Physics 4A
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**Momentum**

**Definition of Force:**

\[ \sum \vec{F} = m \vec{a} = m \frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} \]

Define new observable : Momentum \[ \vec{p} = m\vec{v} \]

**Second Law Reloaded** : \[ \sum \vec{F} = \frac{d\vec{p}}{dt} \]

Rapid change in momentum requires \hspace{1cm} \text{LARGE force}

but a more gradual change in \( p \) requires \hspace{1cm} \text{less force}
Center of Mass (cm) of a system of particles is the point that moves as though (1) all of system's mass was concentrated there and (2) all external forces were applied there.

\[
\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \sum_i^{\infty} \frac{m_i \vec{r}_i}{\sum_i^{\infty} m_i}
\]

\[r_{cm} = \text{Mass weighted average position of particles}\]
Motion Of Center of Mass of a System

\[ \vec{v}_{cm} = \frac{m_1 \vec{v}_1 + m_1 \vec{v}_1 + m_1 \vec{v}_1 \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\sum m_i \vec{v}_i}{M} \Rightarrow M \vec{v}_{cm} = \sum_i m_i \vec{v}_i = \vec{P} \]

Total momentum = total mass x velocity of center of mass

⇒ When no net force acts on a system of particles, \textit{velocity of center of mass} remains unchanged

Wrench under F ≠ 0, spins on a horizontal surface

Center of mass (white dot) moves with \( \vec{v}_{cm} = \text{const} \)
External Forces & Center of Mass Motion

Define $\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt}$

$\sum \vec{F} = \sum \vec{F}_{ext} + \sum \vec{F}_{int}$

$M\vec{a}_{cm} = \sum \vec{F}_{ext} + \sum \vec{F}_{int} = \sum \vec{F}_{ext} + 0 \iff 3^{rd} \text{ law}$

When system of particles acted upon by external forces, center of mass moves as though all mass was at that point and it were acted upon by a net force $= \Sigma F_{ext}$

$M\vec{a}_{cm} = M \frac{d\vec{v}_{cm}}{dt} = \frac{d(M\vec{v}_{cm})}{dt} = \frac{d\vec{P}}{dt} = \sum \vec{F}_{ext}$

if $\sum \vec{F}_{ext} = 0 \Rightarrow \vec{P} = \text{constant}$
Motion of Center Of Mass Under A Force

Center of mass of diver and baseball follow parabolic path although all other points follow more complicated path
External Force & CM Motion

- Shell explodes into two parts
- Fragments follow individual parabolic paths
- Center of mass continues on shell’s original trajectory
Firework!
**Impulse Of A Force**

If a **constant** net force acts over a time interval $\Delta t$

Impulse $\vec{J} = \sum \vec{F} \Delta t$

$\sum \vec{F} = \frac{\vec{p}_2 - \vec{p}_1}{t_2 - t_1}$

$\sum \vec{F} \Delta t = \vec{p}_2 - \vec{p}_1$

$\Rightarrow$ Impulse $\vec{J} = \vec{p}_2 - \vec{p}_1$

If a variable force acts over time interval $\Delta t$

Impulse $\vec{J} = \int_{t_1}^{t_2} \sum \vec{F} dt$

Impulse $\vec{J} = \vec{p}_2 - \vec{p}_1 = F_{\text{ave}} \Delta t$

Change in momentum of body during a time $\Delta t$
equals the impulse of force that acts on body in that time interval
Kinetic Energy, Momentum & Impulse

A body of mass m, initially at rest, is acted upon by a force \( \vec{F} \) over a distance \( \vec{d} \) such that its final speed is \( v \).

Impulse \( \vec{J} = \) gain in momentum

\[ = \vec{p}_2 - \vec{p}_1 = \sum \vec{F} \Delta t \]

Kinetic Energy gained

\[ E = \sum \vec{F} \cdot \vec{s} = \frac{mv^2}{2} = \frac{p^2}{2m} \]
Quarterback Sack: You Pick!

Mike (m=50kg, v=8m/s) & Bubba (m=200kg, v=2m/s) coming to get you!

By whom would, you, the quarterback, prefer to be sacked by?

Both have same momentum (p=400kgm/s), will require same impulse $\Delta p$ to be brought to rest (by you!). But Mike has 4 times the kinetic energy of Bubba! ($K=mv^2/2$)

For a given force that you exert with your body, it takes same amount of time to stop either guy but your body is pushed back $\times 4$ more by faster guy! (work done)

To avoid the instantaneous strain on body, Let Bubba get you!
Principle of Conservation Of Momentum

Consider two isolated objects interacting with each other

\[ \vec{F}_{1\text{on}2} = -\vec{F}_{2\text{on}1} \quad \Leftrightarrow \quad \frac{d\vec{p}_2}{dt} = -\frac{d\vec{p}_1}{dt} \]

\[ \Rightarrow \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0 \Rightarrow \frac{d(\vec{p}_1 + \vec{p}_2)}{dt} = 0 \Rightarrow \vec{p}_1 + \vec{p}_2 = \text{const} \]

Sum of momenta of an isolated system of objects is constant, no matter what forces act between the objects making up the system.

If the vector sum of all external forces on a system is zero, then the total momentum of system is constant.
Recoil Of A Rifle

Assume Hunter+Rifle is an isolated system: \( \sum \vec{F} = 0 \)
⇒ momentum is conserved before & after the shot

Conservation of P before & after ⇒ 0 = \( m_B v_B + m_R v_R \)
⇒ Recoil vel. \( v_R = -\frac{m_B}{m_R} v_B = -0.5 m/s \), \( p_R = m_R v_R = -1.5 \text{ kg.m/s} \)

\( K_R = \frac{1}{2} m_R v_R^2 = 0.375 \text{ J} \); \( K_B = \frac{1}{2} m_B v_B^2 = 225 \text{ J} \)
⇒ work done by target to stop the bullet is 600 times larger than that by the hunter to absorb the kinetic energy in Rifle recoil
Physics Of Collision

In collision of two objects, both objects are momentarily or permanently deformed due to large force acting on them. At collision time, the force jumps from zero at the moment of contact to a very large value within a short time, then returns to zero again.
Two Types of Collision

**Elastic Collision:** If forces between interacting bodies are *conservative* then mechanical energy is conserved. Example: pool balls colliding

**Inelastic Collision:** Involves *non-conservative* forces, net mechanical energy after collision is less that before. Example: Bullet embedding into a block of wood

**Completely Inelastic Collision:** when colliding bodies stick together and move as one body after collision

In collision of isolated bodies (net external force = 0) momentum is conserved *before & after collision*

But only in ELASTIC collision is the *mechanical energy* of the system conserved (K1+U1=K2+U2)
In inelastic collision, kinetic energy is not conserved, its transformed into other types of energy such as Thermal or Potential energy $\Rightarrow K_{after} < K_{initial}$.

*Inverse* of an inelastic collision is an *Explosion* where potential energy (chemical or nuclear) is released leading to increase in Kinetic energy of the fragments.

Typical macroscopic collisions (like cars crashing) are inelastic. Cars are designed such that kinetic energy before crash is absorbed in the structure of the cars.
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Completely Inelastic Collision: Ballistic Pendulum

Ballistic pendulum is a device used to measure the speed of bullets. A bullet of mass \( m \) fired into a pendulum with a block of mass \( M \).

After collision, the block-bullet system swings up to a maximum height \( h \).

What is the relationship between bullet velocity \( v \) and \( h \)?

Analysis in 2 parts:
(a) collision itself
(b) subsequent motion of pendulum \( \rightarrow h \)

(a) Collision happens over a short time, the bullet comes to rest in the block before the block moves "too much" from its position at impact.

No net external force \( \Rightarrow \) momentum is conserved
Ballistic Pendulum

(b) after impact, bullet+block moves up
Net external force ⇒ gravity pulling system down
Cannot use momentum conservation, but can use mechanical energy conservation: \( K_1 + U_1 = K_2 + U_2 \)

(Situation a) \( \Rightarrow \) \( mv = (m+M)v' \)

(Situation b) \( \Rightarrow \) Mech. Energy Conservation
\[
\frac{1}{2} (m + M)v'^2 + 0 = 0 + (m + M)gh \Rightarrow v' = \sqrt{2gh}
\]

substituting in (a) \( \Rightarrow \) \( v = \frac{m + M}{m} v' \),
\[ \Rightarrow v = \frac{m + M}{m} \sqrt{2gh} \]

Can measure bullet speed by watching how much block rises after impact!
Elastic Collision: K & p Are Conserved

E conservation

\[ \frac{1}{2} m_A v_{A1}^2 + \frac{1}{2} m_B v_{B1}^2 = \frac{1}{2} m_A v_{A2}^2 + \frac{1}{2} m_B v_{B2}^2 \]

p conservation

\[ m_A v_{A1} + m_B v_{B1} = m_A v_{A2} + m_B v_{B2} \]

Usually masses and initial velocities are known.
Final velocities obtained from 2 eq. and 2 unknowns

Next, consider some interesting situations
Quick Question

• A pool ball rolling in a straight line with velocity $v$ collides with another stationary ball of the same mass. Neglecting friction and air resistance and assuming an elastic collision which of the following will happen:
  • (A) Both balls will move forward together with velocity $v/2$
  • (B) The first ball will bounce back with velocity $-v/2$ and the second will move forward with velocity $v/2$
  • (C) The first ball will stop and the second ball will move forward with velocity $v$
  • (D) The first ball will bounce back with velocity $-v/2$ and the second will move forward with velocity $3v/2$
Bambi Meets Godzilla & Vice Verca!

\[
\frac{1}{2} m_A v^2 \quad \text{before} \quad \Rightarrow \quad m_A v \quad \text{before} \quad = \quad m_A v_A + m_B v_B \quad \text{after}
\]

Rearrange:

\[
\Rightarrow m_B v_B^2 = m_A (v^2 - v_A^2) = m_A (v - v_A)(v + v_A)
\]

\[\begin{array}{c}
\Rightarrow m_B v_B = m_A (v - v_A) \\
\end{array}\]

\[\begin{array}{c}
\text{dividing 2 eqn} \Rightarrow v_B = v + v_A \\
\end{array}\]

\[\begin{array}{c}
\text{substitute in mom-conserv. eq. to eliminate } v_B \\
\Rightarrow m_B (v + v_A) = m_A (v - v_A) \Rightarrow \\
\end{array}\]

\[
\begin{array}{c}
v_A = \frac{m_A - m_B}{m_A + m_B} v \quad \text{and} \quad v_B = \frac{2m_A}{m_A + m_B} v \\
\end{array}\]

Explain when \(m_A << m_B \) & \(m_A >> m_B\)
Pool Table Physics: \( m_A = m_B \)

\[
\begin{align*}
    v_A &= \frac{m_A - m_B}{m_A + m_B} v \\
    v_B &= \frac{2m_A}{m_A + m_B} v
\end{align*}
\]

⇒ Final Velocity \( v_A = 0 \) & \( v_B = v \); Striking ball stops & gives up all its momentum & Kinetic energy → now at rest

In an elastic collision, relative velocity of the two bodies has the same magnitude before & after impact.
Elastic Collision & Relative Velocity

From Energy & momentum conservation, we obtained $v_B = v + v_A$

Vel. of B relative to A after collision

= negative of vel. of B relative to A before collision

In another Inertial Frame of Reference: A & B have different velocities but their relative velocity is the same

⇒ Relative velocity has same magnitude but opposite direction before and after collision

$$v_{B2} - v_{A2} = -(v_{B1} - v_{A1})$$
Gravitational Slingshot Effect!

Spacecraft with $M_A = 2150\,\text{kg}$ moving with $v_{A1}=+10.4\,\text{km/s}$ (w.r.t sun) approaches Saturn ($M_B = 5.69\times10^{26}\,\text{kg}$, moving with $v=-9.6\,\text{m/s}$). Gravitational attraction of Saturn causes spacecraft to swing around it and head off in opposite direction.

Find final speed of spacecraft after it is out of the range of Saturn's gravity.

Collision? Where?

Grav. Interaction is the same thing

*Elastic collision* since the interaction force (gravity) is conservative
Craft = A  \[ \text{Since } M_B \gg M_A \Rightarrow V_{B2} = V_{B1} = -9.6 \text{ km/s} \]
Saturn = B

Take Craft's original direction as along +x.

Rel. Vel: \( \text{Use } \frac{v_{B2} - v_{A2}}{v_{B1} - v_{A1}} = \text{Elastic collision} \)

\[ \Rightarrow V_{A2} = V_{B2} + V_{B1} - V_{A1} \]
\[= [(-9.6)+(-9.6)-10.4]\text{km/s} = -29.6 \text{ km/s} \]

Craft's speed is x3 larger after "collision"

& Craft's Kinetic energy is x 8.1 larger after interaction!

Grav. slingshot effect!

like ball being hit by a swinging baseball→ homerun
Elastic Collision of Pool Balls in 2D

Both balls have same mass; *glancing* collision

What are \( v'_1 \) & \( v'_2 \)?

Use Energy & Momentum Conservation Laws:

K Energy: \[
\frac{1}{2} m v_1^2 + 0 = \frac{1}{2} m v'_1^2 + \frac{1}{2} m v'_2^2
\]

\[
\sum P_x \Rightarrow m v_1 + 0 = m v'_{1x} + m v'_{2x} = m v'_1 \cos 45 + m v'_2 \cos 30
\]

\[
\sum P_y \Rightarrow 0 + 0 = m v'_{1y} + m v'_{2y} = m v'_1 \sin 45 + m v'_2 \sin 30
\]

Solve for \( v'_1 \) & \( v'_2 \) using last 2 eqns.
Rocket & Jet Propulsion

Rocket or Jet Plane: System’s mass changes with time as burnt fuel mass expelled
Rocket Propulsion In Absence Of Gravity

Consider Rocket as **isolated system** in space, no gravity

\[ P_1 = mv \]

\[ P_2 = ? \]

In time \( dt \), rocket ejects burnt mass \(-dm\) with rel. velocity \( v_{ex} \)

According to observer, \( v_{fuel} = v - v_{ex} \) & \( P_{fuel} = (-dm)(v - v_{ex}) \)

after time \( dt \), rocket's mass \( m \rightarrow m + dm \); \( v \rightarrow v + dv \)

rocket's momentum now is \( P_r = (m + dm)(v + dv) \)

Total mom. \( P_2 = P_r + P_{fuel} = (m + dm)(v + dv) + (-dm)(v - v_{ex}) \)
Rocket Propulsion In Absence Of Gravity

Rocket = isolated
No net force on it
Momentum conserved
\[ P_1 = P_2 \]

\[ mv = (m + dm)(v + dv) + (-dm)(v - v_{ex}) \Rightarrow m dv = -dm v - dm dv \]

\[ \Rightarrow m \frac{dv}{dt} = -v_{ex} \frac{dm}{dt} \Rightarrow ma = \frac{dm}{dt} \]

\[ F_{thrust} = -v_{ex} \frac{dm}{dt} \]

\[ a_{rocket} = -\frac{v_{ex}}{m} \frac{dm}{dt} > 0 \text{ since } dm < 0 \]

If \( v_{ex} \) & \( \frac{dm}{dt} \) are held constant then rocket
acceleration increases till all fuel is gone!
Rocket Launched Under Gravity: $F_{\text{ext}} = mg$

$$a_{\text{rocket}} = \frac{dv}{dt} = \frac{F_{\text{ext}}}{m} - \frac{v_{\text{ex}}}{m} \frac{dm}{dt}$$

$$\Rightarrow \int_{v_0}^{v} dv' = \int_{t=0}^{t} \frac{F_{\text{ext}}}{m} dt - v_{\text{ex}} \int_{m_0}^{m} \frac{dm'}{m'}$$

$$\Rightarrow v - v_0 = -gt + v_{\text{ex}} \ln \frac{m_0}{m}$$

Velocity change of Rocket fired from Earth
A “Rocket like” Problem

What happens to Boat’s motion after throwing package?