Department of Physics, UCSD
Physics 225B, General Relativity
Winter 2015
Homework 3, solutions

1. We start by recalling the Friedmann equations:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{1}$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{2}$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \tag{3}$$

(i) Use $p = w\rho$ above, with $w = 0$ and $w = 1/3$ for matter and radiation domination, respectively. In terms fo q and H the LHS of Eq. (1) is

$$\frac{\ddot{a}}{a} = -\left(-\frac{\ddot{a}a}{\dot{a}^2}\right)\frac{\dot{a}^2}{a^2} = -qH^2$$

Hence the second of the desired relations is just a simple rewriting of (1):

$$-qH^2 = -\frac{4\pi G}{3}(\rho + 3p) = -\frac{4\pi G}{3}(1 + 3w)\rho$$

(from which the result follows by multiplying by -2 for $w = 0$ and by -1 for $w = 1/3$). Using this result in (2) to eliminate ρ, the first of the desired relations follows:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{2}{1+3w}qH^2, \qquad \text{or simply:} \qquad \frac{k}{a^2} = \left(\frac{2}{1+3w}q - 1\right)H^2.$$

(ii) For $k = 0$ the metric is invariant under $a \to \xi a$ and the spatial coordinate $\chi \to \xi^{-1}\chi$. This means that we can change what we mean by a by changing units of distance measurement in a flat universe. This cannot be done for $k \neq 0$ The most obvious case is $k = +1$ where χ is an angular variable.

(iii) The equation is

$$\dot{a}^2 = \frac{8\pi G\rho_0 a_0^3}{3}\frac{1}{a} - k = \frac{\kappa}{a} - k$$

where we have defined

$$\kappa = \frac{8\pi G\rho_0 a_0^3}{3} = 2q_0 H_0^2 a_0^3 = \begin{cases} 2|q_0|H_0^{-1}|2q_0 - 1|^{-3/2} & k = \pm 1 \\ 2q_0 H_0^2 a_0^3 & k = 0 \end{cases}$$

So we need to integrate

$$t = \int \frac{da}{\pm\sqrt{\kappa/a - k}}$$

But the right hand side is, in the $k = 1$ case,

$$\sqrt{\frac{\kappa}{a} - 1a} + \frac{\kappa}{2}\arctan\left(\frac{\kappa/a - 2}{2\sqrt{\kappa/a - 2}}\right)$$

which is not easily inverted. Changing variables to conformal time, $da/dt = (da/d\eta)(1/a) \equiv a'/a$ the Friedmann equation is

$$(a')^2 = \kappa a - ka^2 \quad \Rightarrow \quad \eta = \int \frac{da}{\pm\sqrt{\kappa a - ka^2}}$$

or

$$\eta = \begin{cases} 2\arcsin\sqrt{a/\kappa} & k = +1 \\ 2\sqrt{a/\kappa} & k = 0 \\ 2\,\text{arcsinh}\sqrt{a/\kappa} & k = -1 \end{cases}$$

This is inverted straightforwardly,

$$a(\eta) = \begin{cases} \kappa\sin^2\eta/2 = \frac{1}{2}\kappa(1 - \cos\eta) & k = +1 \\ \kappa(\eta/2)^2 & k = 0 \\ \kappa\sinh^2\eta/2 = \frac{1}{2}\kappa(\cosh\eta - 1) & k = -1 \end{cases}$$

We can then obtain a relation between η and t by integrating $dt = a(\eta)d\eta$:

$$t = \begin{cases} \frac{1}{2}\kappa(\eta - \sin\eta) & k = +1 \\ \frac{1}{12}\kappa\eta^3 & k = 0 \\ \frac{1}{2}\kappa(\sinh\eta - \eta) & k = -1 \end{cases}$$

Before moving on to the next question it worth pondering a bit over the form of the solution. For definiteness let's concentrate on the closed universe case, $k = +1$. That $a(0) = a(2\pi) = 0$ is no surprise, since from the mechanical analogy (particle in a potential) we knew the solution must expand from $a = 0$, reach a maximum (here at $a(\pi) = \kappa$ and then re-colapse to $a = 0$. But with the exact solution we can determine, for example, the life-span of the universe, $t_{\text{life}} = \pi\kappa$.

(iv) Of course you should understand "the volume of the universe today," V_0, as the volume of one of the 3-dimensional hypersurfaces, Σ_0, we used in the foliation that defined the isotropic spacetime, namely the one for which the time coordinate corresponds to today:

$$V_0 = \int_{\Sigma_0} d^3x\sqrt{g^{(3)}}$$

Here the metric on the three surface, $g^{(3)}$, is inherited from that of the ambient the spacetime in an obvious way ($t =$ const, or, if you want to get fancy, if $\phi : \Sigma \to \mathcal{M}$ is the embedding of the hypersurface Σ in the spacetime $\mathcal{M}$, then the metric is the pullback $\phi^* g$ of the metric g defined on $\mathcal{M}$). In our case,

$$ds^2 = -dt^2 + a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]$$

so that

$$\det g^{(3)} = a_0^6 \sin^4\chi \sin^2\theta$$

Here a_0 is the scale factor today, which can (and will) be expressed in terms of observables (H_0 and q_0). So we have

$$V_0 = a_0^3 \int_0^\pi d\chi \sin^2\chi \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \tag{4}$$

$$= 2\pi^2 a_0^3 \tag{5}$$

$$= 2\pi^2[(2q_0 - 1)H_0^2]^{-3/2}. \tag{6}$$

(v) Without loss of generality we place ourselves at $\chi = 0$. Then photons coming from a spherical shell between χ and $\chi + d\chi$ originate from a comoving volume $4\pi \sin^2\chi d\chi$. So the proper volume of the universe we can see, V_f, is

$$V_f = \int_0^\pi d\chi 4\pi a^3(\chi) \sin^2\chi, \tag{7}$$

where $a(\chi) = a(t(\chi))$ is the scale factor at the time the photons were emitted from χ. To figure what this is we have to compute the time $t(\chi)$ it took the photons to arrive from χ to $\chi = 0$. The null geodesic satisfies

$$0 = -dt^2 + a^2(t)d\chi^2.$$

If we have the form of $a(t)$ we can integrate this to give $t(\chi)$. But we already computed this in part (iii). It makes sense to use conformal time, so that the null geodesic has

$$0 = a^2[-d\eta^2 + d\chi^2]$$

Note that the expression for the volume of the shell, and therefore of V, does not change when we use conformal time. So if η_0 is the conformal time today we have

$$\eta_0 - \eta = \chi. \tag{8}$$

Using the result in (iii), $a(\eta) = \frac{1}{2}\kappa(1 - \cos\eta)$, we have

$$V_f = 4\pi \int_0^{\pi} d\chi \sin^2\chi [\tfrac{1}{2}\kappa(1 - \cos(\eta_0 - \chi)]^3. \tag{9}$$

$$= \tfrac{1}{2}\pi\kappa^3 \left(\frac{5\pi}{4} - \frac{3\pi}{8}\cos(2\eta_0) - 5\sin(\eta_0) + \frac{1}{15}\sin(3\eta_0) \right). \tag{10}$$

This can be written in terms of $a_0 = \frac{1}{2}\kappa(1 - \cos\eta_0)$ by solving for $\cos\eta_0$ in terms of a_0, and then a_0 can be written in terms of q_0 and H_0. But in practice it is easier to find the value of η_0 given q_0 and H_0 and plug in, so we will leave it as that.

(vi) The farthest distance we can see today corresponds to setting $\eta = 0$ in Eq. (8). So the volume is

$$V_{f0} = 4\pi a_0^3 \int_0^{\eta_0} d\chi \sin^2\chi = 2\pi a_0^3(\eta_0 - \cos\eta_0 \sin\eta_0).$$

Before we declare victory, notice something strange about this expression. The expectation that $V_{f0} \leqslant V_0$ is violated for $\eta_0 > \pi$. The reason is clear: Eq. (8) says that at $\eta_0 = \pi$ the farthest we can see, that is the photons coming from the big bang, $\eta = 0$, originate from $\chi = \pi$. But that is the opposite pole on the sphere! This means that from that instant on we see the whole universe. The expression for today's proper volume of the seeable universe is

$$V_{f0} = \begin{cases} 2\pi a_0^3(\eta_0 - \cos\eta_0 \sin\eta_0) & \eta_0 < \pi \\ 2\pi^2 a_0^3 & \eta_0 \geqslant \pi \end{cases} \tag{11}$$

2. In class we showed that the angular diameter distance, $d_A = D/\theta$ is related to the luminosity distance (which we computed) by $d_A = d_L/(1+z)^2$. So we have $\theta(z) = D/d_A = D(1+z)^2/d_L(z)$. Now recall

$$d_L = \frac{(1+z)}{H_0\sqrt{|1-\Omega_0|}} S_k\left(\sqrt{|1-\Omega_0|}\int_0^z \frac{dz'}{E(z')}\right).$$

Specializing to the assumptions in this problem (matter dominance and flat universe) we use here $S_0(x) = x$, we take Ω_0 as the matter density relative to the critical density today and $E^2(z) = \Omega_0(1+z)^3$. Note that the universe is flat for $\Omega_0 = 1$, so we expect the factors of $\Omega_0 - 1$ to drop from the computation. For now we keep Ω_0 as if it were arbitrary. Computing:

$$\int_0^z \frac{dz'}{E(z')} = \Omega_0^{-1/2}\int_0^z \frac{dz'}{(1+z')^{3/2}} = 2\Omega_0^{-1/2}\left(1 - \frac{1}{\sqrt{1+z}}\right).$$

and finally we have

$$\theta(z) = \frac{DH_0\Omega_0^{1/2}(1+z)}{2\left(1-\frac{1}{\sqrt{1+z}}\right)}$$

As expected we can now safely replace $\Omega_0 = 1$.

That the function $\theta(z)/DH_0$ has a minimum is easy to see from a plot, or from noting that it diverges to positive infinity both as $z \to 0$ and as $z \to \infty$ (so it must turn around at a minimum in between). Or simply by taking a derivative, we find $\theta_{\min} = (3/2)^3 DH_0$ at $z_{\min} = 5/4$.

Why? We will understand this if we understand why $\theta(z)$ diverges both as $z \to 0$ and $z \to \infty$. Now the behavior as $z \to 0$ is physically obvious: an object of fixed size subtends ever larger angles as it comes closer to the observer. The interesting question is why the behavior as $z \to \infty$ which is diametrically opposite the flat-Euclidean spacetime familiar behavior of θ decreasing with increasing distance. The reason this happens is that the null geodesics that are defining the angle are affected by curvature. An analogy may be useful. Consider the geometry of the sphere: think of S^2, in fact of the globe in your grandfather's office, and place an object of fixed diameter D on it, say a coin. Now imagine yourself as an observer at the north pole and move the coin around, drawing rays (sections of great circles) from the north pole to the coin: the envelope of these rays is a curvilinear cone that subtends minimum angle when the coin is at the equator! The distance from the north-pole in this example is in one to one correspondence with redshift. Of course the example is misleading, the "expansion" from the south pole to the equator is analogous enough but then there should be no contraction but rather a slow turnover to a flatter space. On this flat space our common intuition of θ decreasing with z applies, but once we reach the equator and the universe starts contracting (we are going backwards in time) then θ increases again.

Numerics. With $D = 10$ kpc and $H_0 = 100h$ km/s/Mpc we have (using $c = 3.0\times10^5$km/s): $\theta_{\rm min} = 1.1 \times 10^{-8}h$ rad or $2.3h$ milliarcsec.

The solution ends here, but it is fun to play with other geometries. Specializing to the assumption that the universe is closed instead of flat, but still matter dominated, we now use $S_{+1}(x) = \sin(x)$ and $\Omega_0 > 1$. This yields the awful looking expression

$$\theta(z) = DH_0\sqrt{\Omega_0 - 1}(1+z)\csc\left[2\sqrt{\frac{\Omega_0-1}{\Omega_0}}\left(1-\frac{1}{\sqrt{1+z}}\right)\right].$$

Now, to show that the function has a minimum we can either take a derivative, find a stationary point and then check that it is a minimum by evaluating the second derivative. Or we can inspect the function and find where it may have singularities and what the asymptotic behavior is as we approach the singularities. Now, $\csc(x)$ is singular at $x = 0 \bmod(\pi)$, and $\csc(x) = 1/x + x/3! + \cdots$. So as $z \to 0$ we find that $\theta(z)$ diverges $\theta(z) = DH_0\Omega_0^{1/2}\frac{1}{z} + \cdots$ Of course, this is physically trivial: as $z \to 0$ the distance to the object in question decreases to zero, so the angle subtended by an object of fixed diamater D diverges. What is more interesting is that there is another divergence when the argument of the csc is π:

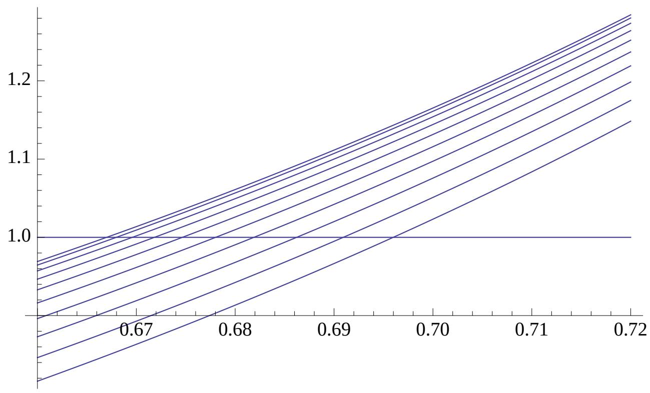
$$\left[2\sqrt{\frac{\Omega_0-1}{\Omega_0}}\left(1-\frac{1}{\sqrt{1+z_{\rm max}}}\right)\right] = \pi$$

And $\sin(x) > 0$ for $0 < x < \pi$ the function $\theta(z)$ stays positive throughout the region $0 < z < z_{\rm max}$. With this and the fact that $\theta(z)$ diverges at $z = 0$ and $z = z_{\rm max}$ we have shown that there is a minimum of $\theta(z)$ in that interval.

The redshift for which the minimum occurs, $z_{\rm min}$, cannot be found analytically in closed form but we can give a simple equation for it. Taking one derivative, dividing through by non-vanishing quantities (like the csc) and equating to zero we have

$$\sqrt{\frac{\Omega_0-1}{\Omega_0}}\cot\left[2\sqrt{\frac{\Omega_0-1}{\Omega_0}}\left(1-\frac{1}{\sqrt{1+z_{\rm min}}}\right)\right] = \sqrt{1+z_{\rm min}}.$$

Let $\omega = \sqrt{1-\Omega_0^{-1}}$ and $x = 1/\sqrt{1+z_{\rm min}}$ so that the equation we are trying to solve takes the form $f(x,\omega) = 1$ where $f(x,\omega) \equiv \omega x\cot(2\omega(1-x))$. You can see that as $\omega \to 0$ the solution is $x = 2/3$ or $z_{\rm min} = 5/4$, which correctly reproduces the flat case, above. As ω increases, so does $z_{\rm min}$. Keep in mind that $\omega < 1$ the limiting value correpsonding to $\Omega_0 \to \infty$. Here is a plot of $f(x,\omega)$ vs x for several values of ω:



You can see the functions for small ω all crossing $f = 1$ at $x = 2/3$ and then as ω increases so does x that solves $f = 1$. The case $\omega = 1$ is solved by $x \approx 0.696$. Of course, this value is non-sensical because it corresponds to infinite density. But it does show that all solutions are close to $x = 2/3$, that is $z = 5/4$ and we obtain an estimate for this minimum in terms of the flat space case:

$$\theta_{\rm min} = \tfrac{2}{3}\theta_{\rm min,flat}\Omega_0^{1/2}\omega\csc(\tfrac{2}{3}\omega)$$

The factor $\omega\csc(\frac{2}{3}\omega)$ starts as $3/2$ at $\omega = 0$ and increases smoothly by about 10% by $\omega = 1$. So roughly speaking $\theta_{\rm min}/\theta_{\rm min,flat} \approx \Omega_0^{1/2}$.

3. In problem 1, part (v) we found the proper volume of the universe we can see. Let's adapt that question into what we need here. In (7) we computed the total volume we can see by adding over volumes of thin concentric shells. We adapt (7) to the problem at present by multiplying the integrand by an appropriate factor that counts how much energy we get from that shell. Now, when in class we computed the luminosity distance we had

$$\frac{F}{L} = \left(\frac{a}{a_0}\right)^2 \frac{1}{4\pi a_0^2 \chi^2}.$$

This is what we get from one source in the shell a comoving distance χ away. There are $n = n_0(a_0/a)^3$ such sources per unit proper volume. Putting it all together the total flux here, today is

$$F_T = L \int_0^{\chi_0} d\chi 4\pi\chi^2 a^3(\chi) \times n_0 \left(\frac{a_0}{a}\right)^3 \times \left(\frac{a}{a_0}\right)^2 \frac{1}{4\pi a_0^2 \chi^2}.$$

Recall, asin problem 1.(v), that $\chi = \chi(t)$, is the comoving distance of a photon that has traveld time t. Since $0 = -dt^2 + a^2 d\chi^2$ it is easiest to convert the integral into an integral over time:

$$F_T = n_0 L \int_0^{t_0} \frac{a(t)}{a_0}\, dt.$$

For a $k = 0$ mater dominated universe

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G\rho}{3} = \frac{8\pi G\rho_0}{3}\left(\frac{a_0}{a}\right)^3$$

or

$$\sqrt{\frac{8\pi G\rho_0 a_0^3}{3}} \int dt = \int \sqrt{a}\, da \qquad \Rightarrow \qquad \frac{a(t)}{a(t_0)} = \left(\tfrac{3}{2}H_0 t\right)^{\frac{2}{3}}$$

Using this above we have, the brightness

$$B = \frac{1}{4\pi} F_T = \frac{1}{4\pi} nL \int_0^{t_0} dt \left(\tfrac{3}{2}H_0 t\right)^{\frac{2}{3}} = \frac{1}{4\pi} nL \left(\tfrac{3}{2}H_0\right)^{\frac{2}{3}} \tfrac{3}{5} t_0^{\frac{5}{3}}$$

We can eliminate t_0 in favor of H_0 (as seen in class) from the equation above $a_0/a = (\frac{3}{2}H_0 t)^{2/3}$ evaluated today, $t_0 = \frac{2}{3}H_0^{-1}$. We obtain finally

$$B = \frac{3}{20\pi} nLt_0 = \frac{1}{10\pi} nLH_0^{-1}\ .$$

Here is an alternative method. The strategy is to compute the total energy today that was radiated by any one star since the beginning of time (!). This times the density of stars today is the energy density today of what was radiated by all stars. The brightness is this times $c/4\pi$ ($= 1/4\pi$ in our units).

The energy emitted by a star between time t and δt is $L\delta t$ (L is energy radiated per unit proper time and the coordinate time t measures proper time of comoving observers; we assume the stars are comoving). This energy today is redshifted by a factor or $a(t)/a(t_0)$. The total amount of energy today radiated by one star over its history is

$$E = \int_0^{t_0} dt\, \frac{a(t)}{a(t_0)} L$$

The rest follows.

4. (i) Use $a(t) = e^{t/\alpha}$. Also from the form of the line element, $d\vec{x}^2 = d\chi^2 + \chi^2 d\Omega_2^2$, we see that $S_k(\chi) = \chi$, which means $k = 0$. Then $H = \dot{a}/a = 1/\alpha$ and we have

$$H^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} \qquad \Rightarrow \qquad \frac{1}{\alpha^2} = \frac{8\pi G\rho}{3}$$

So $\rho =$ constant. Then $\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0$ give $p = -\rho$. This is a cosmological constant solution with $\Lambda = 8\pi G\rho = -8\pi G p = 3/\alpha^2$.

(ii) Since the metric is independent of $\hat{x}, \hat{y}, \hat{z}$ (or simply $\hat{x}^i$) we get three conservation laws,

$$\frac{d\hat{x}^i}{d\lambda} = \frac{v^i}{g_{ii}} = e^{-2\hat{t}/\alpha} v^i,$$

where $v^i =$ constant. Then $u^\mu u_\mu = 0$ for null geodesics gives

$$\frac{d\hat{t}}{d\lambda} = \pm|\vec{v}|\, e^{-\hat{t}/\alpha} \qquad \Rightarrow \qquad \lambda = (\alpha/|\vec{v}|) e^{\hat{t}/\alpha}.$$

This shows that $\hat{t} \to -\infty$ as $\lambda \to 0$, that is, along a geodesic we reach the end of the spacetime given by these coordinates in finite affine parameter. It is easy to integrate $u^\mu u_\mu = -1$, for time-like geodesics and obtain the same result — that asymptotically $\tau \to$ constant as $\hat{t} \to -\infty$. Explicitly, $u^\mu u_\mu = -1$ gives

$$\left(\frac{d\hat{t}}{d\tau}\right)^2 - e^{-2\hat{t}/\alpha}\vec{v}^2 = 1$$

which gives

$$\tau = \int \frac{d\hat{t}}{\sqrt{1 + e^{-2\hat{t}/\alpha}\vec{v}^2}} = \hat{t} + \alpha \ln\left(1 + \sqrt{1 + e^{-2\hat{t}/\alpha}\vec{v}^2}\right).$$

So as $\hat{t} \to -\infty$ we find $\tau = \alpha \ln|\vec{v}| + (\alpha/|\vec{v}|)e^{\hat{t}/\alpha} + \cdots$, while for $\hat{t} \to +\infty$ we find $\tau = \hat{t} + \alpha \ln 2 + \cdots$.