Reissner-Nordstrom: Charged Black Hole.

Look for spherically symmetric and static (or nearly so). 
\( ds^2 = -T(r,t) dt^2 + R(r) dr^2 + r^2 d\Omega^2 \)

Solution to Einstein's Equations

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu} \]

with matter given by electromagnetic field.

Recall \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) and \( J = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \)

(For contact with conventional non-relativistic notation in Minkowski space. \( E^i = -\frac{\partial A^i}{\partial x^0} - \nabla \Phi \), \( \Phi = A^0 \) and \( \Phi = \partial_i \). With low indices,

\[ E^i = -\partial_0 A_i + \partial_i A_0 = -F_{0i} \]

Similarly \( \mathcal{B} = (\nabla x A) \) or \( B_i = \varepsilon_{ijk} \frac{\partial_j A_k}{2} = \varepsilon_{ijk} F_{jk} \)

Note that \( J = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (E^2 - B^2) \) as it should.

Then, as we saw earlier,

\[ T_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta L}{\delta g_{\mu\nu}} \]

\[ L = \int d^4 x \sqrt{-g} L_m \]

\[ T_{\mu\nu} = \int d^4 x \left[ \frac{1}{\sqrt{-g}} \frac{\delta L}{\delta g_{\mu\nu}} L_m + \frac{\delta L}{\delta g_{\mu\nu}} \right] \]

The first term requires \( \delta (\text{det} A) = \delta \text{Tr} \lambda = \delta \text{Tr} \lambda = \delta \right| \right| \lambda \right| \right| \lambda = e \left( 1 - \lambda \right) = \delta \right| \right| \lambda \right| \right| \lambda \) or \( \delta g = g g^{\mu\nu} \delta g_{\mu\nu} \), so \( \sqrt{-g} \delta L_m = \frac{1}{2} g^{\mu\nu} \delta g_{\mu\nu} \)
For the second term use

\[ L = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} g^{\rho \sigma} \]

so that there is no implicit dependence on \( g_{\mu \nu} \) in \( F_{\mu \nu} \), and then

\[ \frac{\delta g^{\mu \nu}}{\delta g_{\mu \nu}} = -g^{\mu \sigma} g^{\nu \rho} - g^{\nu \sigma} g^{\mu \rho}, \text{ so} \]

\[ \frac{\delta L}{\delta g_{\mu \nu}} = F_{\mu \rho} F^{\rho \nu} g^{\lambda \sigma} \]

or

\[ T^{\mu \nu} = F_{\mu \rho} F^{\nu \rho} g^{\lambda \sigma} - \frac{1}{4} g^{\mu \nu} F_{\rho \lambda} F^{\rho \sigma} \]

or

\[ T_{\mu \nu} = F_{\mu \nu} F^{\rho \sigma} - \frac{1}{2} g_{\mu \nu} F_{\rho \lambda} F^{\rho \lambda} \]

For spherical symmetry need radial \( E \) (and possibly \( B \)),
so in radial coordinates we have \( E_{t \theta} = E_{t \phi} = 0 \) and

\[ F_{t r} = - F_{r t} = f(t, r) \]

For \( B \) to be radial we need to generalize \( B_{r} = \frac{1}{2} E_{ij} E_{j k} \):

\[ B_{r} = \frac{1}{2} E_{00} E_{11} F^{11} \] and use \( E_{\mu \nu \rho} = \frac{1}{\sqrt{-g}} E_{\mu \nu \rho} \). Or, more directly, but pedestrianaly, go back to cartesian \( x, y, z \). Then

\[ F_{\theta \phi} = E_{ij} \frac{\partial x^{i}}{\partial \theta} \frac{\partial x^{j}}{\partial \phi} \]

and \( E_{ij} = E_{ij} B_{k}^{k} \) or \( E_{ijk} x^{k} \) for radial.

But then \( F_{\theta \phi} = g(r) E_{ij} \frac{\partial x^{i}}{\partial \theta} \frac{\partial x^{j}}{\partial \phi} x^{k} = \sin \theta g(r) \)

(The factor \( E_{ij} \frac{\partial x^{i}}{\partial \theta} \frac{\partial x^{j}}{\partial \phi} x^{k} \) is just the determinant

\[ \begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} & x \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} & y \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} & z \end{vmatrix} \]

which is the measure for the volume integral at \( r = 1, \sin \theta \).)


So we take
\[ ds^2 = -T(X, r) dt^2 + R(r) dr^2 + r^2 d\Omega^2 \]
\[ F_{tr} = f(t) \]
\[ F_{t\phi} = g(t) \sin \Theta \]
and plug into Einstein's.

Quick computation
\[ \Gamma_{\mu\nu\lambda} = \frac{1}{2} g_{\mu\lambda,\nu} + g_{\mu\nu,\lambda} - g_{\nu\lambda,\mu} \]

\[ \Gamma_{tet} = -\frac{1}{2} \frac{\partial T}{\partial r} \]
\[ \Gamma_{t\theta\theta} = -\frac{1}{2} \frac{\partial T}{\partial r} R' \]
\[ \Gamma_{r\theta\theta} = \frac{1}{2} \frac{\partial R}{\partial r} \]
\[ \Gamma_{\phi\phi\phi} = -\frac{1}{2} \frac{\partial \phi}{\partial r} \]
\[ \Gamma_{\phi\phi\theta} = -\frac{1}{2} \frac{\partial \phi}{\partial r} \]
\[ \Gamma_{\phi\phi\psi} = \frac{1}{2} \frac{\partial \phi}{\partial r} \]

\[ R_{\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\rho} - \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\rho} \]

\[ R_{tet} = \frac{1}{2} \frac{T''}{R} - \frac{1}{2} \frac{R''}{R^2} + \left( \frac{1}{2} \frac{T'}{R} + \frac{1}{2} \frac{R'}{R} + \frac{2}{r} \right) \left( \frac{1}{2} \frac{T'}{R} - \frac{1}{2} \frac{R'}{R} \right) \]

\[ R_{t\theta\theta} = \frac{1}{2} \frac{R''}{R} - \frac{1}{2} \frac{R''}{R^2} - \frac{1}{2} \frac{T'}{R} \]

\[ R_{\phi\phi\phi} = \frac{1}{2} \frac{R''}{R} - \frac{1}{2} \frac{R''}{R^2} - \frac{1}{2} \frac{T'}{R} \]

\[ R_{\phi\phi\theta} = \frac{1}{2} \frac{R''}{R} - \frac{1}{2} \frac{R''}{R^2} - \frac{1}{2} \frac{T'}{R} \]

\[ R_{\phi\phi\psi} = -\frac{1}{2} \frac{T''}{R} + \frac{1}{4} \frac{T'^2}{R^2} - \frac{1}{4} \frac{R'^2}{R^2} + \frac{R}{R^2} \]
\[ R_{\ell t} = R_{t\ell} = \frac{2}{(T^2 - t^2)} + \frac{1}{T} - 0 = 0 \]

\[ R_{\theta\theta} = \partial_r \left( -\frac{r^2}{R} \right)^2 + \left( \frac{1}{2} \frac{1}{T^2} + \frac{1}{2} \frac{\theta^2}{R^2} + \frac{\theta^2}{T} \right) \left( -\frac{r^2}{R} \right) - 2 \left( -\frac{r}{R} \right) \left( \frac{1}{T} \right) - \left( \frac{\cos^2 \theta}{\sin^2 \theta} \right)^2 \]

\[ = \frac{1}{2} \frac{r R^2}{R^2} - \frac{1}{R} - \frac{1}{2} \frac{T^2}{T R} + \frac{1}{4} d\theta \theta \]

\[ R_{\phi\phi} = \partial_r \left( -\frac{\sin^2 \theta}{R} \right) \left[ \frac{1}{2} \frac{1}{T^2} + \frac{1}{2} \frac{\theta^2}{R^2} + \frac{\theta^2}{T} \right] \left( -\frac{\sin^2 \theta}{R} \right) + \left( \frac{\cos \theta}{\sin \theta} \right) \left( -\sin \theta \cos \theta \right) \]

\[ \quad - 2 \left( \frac{1}{T} \right) \left( -\frac{r \sin^2 \theta}{R} \right) - 2 \left( -\sin \theta \cos \theta \right) \left( \frac{\cos \theta}{\sin \theta} \right) \]

\[ = \sin^2 \theta \left[ \frac{r R^2}{2 R^2} - \frac{1}{R} - \frac{1}{2} \frac{T^2}{T R} + \frac{1}{4} d\theta \theta \right] = \sin^2 \theta \ R_{\theta\theta} \]

Ricci scalar:

\[ R = g^{\mu\nu} R_{\mu\nu} = \ldots \text{ better use trace of } T \]

So we have

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8 \pi G T_{\mu\nu} \]

\[ \Rightarrow \quad \frac{-R}{8 \pi G} = T_{\mu\nu} \]

\[ \Rightarrow R_{\mu\nu} = 8 \pi G \ T_{\mu\nu} + \frac{1}{4} g_{\mu\nu} \left( -8 \pi G T \right) = 8 \pi G \left( T_{\mu\nu} - \frac{1}{4} g_{\mu\nu} T \right) \]

Now, compute \( T_{\mu\nu} \): \[ T_{\ell t} = F_{\ell\theta} F^{\ell\theta} - \frac{1}{4} g_{\ell t} F_{\mu
u} F^{\mu\nu} = \frac{1}{R} f^2 + \frac{1}{2} T \left( -\frac{1}{R^2} f^2 + \frac{g^2}{T^2} \right) \]

\[ = \frac{1}{2} \frac{f^2}{R} + \frac{1}{2} T \frac{g^2}{T^2} \]

\[ T_{\theta\theta} = \frac{1}{4} g_{\theta\theta} F_{\mu
u} F^{\mu\nu} = -\frac{1}{4} T \left( -\frac{1}{R^2} f^2 + \frac{g^2}{T^2} \right) \]

\[ = -\frac{1}{2} \frac{f^2}{T} - \frac{g^2 R}{2 T} \]

\[ T_{\ell\theta} = F_{\ell\theta} F^{\ell\theta} - \frac{1}{4} g_{\ell\theta} F_{\mu
u} F^{\mu\nu} = \frac{1}{R^2 \sin^2 \theta} \left( \sin^2 \theta - \frac{1}{2} r^2 \left( -\frac{1}{R^2} f^2 + \frac{g^2}{T^2} \right) \right) \]

\[ = \frac{1}{2} \frac{g^2}{R^2} + \frac{1}{2} \frac{r^2 s^2}{T} \]

\[ T_{\theta\theta} = \sin^2 \theta \ T_{\ell\theta} \]
So \[ T = 9^{\mu \nu} T_{\mu \nu} = \frac{1}{f} \left( \frac{1}{2} \frac{T^2}{R^2} + \frac{1}{2} \frac{T^2 g^2}{R^2} \right) + \frac{1}{R} \left( \frac{1}{2} \frac{T^2}{f^2} - \frac{g^{2 R}}{2 R^2} \right) \]

\[ + \frac{2}{f} \left( \frac{1}{2} \frac{g^{2 R}}{f^2} + \frac{1}{2} \frac{g^{2 R}}{R^2} \right) \]

And we have

\[ \text{L} \:: \quad \frac{1}{2} \frac{T^2}{R^2} - \frac{1}{4} \frac{T^1 T^1}{R^2} - \frac{1}{4} \frac{T^2}{R^2} + \frac{1}{R} \frac{T^1}{R} = \left( \frac{1}{2} \frac{f^2}{R^2} + \frac{1}{2} \frac{T^2}{R^2} \right) 8 \pi 6 \]

\[ \text{r} \:: \quad -\frac{1}{2} \frac{T^2}{r^2} + \frac{1}{4} \frac{T^1}{r^2} + \frac{1}{4} \frac{T^1}{r^2} + \frac{1}{r} = \left( \frac{1}{2} \frac{f^2}{r^2} - \frac{g^{2 R}}{2 r^2} \right) 8 \pi 6 \]

\[ \text{a} \:: \quad \frac{1}{2} \frac{T^2}{R^2} - \frac{1}{R} \frac{T^1 T^1}{R^2} + 1 = \left( \frac{1}{2} \frac{f^2}{R^2} + \frac{1}{2} \frac{T^2}{R^2} \right) 8 \pi 6 \]

\[ \text{L} \] and \[ \text{r} \] unknowns; need more: Maxwell's Equations

\[ g^{\mu \nu} \partial_\mu F_{\nu \lambda} = 0 \quad \text{and} \quad \partial_\mu F_{\nu \lambda} = 0 \]

Recall \[ \partial_\mu F_{\nu \lambda} = \partial_\nu F_{\mu \lambda} - \tilde{\Gamma}^\rho_{\mu \nu} F_{\rho \lambda} - \tilde{\Gamma}^\rho_{\nu \lambda} F_{\mu \rho} \]

So, in components

\[ g^{\mu \nu} \partial_\mu F_{\nu \lambda} = -\frac{1}{4} \left[ -\frac{T^1}{R} \right] + \frac{1}{R} \left( -\frac{1}{2} \frac{E^1}{R} \right) \]

\[ + \frac{1}{R^2} \left[ - (-\frac{E^1}{R}) \right] + r^2 \sin^2 \theta \left[ - (-\frac{E^1}{R}) \right] \]

\[ = -\frac{T^1}{R} + \frac{1}{2} \frac{E^1}{R} + \frac{1}{2} \frac{T^1}{R} - \frac{2E}{r R} \]

\[ g^{\mu \nu} \partial_\mu F_{\nu \lambda} = -\frac{1}{4} \left[ 0 \right] + \frac{1}{R} \left( 0 \right) + \frac{1}{R} \left( 0 \right) = 0 \quad \text{autonomous} \]

\[ g^{\mu \nu} \partial_\mu F_{\nu \theta} = -\frac{1}{4} \left( 0 \right) + \frac{1}{R} \left( 0 \right) + \frac{1}{r^2} \left[ 0 \right] + \frac{1}{r^2} \left[ 0 \right] = 0 \]

\[ g^{\mu \nu} \partial_\mu F_{\nu \phi} = -\frac{1}{4} \left( 0 \right) + \frac{1}{R} \left( 0 \right) + \frac{1}{r^2} \left[ \cos \theta \sin \theta \right] + \frac{1}{r^2} \left( 0 \right) = 0 \]
and for the 2nd equation take

\[ \nabla_r F_{\phi} + \nabla_\theta F_{\phi} + \nabla_\phi F_{\phi} \]

\[ = (\sin \theta g' - \frac{\theta}{r} \sin \theta g) + (-\frac{1}{r}(-g\sin \theta)) + (-\frac{1}{r}(g\sin \theta)) \]

\[ = \sin \theta g' \]

so \( \nabla_r F_{\phi} = 0 \Rightarrow \sin \theta g' = 0 \Rightarrow g = \text{constant} \).

And Maxwell's equation gives

\[ \frac{f}{R} = \left[ \frac{1}{1} \frac{R}{r} + \frac{1}{2} \frac{T}{T} - \frac{2}{r} \frac{R}{r} \right] \]

Look for a solution of the form \( T = \frac{k}{r} \)

\[ \Rightarrow \frac{d}{dr} \left( \frac{T}{r} \right) = -2 \frac{d}{dr} \frac{k}{r^2} \Rightarrow f = \frac{k}{r^2} \]

and

\[ \frac{\frac{R}{r^2}}{R} = \frac{1}{r} + 1 = \frac{4\pi G}{r^2} \left( \frac{g^2 + k^2}{2} \right) \]

and

\[ -\frac{1}{2} \frac{T''}{T} - \frac{T'}{Tr} = \frac{4\pi G}{r^2} \left( \frac{k^2 + g^2}{2} \right) \]

or

\[ T'' + \frac{2}{r} T' = \frac{4\pi G}{r^2} (k^2 + g^2) \]

\[ \Rightarrow \frac{r^2}{4} (r^2 T')' = \frac{4\pi G}{r^2} (k^2 + g^2) \Rightarrow r^2 T' = \frac{4\pi G}{r^2} (k^2 + g^2) \]

\[ T = 1 - \frac{2GM}{r} + \frac{4\pi G}{r^2} (g^2 + k^2) \]

Check (10), \[ -\frac{T'}{TR} = T + 1 \Rightarrow -T' = T + 1 = \left[ \frac{4\pi G (k^2 + g^2)}{r^2} - \frac{2GM}{r} \right] + \frac{16GM}{r^2} \frac{4\pi G}{r^2} \]

\[ \Rightarrow \frac{4\pi G (k^2 + g^2)}{r^2} \left( \text{LHS} \right) \]
The solution is \[ ds^2 = -T dt^2 + R dr^2 + r^2 d\Omega^2 \]

\[ T = \frac{1}{R} = 1 - \frac{2GM}{r} + \frac{4\pi G (\rho^2 + q^2)}{r^2} \]

and \[ F_{tr} = \frac{q}{r^2}, \quad F_{\theta \phi} = p \sin \theta \]

(Note that \( E_r = F_{tr} = \frac{q}{r^2} \) and \( B_r = F_{\theta \phi} = \frac{p}{r \sin \theta} \).

So \( q, p \) are electric and magnetic charges, and the notation \((q, p)\) is standard for "dyons".)

Singularity as \( r = 0 \). Horizon singularities (see below) arecoordinate effects.

**Event horizons?** In a static space-time (Killing vector \( \partial_t \))

asymptotically flat-like, \( \partial_t \partial_r = 0 \) choose coordinates \((r, \theta, \phi)\) so metric looks Minkowski as \( r \to \infty \).

Hypersurface \( r = \text{const} \) : timelike "cylinder" (topology \( S^2 \times \mathbb{R} \)) as \( r \to \infty \).

Now decrease \( r \) from infinity to some \( r_H \) where the surface becomes null, an event horizon.

![Diagram](image)

Event horizon: path, crossing \( r_H \), unable to escape for \( r > r_H \).

How to determine \( r_H \)? \( \partial_t r \) is a 1-form normal to \( r = \text{const} \) hypersurface, with norm

\[ g_{\mu \nu} (\partial_\mu r)(\partial_\nu r) = g^{rr} \]

We want this to vanish, so \( g^{rr}(r_H) = 0 \).

This method is very restrictive (to spaces that are static e.g. with coordinates \((t, \theta, \varphi)\) as above, found).
Method applies for RN - metric. So

$$\Gamma^+_\pm = 1 - \frac{2GM}{r} + 4\pi c^2 (q^1 q^2) = 0$$

Pure or no solutions (at most):

$$\Gamma^+_\pm = \frac{2GM \pm \sqrt{(2GM)^2 - 4\pi c^2 (p^1 + q^2)}}{2}$$

$$= GM \pm \sqrt{(GM)^2 - 4\pi c^2 (p^1 + q^2)}$$

Cases:

1) $$4\pi (p^1 + q^2) > GM^2$$

No solutions $$\Rightarrow$$ no event horizon,

"Naked Singularity"

**Example:**

Cosmic Censorship Conjecture (Penrose): Nature abhors a naked singularity, or

Naked singularities cannot form in gravitational collapse from generic, initially nonsingular states in an asymptotically flat spacetime obeying the dominant energy condition:

For all timelike vectors $$\xi^a$$, $$\Sigma_a \xi^a \xi^b > 0$$ (for "weak energy condition")

And $$\Sigma_a t^a$$ is a non-spacelike vector

(Basically, $$p > 1$$).
(vi) $4\pi (\rho^2 q^2) > 6 M^2$

Two distinct solutions, with $r \leq r_+$

In $r_- < r < r_+$, $\text{d}r \text{d}t$ is timelike and $\text{d}r \text{d}t$ is spacelike.

But both for $r > r_+$ and $r < r_-$, $\text{d}r \text{d}t$ is spacelike and $\text{d}t \text{d}t$ is timelike.

If you fall into this black hole within a spaceship full of gas, once you get to $r = r_+$ you must continue falling towards lesser $r$, but once you come out to $r < r_-$ you can turn on your thrust engines, turn around before you hit $r = 0$, go back to $r = r_-$. Then you must continue, with $t > 0$, until you come out to $r = r_+$. You can then decide to continue out to $r = 0$ or turn around and "re-enter" the black hole.

$r = 0$ singularity is timelike (recall, for Schwarzschild, spacelike)
Conformal diagram

MTW has a step by step on how to derive this.
(iii) \( GM^2 = 4\pi (q^2 + p^2) \)

"Extreme" RN-solution.

In this case \( r_e = r = GM \), and

\[ g^{rr} = \left(1 - \frac{GM}{r}\right)^2 \]

In fact

\[ ds^2 = -\left(1 - \frac{GM}{r}\right)^2 dt^2 + \left(1 - \frac{GM}{r}\right)^{-2} dr^2 + r^2 d\Omega^2 \]

So, there is a horizon at \( r = GM \), but it is never timelike.

The singularity is at \( r = 0 \) and it is timelike.

Penrose diagram

\[ r = 0 \]

\[ r = GM \]
Solutions with many extreme RN black holes: remarkably, we can produce metrics which are exact solutions of Einstein's equations in empty space with as many RN black holes as we want.

\[
\text{In} \quad ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2
\]

let

\[
\rho = r - GM
\]

so

\[
r^2 = (\rho + GM)^2 = \rho^2 H^2(\rho)
\]

where \( H(\rho) = 1 + \frac{GM}{\rho} \)

Also \[
1 - \frac{2GM}{\rho} = 1 - \frac{2GM}{\rho + GM} = \frac{\rho}{\rho + GM} = \frac{1}{1 + \frac{GM}{\rho}} = H^{-1}
\]

so

\[
ds^2 = -H^{-2}(\rho) dt^2 + H^2(\rho) \left[ d\rho^2 + \rho^2 d\Omega^2 \right]
\]

Notice, the term in \( \Omega^2 \) is just the metric of flat Euclidean 3-space in spherical coordinates, so we can write

\[
ds^2 = -H^{-2}(\rho) dt^2 + H^2(\rho) \left[ dx^2 + dy^2 + dz^2 \right]
\]

where \( 1x^2 = x^2 + y^2 + z^2 \)

If we take the metric (1) as a ansatz and plug it into Einstein's equations, we find it is a solution provided \( \nabla^2 \phi = 0 \). To be precise, we need an EM field too. To motivate it, we have to write the extended RN metric

\[
F_{tr} = \frac{q}{r^2} = -\partial_r A_t + \partial_\theta A_r
\]

so \( A_t = \frac{q}{r} = \frac{GM}{\sqrt{4\pi}} \frac{1}{\rho + GM} \) but \( 1 - H^{-1} = \frac{1}{\rho} = \frac{1}{\sqrt{4\pi}} \frac{1}{\rho + GM} \)

with \( H(\rho) \)}
\[ A_* = \sqrt{\frac{GM}{\eta}} \frac{1 - H^{-1}}{GM} = \sqrt{\frac{\alpha}{4\pi}} (1 - H^{-1}) \] (24)

So our looks for solutions with \( \phi \) and \( \psi \). Then \( \psi \) will satisfy
\[ \nabla^2 \psi = 0 \quad \text{where} \quad \nabla^2 = \partial^2 \partial x + \partial^2 \partial y + \partial^2 \partial z \]

The most general solution with \( \psi \to 1 \) as \( \eta \to 0 \) is
\[ \psi = 1 + \frac{N}{2} \sum_{k=1}^{\infty} \frac{G r_k}{|\mathbf{r} - \mathbf{r}_k|} \]

(Actually, I guess
\[ \psi = 1 + \int \frac{\rho(x) \, dx}{|\mathbf{r} - \mathbf{r}'|} \]

works too, provided \( \rho \) has support in a finite region, \( |\mathbf{r}| < R \).

But, \( \psi \to 1 \) \( \Rightarrow \) \( \phi \to 0 \) \( \Rightarrow \) \( \frac{\partial \phi}{\partial r} \to 0 \) \( \Rightarrow \) \( \phi \to 0 \) \( \Rightarrow \) \( \phi \to 0 \).

I think, however, these solutions are inconsistent with the electric forces between the charges! I don't know what to do.

Note that the electric repulsion between holes cancels the gravitational attraction:

\[ F_{12} = -\frac{GM_1M_2}{r^2} + \frac{q_1q_2}{r^2} = 0 \quad \Rightarrow \quad \frac{q_1q_2}{r^2} = \frac{GM_1M_2}{r^2} \]

or \( q_1 = \sqrt{\frac{GM_1}{r^2}} \) \( q_2 = \sqrt{\frac{GM_2}{r^2}} \)

and we are off by a \( \sqrt{r} \).
Note: Verify solutions:
\[ ds^2 = -H^{-2}dt^2 + H^2 (dx^2 + dy^2 + dz^2) \]

\[
\Gamma_{itt} = -\frac{1}{2} g_{tt} \partial_t = -\frac{1}{2} (-1)(-2H^{-3}) \partial_t H = -H^{-3} \partial_t H
\]
\[
\Gamma_{tt} = -H^{-3} \partial_t H
\]
\[
\Gamma_{t} = -H^{-1} \partial_t H
\]
\[
\Gamma_{\alpha j} = \frac{1}{2} \left( (H^{-1} \partial_j H_k) + (H^{-1} \partial_k H_j) - (H^{-1} \partial_k H_j) \right) = H \left( \partial_j H_k + \partial_k H_j - \delta_{jk} H \right)
\]
\[
\Gamma_{\alpha j} = H^{-1} \left( \partial_j H_k + \partial_k H_j - \delta_{jk} H \right)
\]
\[
\mathcal{R}_{\mu
u} = \partial_\mu \Gamma^\rho_{\nu\rho} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\rho} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\rho}
\]
\[
R_{tt} = \partial_t (-H^{-5} \partial_t H) + \left( -H^{-1} \partial_t H + 3H^{-1} \partial_t H \right) + \left( -H^{-5} \partial_t H \right) - 2 \left( H^{-5} \partial_t H \right) \left( H^{-5} \partial_t H \right)
\]
\[
= 5H^{-6} \partial_t H^2 - H^{-5} \partial_t H - 4H^{-6} \partial_t H^2 = H^{-6} \partial_t H^2 - H^{-5} \partial_t H
\]
\[
R_{ij} = \partial_i \left( H^{-1} \left( \delta_{ij} H - \delta_{ij} H \right) \right) - \partial_j \left( 2H^{-1} \partial_t H \right)
\]
\[
+ [2H^{-1} \partial_t H \partial_i H^{-1} \left( \delta_{ij} H - \delta_{ij} H \right) - H^{-2} \left( \delta_{ij} H_d + \delta_{ik} H_j - \delta_{ij} H_k \right)]
\]
\[
- \left( H^{-1} \partial_t H \right) \left( H^{-1} \partial_t H \right) - H^{-2} \left( \delta_{ij} H_d + \delta_{ik} H_j - \delta_{ij} H_k \right)
\]
\[
= \partial_i \left( \delta_{ij} H - \delta_{ij} H \right) - \partial_j \left( \delta_{ij} H - \delta_{ij} H \right)
\]
\[
= -H^{-2} \left( \delta_{ij} H - \delta_{ij} H \right) + H^{-1} \left( \delta_{ij} H - \delta_{ij} H \right) + 2H^{-2} H_{ij} H_{ij} - 2H^{-2} H_{ij} H_{ij}
\]
\[
= -\delta_{ij} H^{-1} \partial_t H + H^{-2} \delta_{ij} H_{ij} - 2H^{-2} H_{ij} H_{ij}
\]

\[
T_{\mu \nu} = F_{\mu \nu} F_{\nu} - \frac{1}{4} g_{\mu \nu} F_{\nu} F_{\nu}
\]

\[
T_{tt} = F_{tt} F_{t}^2 - \frac{1}{4} g_{tt} \partial_t F_{tt} F_{tt}^2 = H^{-2} \frac{1}{4H^2} \left( H^{-2} H_{tt} \right)^2 - H^{-2} \left( 1 - H^2 \right) \left( 1 - H^2 \right)
\]
\[
= \frac{1}{800} H^{-2} H_{tt}
\]

\[
R_{tt} = \gamma_{tt} \Rightarrow H^{-2} H_{tt} - H^{-5} \partial_t H = \gamma_{tt} \left( \delta_{tt} H_{tt} \right)
\]

(\text{Only for } g_{tt} \neq 0).

\[
\Rightarrow \partial_t^2 H = 0
\]
Kerr Metric: Rotating Black Holes

No hair "theorem": stationary, asymptotically flat solutions to Einstein's + Maxwell's are fully characterized by $M, Q, (\rho, \theta)$ and $J = aM$

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 - \frac{2GMa \sin^2 \theta}{\rho^2} (dtd\varphi + d\varphi dt)$$

$$+ \frac{\rho^2}{\Delta} d\rho^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} \left[ (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta \right] d\varphi^2$$

where $\Delta(r) = r^2 - 2GMr + a^2$

$$\rho^2(\rho, \theta) = r^2 + a^2 \cos^2 \theta$$

Note: Include charge by changing $2GMr \rightarrow 2GMr - Q^2 \Delta_0 (\rho^2 + \rho^2)$.

And $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ with

$$A_t = \frac{Qr - \rho a \sin \theta}{\rho^2}$$

$$A_\varphi = -\frac{Qar \sin \theta + \rho (r^2 + a^2) \cos \theta}{\rho^2}$$

(Kerr-Newman)

The novel feature is $J = aM \neq 0$, so let's simply set $a = 1 = 0$ and study Kerr's solution.

Note that for $M = 0$ we have flat space but in weird coordinates (called Boyer-Lindquist coordinates):

$$ds^2 = -dt^2 + \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} d\rho^2 + \left( r^2 + a^2 \sin^2 \theta \right) d\theta^2 + \left( r^2 + a^2 \right) \sin^2 \theta d\varphi^2$$

Ellipsoidal coordinates

$$x = \sqrt{r^2 + a^2} \sin \theta \cos \varphi$$

$$y = \sqrt{r^2 + a^2} \sin \theta \sin \varphi$$

$$z = r \cos \theta$$
Killing vectors: $\theta^t$ and $\theta^\phi$

Also a killing tensor $K_{\mu\nu} = \frac{\partial}{\partial \lambda} (l^\mu n^\nu + l^\nu n^\mu) + r^2 g_{\mu\nu}$

with $l^\mu = \frac{1}{A} (r^2 + \theta^2, 0, 0, 0)$ and $n^\mu = \frac{1}{2r^2} (r^2 + \theta^2, -\Delta, 0, 0)$

They have $l^2 = n^2 = 0$ $\quad l \cdot n = -1$.

So geodesics are easy to find: three constants (not just per unit mass)

$$\frac{d}{d\lambda} l^\mu = \frac{\partial}{\partial \lambda} (l^\mu n^\nu + l^\nu n^\mu)$$

plus

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = -1 \quad \text{or} \quad 0 \quad \text{for timelike or null}$$

Note that

$$\frac{dE}{d\lambda} = \frac{1}{2} (r^2 + \theta^2) \frac{dt}{d\lambda} + \frac{2GM}{\rho^2} \frac{d\phi}{d\lambda} = -g_{tt} \frac{dt}{d\lambda} - g_{t\phi} \frac{d\phi}{d\lambda}$$

$$\frac{dL}{d\lambda} = g_{t\phi} \frac{dt}{d\lambda} + g_{\phi\phi} \frac{d\phi}{d\lambda}$$

and

$$\frac{\mathcal{L}}{\dot{\mathcal{L}}} = -g_{tt} - g_{t\phi} \frac{dt}{d\lambda} \frac{d\phi}{d\lambda}$$

so if $\mathcal{L} \equiv d\mathcal{L}/d\lambda$ we have

$$g_{t\phi} + c_0 g_{\phi\phi} + \frac{\mathcal{L}}{\dot{\mathcal{L}}} (g_{tt} + g_{t\phi} \frac{dt}{d\lambda}) = 0$$

$$\omega = \frac{-c_0 (g_{tt} + g_{t\phi} \frac{dt}{d\lambda})}{g_{t\phi} + \frac{\mathcal{L}}{\dot{\mathcal{L}}} g_{\phi\phi}}$$

so in particular, even if $L = 0$ we can have $\omega \neq 0$ ($\omega = -g_{tt}/g_{t\phi}$) or with $\omega = 0$ we can have $L \neq 0$. 
Horizon: \( g^{rr} = 0 \iff \frac{A}{r^2} = 0 \). Since \( p^2 > 0 \), this is \( A = 0 \) or
\[
R^2 - 2GM + a^2 = 0
\]
or
\[
R = R_+ = GM \pm \sqrt{(GM)^2 - a^2} \quad (GM > |a|)
\]

Stationary Limit Surface: by definition, this is a surface where \( \delta c \)
becomes null:
\[
g_{\nu} \left( \frac{\partial}{\partial \nu} \right) \delta c = 0 \iff 1 - \frac{2GM}{r} = 0
\]
or
\[
r^2 + a^2 \cos^2 \theta - 2GM = 0 \quad (\theta = \Delta t = 0)
\]

\( \delta c \) is spacelike outside the outer horizon! It is the "ergosphere".
Moreover, at \( r = r_+ \)
\[
g_{\nu} \left( \frac{\partial}{\partial \nu} \right) \delta c = \frac{r^2 + a^2 \cos^2 \theta - 2GM}{r^2 + a^2 \cos^2 \theta} = \frac{a^2 \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \geq 0
\]
and this equality holds only at \( a = 0 \) where the stationary limit surface
and \( r = r_+ \) coincide.

The ergosphere is quite peculiar. Consider for simplicity the \( \theta = \frac{\pi}{2} \)
null lines have (with \( r^2 \) constant) look at tangential emission:
\[
0 = g^t_t \delta t^2 + 2g^{t\theta} \delta t \delta \theta + g^{\theta\theta} \delta \theta^2
\]
\[
\therefore \omega = \frac{d \delta t}{d \theta} = \frac{g_{tt}}{g_{\theta\theta}} + \sqrt{\left( \frac{g_{tt}}{g_{\theta\theta}} \right)^2 - \left( \frac{g_{tt}}{g_{\theta\theta}} \right)^2}
\]
Ergosphere! \( g_{tt} > 0 \) so
\[
\sqrt{\left( \frac{g_{tt}}{g_{\theta\theta}} \right)^2 - \left( \frac{g_{tt}}{g_{\theta\theta}} \right)^2} \leq \left| \frac{g_{tt}}{g_{\theta\theta}} \right| \quad \therefore \text{both solutions } k \text{ have same sign!}
\]
and at stationary limit surface one solution has \( \omega = 0 \).
In fact $\frac{g_{tt}}{g_{rr}} = \frac{2GM_{ar} \sin^2 \theta}{\sin^2 \theta \left[ (r^2 + a^2)^2 - a^2 r^2 \sin^2 \theta \right]} = c^2$

The $K$ sign determined by $a = J/M$.

$\Rightarrow$ photons emitted tangentially (with $r=0$ and $\theta = 0$) from the ergosphere move in same direction as rotation of black hole.
Null geodesics in more detail:

We had

\[ E = -g_{tt} \frac{dt}{d\lambda} - g_{\theta\theta} \frac{d\theta}{d\lambda} \quad L = g_{\phi\phi} \frac{d\phi}{d\lambda} + g_{rr} \frac{dr}{d\lambda} \]  

(1)

Instead of doing most general, why not we limit ourselves to \( \theta = \frac{\pi}{2} \) trajectories. Then

\[ g_{\nu\mu} \frac{dx^\nu}{d\lambda} \frac{dx^\mu}{d\lambda} = 0 \]

\[ g_{tt} \left( \frac{dt}{d\lambda} \right)^2 + 2 g_{\phi\phi} \left( \frac{d\phi}{d\lambda} \right) \left( \frac{dt}{d\lambda} \right) + g_{\phi\phi} \left( \frac{d\phi}{d\lambda} \right)^2 + g_{rr} \left( \frac{dr}{d\lambda} \right)^2 = 0 \]

Solve (1) above for \( \frac{dt}{d\lambda} \) and \( \frac{d\phi}{d\lambda} \), write

\[ M \left( \frac{d\phi}{d\lambda} \right) = \left( \begin{array}{c} -E \\ L \end{array} \right) \quad \text{where} \quad M = \begin{pmatrix} +g_{\phi\phi} & -g_{\phi\theta} \\ g_{\phi\theta} & g_{rr} \end{pmatrix} \]

we need \( M^{-1} \) which is just \( \text{Laplace of the metric} \):

\[ M^{-1} = \frac{1}{g_{tt} g_{\phi\phi} - g_{\phi\theta}^2} \begin{pmatrix} g_{\phi\phi} & -g_{\phi\theta} \\ -g_{\phi\theta} & g_{rr} \end{pmatrix} \]

\[ \left( \frac{dt/d\lambda}{d\phi/d\lambda} \right) = M^{-1} \left( \begin{array}{c} -E \\ L \end{array} \right) = \frac{1}{g_{tt} g_{\phi\phi} - g_{\phi\theta}^2} \begin{pmatrix} -g_{\phi\phi} E - g_{\phi\theta} L \\ g_{\phi\theta} E + g_{rr} L \end{pmatrix} \]

So

\[ g_{tt} \left( \frac{g_{\phi\phi} E + g_{\phi\theta} L}{E + g_{rr} L} \right)^2 \quad \frac{dt/d\lambda}{d\phi/d\lambda}^2 \quad 2 g_{\phi\phi} \frac{1}{D^2} \left( g_{\phi\phi} E + g_{\phi\theta} L \right) \left( g_{\phi\phi} E + g_{\phi\theta} L \right) \]

\[ + \frac{1}{D^2} \left( g_{\phi\phi} \left( g_{\phi\phi} E + g_{\phi\theta} L \right)^2 \right) = + g_{rr} \left( \frac{dr}{d\lambda} \right)^2 = 0 \]

This is of the form

\[ \left( \frac{dr}{d\lambda} \right)^2 + V_{\text{eff}} = 0 \]
\[ \text{Compl: } \left( s + \theta = \frac{3}{2} \right) \quad \left( r^2 = r^2 \right) \quad \left( \Delta = r^2 + a^2 - 2GMr = r^2 - 2GMr \right) \]

\[ D = g_{tt} g_{\theta\theta} - g_{\theta\theta} = - \left( 1 - \frac{2GMr}{r^2} \right) \left( \frac{(r^2 + a^2)^2 - a^2 \theta^2}{r^2} \right) - \left( \frac{2GMr}{r^2} \right)^2 \]

\[ = - \frac{\Delta}{r^2} \left( \rho^2 - a^2 \theta^2 \right) - \left( \frac{2GMr}{r^2} \right)^2 \]

\[ \text{Better yet, since } \omega = -\frac{9\omega}{3\omega}, \quad D = g_{tt} g_{\theta\theta} - \omega^2 g_{\phi\phi} = g_{tt} (g_{tt} - \omega^2 g_{\phi\phi}) \]

\[ V_{,tt} D^2 g_{rr} = g_{tt} \left( g_{\phi\phi} E^2 + 2EL g_{\phi\phi} g_{tt} + L^2 g_{\phi\phi} \right) \]

\[ = g_{tt} \left( g_{\phi\phi} g_{\phi\phi} E^2 + EL \left( g_{\phi\phi} g_{\phi\phi} + g_{tt} g_{tt} \right) \right) \]

\[ + g_{tt} \left( g_{\phi\phi} g_{\phi\phi} E^2 + 7 g_{\phi\phi} g_{tt} EL + g_{tt} L^2 \right) \]

\[ = E^2 \left( g_{tt} g_{\phi\phi} - 2 g_{tt} g_{tt} + g_{tt} g_{tt} \right) \]

\[ + 2EL \left( g_{tt} g_{tt} g_{\phi\phi} - g_{tt} g_{tt} g_{tt} + g_{tt} g_{tt} g_{tt} \right) \]

\[ + L^2 \left( g_{tt} g_{tt} - 2 g_{tt} g_{tt} + g_{tt} g_{tt} \right) \]

\[ = E^2 \left( g_{tt} g_{\phi\phi} - \omega^2 g_{\phi\phi} \right) + 2EL \left( \omega^2 g_{\phi\phi} - \omega^2 g_{\phi\phi} \right) \]

\[ + L^2 \left( g_{tt} g_{tt} - \omega^2 g_{tt} g_{tt} \right) \]

\[ V_{,tt} = \frac{g_{tt}}{g_{rr} D \left[ E^2 - 2EL \omega + L^2 \frac{g_{tt}}{g_{rr}} \right]} \]

\[ \approx \frac{A}{r_1^2} \left[ \frac{(r_1^2 + a^2)^2 - a^2 \theta^2}{r_1^2} \right] \frac{E^2 - 2EL \omega + L^2 \frac{g_{tt}}{g_{rr}}}{r_1^2 (r_1^2 + a^2) - 2GMr a^2} \]

\[ \text{preferably} \quad \frac{1}{r_1^2 + a^2 GMr} \left[ (r_1^2 + a^2)^2 - a^2 \theta^2 - 2GMr a^2 \right] \]

\[ V_{,tt} = \frac{1}{g_{rt} \left( g_{tt} - \omega^2 g_{tt} \right)} \left[ E^2 - 2EL \omega + L^2 \frac{g_{tt}}{g_{rt}} \right] \]
Since \( \left( \frac{d}{dt} \right)^2 > 0 \) solutions only exist for

\[(E-V_+)(E-V_-) > 0 \]

that is both \( V_+ > E \) or both \( V_- < E \).

So if \( V_+ \approx \infty, V_\approx \pm \frac{L}{r} \). Let's take \( L > 0 \).

Since \( V_+ > V_- \), where we should also note that \( L < 0 \), since previously we relabeled \( L \) and \( a \frac{r^3}{M^2} \) as \( \lambda \), and we are assuming \( a > 0 \).

Clearly \( V_+ = V_- + \Delta = 0 \) \( \Rightarrow \) the event horizon \( r = r_+ \).

Therefore

\[ V_+ = V_- = \frac{2GMraL}{(r_+^2 + a^2)^2} = \frac{aL}{2GMr_+} \]

Note that \( V_+ \) has no zeroes, while \( V_- \) has a zero at

\[ 2GMr_0a = r_0^2 \Delta(r_0) \]

\[ = (2GMa)^2 \frac{r_0^2}{(r_0^2 + a^2 - 2GMr_0)} \]

In principle four zeroes, but note that \( q + a = 0 \) three zeroes are on \( r = 0 \) while one is at \( r = 2GM \). So we suspect only one zero is in the region \( r > r_+ \).

\[ L < 0 \]
Now, writing \(-g_{tt} = 1 - \frac{2GMr}{\rho^2} = \frac{\Delta - a^2 \sin^2 \theta}{\rho^2}\),

we have \(a + \theta = \frac{\pi}{2}\)

\[
\frac{g_{tt}}{g_{\phi\phi}} = -\frac{\Delta - a^2}{(\rho^2 + a^2)^2 - a^2 \Delta}, \quad \omega = \frac{2GMra}{(\rho^2 + a^2)^2 - a^2 \Delta}
\]

\[
\omega^2 - \frac{g_{tt}}{g_{\phi\phi}} = \frac{(2GMra)^2 + (\Delta - a^2) [Cr^2 + a^2]}{[Cr^2 + a^2]^2 - a^2 \Delta}
\]

\[
\text{numerator} = (2GMra)^2 + (r^2 - 2GMr) [C(r^2 + a^2) (r^2 + a^2) + 2GMra^2]
\]

\[
= r^2 [(1 + a^2) r^2 + 2GMra^2] - 2GMr [C(r^2 + a^2)] r^2
\]

\[
= (C(r^2 + a^2)) r^4 - 2GMr^5
\]

\[
= r^4 (C(r^2 + a^2) - 2GM) = r^4 \Delta
\]

so

\[
V_\pm = L \left[ \frac{2GMra \pm \frac{1}{2} \sqrt{\Delta}}{(\rho^2 + a^2)^2 - a^2 \Delta} \right]
\]

and we have

\[
\left( \frac{dr}{d\lambda} \right)^2 = \frac{(E - V_+) (E - V_-)}{g_{rr} (g_{tt} - \omega^2 g_{\phi\phi})}
\]

\[
d\psi = -g_{rr} \frac{g_{tt}}{g_{\phi\phi}} (\omega^2 - \frac{g_{tt}}{g_{\phi\phi}})
\]

\[
= \frac{F^2}{\Delta} \rho^2 \frac{\Delta}{[C(r^2 + a^2)^2 - a^2 \Delta]^{\frac{3}{2}}}
\]

\[
\Delta \rho^2 \frac{\Delta}{[C(r^2 + a^2)^2 - a^2 \Delta]^{\frac{3}{2}}} = \frac{r^4}{[C(r^2 + a^2)^2 - a^2 \Delta]^2}
\]

\[
= \frac{(dr)}{d\lambda} = \frac{C(r^2 + a^2)^2 - a^2 \Delta}{r^4} \frac{r^4 \Delta}{(E - V_+) (E - V_-)}
\]
Inside process

\[ \text{energy} = p_{(1)} + p_{(2)} \]

or, consistently with \((\mathcal{D})\)^\mu

\[ E_{(\mathcal{D})} = E^{(1)} + E^{(2)} \]

Clearly, \(E^{(2)} > 0\), but if you push \(E^{(2)}\) hard enough you can arrange \(E^{(2)} < 0\) so

\[ E^{(1)} > E^{(2)} \]

\(\Rightarrow\) come out of ergosphere with more than the original total energy.

Energy comes from black hole \(\Rightarrow\) reduce bh's angular momentum

Crock must be thrown against rotation of bh.

To see this, let's figure out the condition that the rock \(r\) crosses the event horizon \(R_+\). We must be slightly careful since \(r=R_+\) is a null surface.
Killing Horizons: If a Killing vector $\xi^\mu$ is null on a null hypersurface $\Sigma$, then we say $\Sigma$ is a Killing Horizon.

For Kerr, $\partial^\nu$ is not null on the event horizon; it is null on the SCS (stationary limit surface) by def'n.

The event horizon is null, and
$$\Sigma^\mu = \delta^\mu_k + \kappa \xi^\mu$$
is null for some constant $\kappa$. Exercise: show $\kappa = \frac{g}{a^2 + b^2}$

Calculate: $\chi^2 = 0 = \delta^k_k + 2\Omega_\nu \partial^\nu \partial_\nu + \Omega_\nu \partial_\nu$
$$= g_{tt} + 2a \Omega_\nu g_{\nu \theta} + \Omega_\nu g_{\nu \theta}$$

so $\Omega_\nu = -\frac{g_{tt}}{g_{\theta \theta}} + \sqrt{\left(\frac{g_{tt}}{g_{\theta \theta}}\right)^2 - \frac{g_{tt} g_{\nu \theta}}{g_{\theta \theta}}} - \frac{g_{tt} g_{\nu \theta}}{g_{\theta \theta}}$

Now on $r = r_+$ $A = r^2 - 2GM + a^2 = 0$ and
$$g_{tt} = -(1 - \frac{2GM}{r^2}) = -\frac{1}{r^2} \left( r^2 - 2GM \right) = -\frac{1}{r^2} \left( r^2 - a^2 \cos^2 \theta - r^2 - a^2 \right)$$
$$= \frac{1}{r^2} a^2 \sin^2 \theta$$

$$g_{\theta \theta} = \frac{\sin^2 \theta}{r^2} \left[ (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta \right] = \frac{\sin^2 \theta}{r^2} (1 + ta^2)^2$$

$$g_{\phi \phi} = -26Mra \frac{\sin^2 \theta}{r^2} = -\frac{\alpha a (a^2 + l^2) \sin^2 \theta}{r^2}$$

so $\Omega_\nu = \frac{a}{a^2 + b^2} + \sqrt{\frac{a^2}{a^2 + b^2} - \frac{a^2}{a^2 + b^2} = \frac{a}{a^2 + b^2}}$
Exercise: show $r = R_+$ is null.

Note: if $\Sigma$ is defined through $f(x^i) = \text{constant}$, then $\Sigma$ is null $\Leftrightarrow$ $\nabla f$ is null.

[Calculate: $r = R_+$ is the same as $f(t, r, \theta, \phi) = r$. Then
\[
\frac{\partial}{\partial r} \left( g^{rr} \frac{\partial}{\partial r} \right) = 0 \quad \text{since} \quad g^{rr} = 0 \quad \text{at} \quad r = R_+.\]
To see what condition we want to impose on \( p^{(a)} \) (that signals \( p^{(a)} \) crosses \( r^+ \)), look at a null case in Minkowski space first:

![Minkowski diagram](attachment:image)

The surface is defined by \( x-t = \text{constant} \), and \( \nabla_x(x-t) \) is a null vector (both normal and tangent to it).

\[
\nu_x = \nabla_x(x-t) = \nu^\mu (1,1)
\]

Now, if we have a particle moving along \( x^\mu(x) \), then

\[
\nu \cdot \frac{dx}{d\lambda} = -\frac{dt}{d\lambda} + \frac{dx}{d\lambda}
\]

So, if \( \nu \cdot \frac{dx}{d\lambda} < 0 \) then \( \frac{dx}{dt} < \frac{dt}{d\lambda} \) or \( \frac{dx}{dt} < 0 \)

while if \( \nu \cdot \frac{dx}{d\lambda} > 0 \) then \( \frac{dx}{dt} > 0 \)

Going back to our problem, if \( X^\mu \) is a null tangent to \( r=r^+ \) then \( X \cdot p^{(a)} < 0 \) at \( r=r^+ \) signals motion inwards, but moreover since \( X \) is a Killing vector \( X \cdot p^{(a)} = \text{constant} \).

Now \( \bar{X} = \bar{\delta}_t + \frac{p_t}{a} \frac{\partial}{\partial \theta} \) with \( \bar{\delta}_t = \frac{a^2}{a^2 + p^2} \) is a null Killing vector on \( r=r^+ \). \( \bar{\delta}_t \) can be interpreted as the angular velocity of the black hole at the event horizon \( r^+ \), or can be \( a \), since it corresponds to the minimum for a massive particle at \( r=r^+ \).

Then the condition is \( X \cdot p^{(a)} < 0 \)

\[
\Rightarrow X \cdot p^{(a)} = \bar{\delta}_t \cdot p^{(t)} + \bar{r}_t \frac{\partial}{\partial \phi} p^{(a)} = -E^{(a)} + L^{(a)} < 0
\]

\[
\Rightarrow L^{(a)} < \frac{E^{(a)}}{a^2} < 0
\]

So the angular momentum of the black hole decreases by \( L^{(a)} \).
\[ \delta M = L^{(2)} \]
\[ \delta J = L^{(2)} \]

\[ a \cdot \delta J < \frac{\delta M}{\Omega H} \]

Conclusion: Energy is extracted from the black hole. As a result, the black hole loses mass and spin.

However, the process cannot violate the area theorem, that the area of the event horizon is non-decreasing.

The area of the event horizon is

\[ A = 4\pi (R^2 + a^2) \]

2. Calculate from \( ds^2 = g_{ij} dx^i dx^j = ds^2 + dt^2 = 0, r = R_+ \).

Then

\[ A = \int |d\mathbf{s}| d\phi d\theta \]

\[ = \int \sqrt{\left( \frac{\sin\theta}{\rho^2} \left[ (R^2 + a^2)^2 - a^2 \Delta \sin^2 \theta \right] \right)} \, d\theta d\phi \]

At \( r = R_+ \), we have \( \Delta = 0 \) so

\[ A = (R^2 + a^2) \int \sin\theta d\theta d\phi \]

To show that \( A \) is non-decreasing, define the "improved mass" through

\[ M^2_{\text{irr}} = \frac{A}{16\pi G^2} = \frac{1}{16\pi^2} (R^2 + a^2) \]

\[ \Delta = 0 \Rightarrow R_+^2 + a^2 = 2GM \]

\[ R_+ = GM + \sqrt{(GM)^2 - a^2} \]

\[ \frac{1}{4\pi^2} 2GM [GM + \sqrt{(GM)^2 - a^2}] \]

or

\[ M^2_{\text{irr}} = \frac{1}{2} \left[ M^2 + \sqrt{M^4 - (J/c)^2} \right] \quad (J = Ma) \]
Now
\[ 2 \cdot 2M_{\text{irr}} \delta M_{\text{irr}} = 2M \delta M + \frac{4M^2 \delta M - 2 J \delta J / c^2}{\sqrt{M^4 - (J/c)^2}} \]
\[ = \frac{2(M \sqrt{M^4 - (J/c)^2} + J/c)^2 \delta M - J \delta J / c^2}{\sqrt{M^4 - (J/c)^2}} \]
we recognize
\[ M^3 + M \sqrt{M^4 - (J/c)^2} = M \left( M^2 + \sqrt{M^4 - (J/c)^2} \right) \]
\[ = 2M M_{\text{irr}} \]
\[ = 2M \frac{1}{4c^2} (R^2 + a^2) \]
\[ = 2M \frac{a}{4c^2} \frac{1}{\Omega_H} \]
\[ = \frac{J}{4c^2} \Omega_H \]
so
\[ \delta M_{\text{irr}} = \frac{J/c^2}{4M_{\text{irr}} \sqrt{M^4 - (J/c)^2}} \left[ \frac{\delta M}{\Omega_H} - \delta J \right] \]
so our bound that \( \delta J < \delta M / \Omega_H \) implies \( \delta M_{\text{irr}} > 0 \)
Now \( \delta A = 16 \pi c^2 \delta M_{\text{irr}} = \frac{8 \pi J \left( \delta M / \Omega_H - \delta J \right)}{\sqrt{M^4 - J^2/c^2}} \)
or
\[ \delta M = \frac{k}{8 \pi c} \delta A + \Omega_H \delta J \]
where \( k = \frac{\sqrt{G^2 M^2 - J^2/c^2}}{J/M} = \frac{\sqrt{G^2 M^2 - a^2}}{r_e + a^2} \)
or \( k = \frac{\sqrt{G^2 M^2 - a^2}}{2 \sqrt{2GM + \sqrt{(2GM)^2 - a^4}}} \)
[Note: \( \kappa \) is the surface gravity of the Kerr metric. For a Killing horizon with Killing (null) vector \( \mathcal{X} \), the surface gravity is 
\[ \kappa^2 = -\frac{1}{2} (\nabla \mathcal{X}) (\nabla^\mu \mathcal{X}^\mu) \]

study this later]

Now 
\[ \delta M = \frac{\kappa}{8\pi G} \delta A + \mathcal{A} \delta T \]

is just like 
\[ dE = \mathcal{A} T dS - \mathcal{P} dV \]

for a thermodynamic system, with the association
\[ E \equiv M \]
\[ A \equiv S \]
\[ \frac{T}{\mathcal{A}} \equiv \frac{\kappa}{8\pi G} \]

The ambiguity in the association of \( A \equiv T \) (where do we put the \( 8\pi G \)) is settled by Hawking's black hole evaporation.

**Thermodynamics**

**Black Holes**

Ohlaw: \( T \) is constant in thermal equilibrium. Stationary black holes have constant \( T \).

1st Law: Energy conservation

2nd Law: \( \delta S > 0 \)
\[ \delta A > 0 \]

(Generalized 2nd law \( \delta (S + \frac{\mathcal{A}}{8\pi G}) > 0 \)).

Note: To make sense of units, \( S \) is dimensionless (\( k_B = 1 \)) but \( \frac{\mathcal{A}}{8\pi G} \) has units of mass x length, same as \( T \). So it really should be
Stationary axisymmetric space: general observations,

(i) General case: require \( g_{uv} = g_{uv}(r, \theta) \) (not of \( t, \phi \)), and symmetry \( t \to -t, \phi \to -\phi \) (so \( g_{tt} = g_{rr} = 0; g_{\phi \phi} \)).

\[ ds^2 = -\hat{A} dt^2 + B d\phi^2 - 2B \omega dt d\phi + C dr^2 + D d\theta^2 \]

\[ = -A dt^2 + B (d\phi - \omega dt)^2 + C dr^2 + D d\theta^2 \quad \hat{A} = A - B \omega^2 \]

Note that

\[ g^{tt} = \frac{1}{A}, \quad g^{\phi \phi} = \frac{1}{\hat{A}} \quad \text{if} \quad g = \begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{\phi t} & g_{\phi \phi} \end{pmatrix} \Rightarrow \quad g^{-1} = \frac{1}{det \, G} \begin{pmatrix} \frac{\partial \phi}{\partial \phi} & \frac{-\partial \phi}{\partial r} \\ \frac{-\partial \phi}{\partial r} & \frac{\partial \theta}{\partial \theta} \end{pmatrix}, \quad \omega = \frac{g_{t\phi}}{g_{\phi \phi}} = \frac{g_{\phi t}}{g_{t\phi}} \]

\[ \Rightarrow \quad g^{tt} = -\frac{1}{A}, \quad g^{\phi \phi} = \frac{A - B \omega^2}{\hat{A}} \quad g^{\theta \theta} = -\frac{\omega}{A}, \quad \hat{A} = A - B \omega^2 \]

For Kerr, plug into \( \hat{A} = 0 \). luxury resid.

(ii) Killing vectors \( \partial_t, \partial_\phi \Rightarrow \) curved spacetime

\[ L = \rho \hat{g} = m g_{tt} \frac{dx^t}{dt} \quad \text{and} \quad E = \rho \hat{e} = m g_{\phi \phi} \frac{dx^\phi}{dt} \]

\( \omega \), replace \( \gamma \rho \to \lambda \), for massless.

More explicitly

\[ L = g_{\phi \phi} \frac{d\phi}{dt} + \frac{\partial +}{\partial \lambda} \]

\[ E = g_{t \phi} \frac{d\phi}{dt} + \frac{\partial +}{\partial \lambda} \]

\[ L = 0 \quad \frac{d\omega}{dt} = -\frac{\partial \phi}{\partial \theta} = \frac{c'}{c} \quad \text{ANGULAR VELOCITY WITHOUT ANGULAR MOMENTUM} ? \]

Suppose metric is asymptotically flat (as in Kerr). Then “drop” body from \( \infty \)

towards center \( \rho \to 0 \) towards \( r = 0 \) with \( \frac{d\theta}{dt} = 0 \) originally (since \( \cos(\theta) = 1 \)).

Then \( \frac{d\phi}{dt} \) will change as body drops.

“Dragging of inertial frames”: our test body is free falling, so locally it is moving

in straight line \( \rightarrow \) interpret \( \frac{d\phi}{dt} \to 0 \) as moving/accel dry metric frames.
(iii) Stationary limit surface

Consider photon emitted in $\theta$ - direction (from (r, $\theta$, $\phi$))

At emission $d\theta = 0 = dr \Rightarrow ds^2 = 0 = g_{tt} dt^2 + 2 g_{t\theta} dtd\theta + g_{\theta\theta} d\theta^2$

$$\Rightarrow \frac{d\theta}{dt} = \frac{\frac{g_{tt}}{g_{\theta\theta}} \left( \frac{\partial g_{tt}}{\partial t} - \frac{\partial g_{tt}}{\partial \theta} \right)}{\sqrt{\left( \frac{\partial g_{tt}}{\partial t} \right)^2 - \frac{\partial g_{tt}}{\partial \theta} \frac{\partial g_{tt}}{\partial \theta}}} = \omega \sqrt{\frac{\omega}{B}}$$

While $g_{tt}/g_{\theta\theta} < 0$ get $< \frac{d\theta}{dt} > 0 \Rightarrow \theta$ emitted in + $\delta$R

$\frac{d\theta}{dt} < 0 \Rightarrow \theta$ emitted in - $\delta$R

- On a $g_{tt} = 0$ surface $(\frac{A}{B} = \omega)$ $\frac{d\theta}{dt} = \frac{2\omega}{\sqrt{B}}$ two way test

Massive particles all dragged in same direction on $g_{tt} = 0$ surface

"Stationary limit surface" = any surface with $g_{tt} = 0$

Q: Schwarzschild? (leave for student to ponder)

Inside stationary limit surface all bodies and radiation are forced to move in same direction, cannot remain fixed.

To see this
Suppose \( U^\alpha \) is \( \alpha \)-vel of body, \( U^\alpha = 1 \). If we take \( U^\alpha = (U_t,0,0,0) \)
\[ \eta_{tt} = -\frac{1}{2}U^t U^t < 0, \]
which is incompatible with interior of \( \text{lim. surf.} \).
But recalling that \( g_{k1}(U^1) + g_{k2}(U^2 + g_{k3} U^3) = -1 \) since \( g_{k4} = 0 \)
and the relative sign of \( U^1, U^2 \) are not fixed, but if \( \eta_{tt} = 0 \) we neglect \( g_{k1}(U^1) \)
and get \( U^\alpha (U^\alpha - 2U^t U^t) = -\frac{1}{2} \) which is easily satisfied.

(iii) Redshift: recall for comoving observers (fixed coordinates)

\[ \lambda_{rec} = \sqrt{g_{tt}(rec)} \sqrt{g_{tt}(event)} \]

For an observer at stationary limit surface, \( g_{tt}(event) \rightarrow 0 \Rightarrow \lambda_{rec} \rightarrow \infty. \)
This is just as with Schwarzschild.

(iv) Event Horizons. Again we look for null 3-surfaces.
\[ f(x^\alpha) = 0 \]
defines surface

\[ \Delta f = \text{gradient} = \text{normal to surface} = n^\alpha. \]

Tangent \( f(x^\alpha) = 0 \Rightarrow \text{d} f(x^\alpha) = \frac{\partial f}{\partial x^\alpha} \text{d} x^\alpha \Rightarrow \frac{\partial f}{\partial x^\alpha} n^\alpha = 0 \Rightarrow \text{vector} \frac{\partial}{\partial x^\alpha}. \)

In particular, if \( n^\alpha \) is null then \( n^\alpha = g_{\alpha \beta} n^\beta \parallel n_{\alpha} (g_{\alpha \beta} n^\beta = 0) \)
\( \Rightarrow \) look for \( g_{\alpha \beta} \Delta f = 0. \)

Recall in spherically symmetric case we take \( f = f(1) \)
\[ g_{\alpha \beta} \Delta f = 0 \Rightarrow (\Delta f)^2 = 0 \Rightarrow \Delta f = 0. \]
Now with axial symmetry, \( f = f(r, \theta), \)
\[ (\Delta f)^2 = 2 \frac{\partial f}{\partial r} \frac{\partial f}{\partial r} + 2 \frac{\partial f}{\partial \theta} \frac{\partial f}{\partial \theta} + \frac{\partial f}{\partial \phi} \frac{\partial f}{\partial \phi} = 0. \]
We can still look for solutions with \( f = f(1) \) and \( \Delta f = 0. \) Let's
look at this (and others) in Kerr metric.
Back to Kerr

(i) Singularities. From $R^{+\infty}$ B_{Kr} one finds $\rho=0$ is a singularity.

Now $\rho^2 = r^2 + a^2 \sin^2 \theta = 0 \Rightarrow r=0 \quad \theta = \frac{\pi}{2}$

Recall Boyer–Lindquist coordinates in Cartesian:

\[
x = \frac{1}{\rho^2} \sin \theta \cos \phi
\]

\[
y = \frac{1}{\rho^2} \sin \theta \sin \phi
\]

\[
z = r \cos \theta
\]

\[
x^2 + y^2 = a^2 \cos^2 \theta
\]

$\Rightarrow \quad \theta = \frac{\pi}{2}$ in $x^2 + y^2 = a^2$ circle ("equator of")

The singularity is not a point but $\mathbb{S}^2$.

(ii) Event Horizon: look for $\Omega_r = 0$. Now $\Omega_r = \frac{\Delta}{\rho^2}$, so we have $\Delta = 0$.

\[
\Rightarrow \quad r^2 - 2GMr + a^2 = 0 \Rightarrow r_+ = GM \pm \sqrt{(GM)^2 - a^2}
\]

Now, if $|a| < GM \Rightarrow r_+ < a < r_+$ and the singularity is behind the horizon (singularity at $r=0$, horizon $r_+$).

For $|a| > GM$, no horizon, naked singularity.

$|a| = GM - r_+$ is "extreme Kerr B.H." (It is believed, though calculation, that realistic BH's are near extremal Kerr B.H.'s, since aeration increases $a = \lambda / M$. Limited only by accreting matter radiating away some angular momentum. Calculations give $\lambda \approx 0.991GM$ - see text by Hobson, Ellis, Wiltshire, Lavenda, p. 324).

Geometry: take $r = \text{const.}$ ($=r_+$) $t = \text{const.}$ 2-dim surface. Line element is

\[
ds^2 = \frac{\rho^2}{\rho^2} d\theta^2 + \frac{(\rho^2 M \cos \theta)}{\rho^2} \sin^2 \theta \, d\phi^2
\]

Not the geometry of $S^2$ embedded in $R^3$. Rakes a pancake, or more technically, an axisymmetric ellipsoid (embedded in $R^3$).
(iii) Stationary Limit Surface: \( g_{tt} = 0 \)

\[
1 - \frac{2GM}{r^2} = 0 \Rightarrow r^2 + a^2 \cos^2 \theta - 2GM = 0
\]

\[
\Gamma^*_+ (\theta) = GM + \sqrt{(GM)^2 - a^2 \cos^2 \theta}
\]

(Not the same \( r_+ \) as before, exact \( \text{Kerr} \) solution)

Geometry (a) previous

\[
ds^2 = f_+^2 \, d\theta^2 + \frac{2GMr_+ (2M^2 r_+^2 + 2a^2 \sin^2 \theta)}{f_+^2} \, \sin^2 \theta \, d\varphi^2
\]

with \( f_+ \equiv f_+ (\theta) \)

\[
2 \dot{\varphi} = 2GMr_+.
\]

Which is based on: \( r_+ = GM + \sqrt{(GM)^2 - a^2 \cos^2 \theta} \)

\[
\Rightarrow r_+ (\theta) > r_+ > r_0 (\theta), \quad \text{with} \quad r_+ (0, \pi) = r_+ \quad \text{and} \quad r_0 (0, \pi) = r_0 \quad \text{(equal radials)}
\]

Also \( r_0 (\theta) = 0 \) is stationary.

In ergosphere, every mass is forced to move. As before \( u^2 = -1 \) with \( v^0 = (u_t, 0, 0) \)

\[
(\nu_t)^2 / \left( g_{tt} + 2g_{t\phi} \nu_t + g_{\phi\phi} \nu^2 \right) = -1 \quad \text{where} \quad \nu = \frac{v^0}{u^0} = \frac{df}{dt}
\]

\[
u^2 = -1 \Rightarrow g_{tt} + 2g_{t\phi} \nu_t + g_{\phi\phi} \nu^2 < 0 \Rightarrow \nu \in (\nu_-, \nu_+) \quad \text{with}
\]

\[
\nu^2 = -\frac{g_{tt}}{g_{tt} + 2g_{t\phi} \nu_t + g_{\phi\phi} \nu^2} = \nu \pm \sqrt{\frac{A}{B}}
\]

Special cases

(1) \( A = 0 \quad (\nu_+ = 0) \Rightarrow \nu_+ = 0 \quad \nu_+ = 2\omega \quad \text{This occurs at} \quad r = r_{+0} \) (stat. lim. self)

(2) \( A = \text{const.} \quad \omega^2 = \frac{g_{tt}}{g_{t\phi}} = \omega \quad \text{This occurs at} \quad r = r_0 \) so call \( \nu_+ \) weir.
We have \( \Omega_H = \omega(r_H, \theta) = \frac{a}{2GMr_H} \) (from Kerr metric).

At the horizon, the angular velocity is limited to the one value \( \Omega_H \).