \( m_i \) at \( \vec{r}_i \) has \( \vec{v}_i = \vec{v}_{cm} + \vec{v}'_i \) \( \iff \) vel. rel. to CM.

\[
K_i = \frac{1}{2} m_i (\vec{v}_{cm} + \vec{v}'_i) \cdot (\vec{v}_{cm} + \vec{v}'_i)
\]

\( \implies K_i = \frac{1}{2} m_i (v_{cm}^2 + 2\vec{v}_{cm} \cdot \vec{v}'_i + v'^2_i) \)

\( \implies K = \frac{1}{2} \left( \sum_i m_i v_{cm}^2 + \vec{v}_{cm} \cdot (\sum_i m_i \vec{v}'_i) + \sum_i \left( \frac{1}{2} m_i v'^2_i \right) \right) \)

Since \( \sum_i m_i \vec{v}'_i = M \times \) Vel. of CM relative to CM = 0

\( \implies K = \frac{1}{2} M v_{cm}^2 + 0 + \frac{1}{2} I_{cm} \omega^2 \)
Work Done By Torque In Rotational Motion

Tangential force $\vec{F}$ over time $dt$ applied at rim of a disk causes torque $\vec{\tau}$, leads to ang. displacement $d\theta$

Work done $dW = F_{\text{tan}} \, ds = F_{\text{tan}} \, R \, d\theta$

$$\Rightarrow dW = \tau_z \, d\theta \Rightarrow W = \int_{\theta_1}^{\theta_2} \tau_z \, d\theta$$

If applied torque is constant

$$\Rightarrow W = \int_{\theta_1}^{\theta_2} \tau_z \, d\theta = \tau_z (\theta_2 - \theta_1)$$
Work & Power In Rotational Motion

As result of work done by $\bar{\tau}$, kinetic energy changes

Since $\bar{\tau} = I\ddot{\alpha}_z \Rightarrow \tau_z d\theta = I\alpha_z d\theta$

$$\tau_z d\theta = I \frac{d\omega_z}{dt} d\theta = I \frac{d\theta}{dt} d\omega = I \omega_z d\omega_z$$

$$\Rightarrow W = \int_{\theta_1}^{\theta_2} \tau_z d\theta = \int_{\omega_1}^{\omega_2} I \omega_z d\omega_z = \frac{1}{2} I (\omega_2^2 - \omega_1^2) = \Delta K$$

Work-energy theorem for rotating rigid bodies

Power associated with applied external torque:

$$P = \frac{dW}{dt} = \tau_z \frac{d\theta}{dt} = \tau_z \omega_z$$
Cable Unwinding Off A Cylinder

Cable wrapped around cylinder (mass M) is attached to object mass m. As cable unwinds, $U_{\text{grav}}$ converted to kinetic energy. Find speed of object as it hits floor

\[ K_1 + U_1 = K_2 + U_2 \]
\[ 0 + mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + 0 \]

\[ \Rightarrow mgh = \frac{1}{2}mv^2 + \frac{1}{2} \left( \frac{1}{2} MR^2 \right) \left( \frac{v}{R} \right)^2 \]

\[ \Rightarrow v = \sqrt{\frac{2gh}{1 + M/2m}} \]

Cylinder is Solid