Black Holes

Start with Schwarzschild:
As seen briefly in 1st quarter

\[
 ds^2 = -(1 - \frac{2GM}{r}) dt^2 + (1 - \frac{2GM}{r})^{-1} dr^2 + r^2 d\Omega^2
\]

is a solution of Einstein's equations in empty space

\[
 R_{\mu\nu} = 0
\]

that has spherical symmetry and it is static.

Birkhoff's theorem asserts that the Schwarzschild metric is the unique static, spherically symmetric solution of Einstein's equations in empty space.

We won't go over the proof here, but the ingredients are
(a) Define spherical symmetry as having corresponding symmetries: there are three killing vectors that generate the symmetries of the sphere. These are the generators of the Lie Algebra of the group of rotations, \( SO(3) \), that leave the sphere \( S^2 \) invariant.
(b) You are familiar with the algebra, it is just the same as angular momentum in QM:

\[
 [L_i, L_j] = i\epsilon_{ijk} L_k
\]

or, since we use anti-hermitian generators, let \( X_i = i\mathbf{L}_i \Rightarrow \)

\[
 [X_i, X_j] = i\epsilon_{ijk} X_k
\]

Exercise: transforming to spherical coordinates \( x_1 = r\theta_z - z\theta_r, x_2 = r\theta_r - x\theta_z, x_3 = x\theta_r - y\theta_x \) show that these Killing vectors are

\[
 R = \theta_r \quad S = \cos\theta_x = \cos(\varepsilon_{23}) = \sin\theta - \cos\theta \sin\varepsilon \quad T = -\sin\theta \theta_x - \cos\theta \cos\varepsilon \theta_y
\]
(iii) Frobenius theorem then allows one to show the space is foliated by 2-spheres. Basically the theorem says that if you have a set of vector fields that closes under commutation, \([X_i, X_j] = 0\), a linear combination of \(X_i\)'s, then the integral curves form a submanifold of the manifold on which they are defined.

(iii) Put spherical coordinates \(\theta, \phi\) on one sphere. Extend do other neighboring spheres using orthogonal geodesics

![both points same(\theta, \phi)]

and characterize the other spheres by two coordinates, say \(p, q\).

(The space of orthogonal geodesics through one point on a sphere is 4-2 = 2 dimensional). Then by suitable one has

\[
\text{d} s^2 = g_{pp}(p, q) \text{d} p^2 + 2g_{pq}(p, q) \text{d} p \text{d} q + g_{qq}(p, q) \text{d} q^2 + R(p, q) \text{d} \phi^2
\]

and by changing variables one can write

\[
\text{d} s^2 = T(t, r) \text{d} t^2 + R(t, r) \text{d} r^2 + R(t, r) \text{d} \phi^2
\]

(iv) Plug this into Einstein's equations and solve. Impose the condition that the metric is static. This too has to be done with some care. A metric is stationary if it has a timelike Killing vector near infinity, and a stationary metric is static if in addition the timelike Killing vector is orthogonal to a family of hypersurfaces.
Singularities in Schwarzschild.

It is difficult to define in general what is meant by a singularity. One common means of determining whether there is a singularity is to look for infinities in geometric quantities (coordinate independent), such as $R$, $R_{\alpha\beta} R^{\alpha\beta}$, $R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}$, etc.

In the case at hand, the metric

$$ds^2 = -(1 - \frac{2GM}{r}) dt^2 + (1 - \frac{2GM}{r})^{-1} dr^2 + r^2 d\Omega^2$$

is singular at $r = 2GM$ and at $r = 0$. But are these real singularities or artifacts of the metric?

In this case $R = 0$ and $R_{\alpha\beta} = 0$. But $R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \neq 0$ and computing explicitly one finds

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = \frac{48 G^2 M^2}{r^6}$$

Here is no singularity at $r = 2GM$ as far as this invariant can show, but there certainly is one at $r = 0$.

In fact we will introduce coordinate that have a perfectly regular metric at $r = 2GM$.

Another way of defining singularities is by finding inextendible geodesics that terminate at finite affine parameter. Let's study geodesics.
Geodesics

\[ ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2d \theta^2 + \sin^2 \theta d \phi^2 \]

\[ \Gamma^\mu_{\nu \lambda} = g^\mu_{\lambda} \Gamma^\nu_{\mu \lambda} - \frac{1}{2} g^\mu_{\lambda} (g_{\nu \lambda, \rho} + g_{\nu \rho, \lambda} - g_{\rho \lambda, \nu}) \]

\[ \Gamma^r_{tt} = -\frac{1}{2} g_{tt, r} = \frac{GM}{r^2} \]
\[ \Gamma^r_{tr} = \Gamma^r_{rt} = -\frac{GM}{r^2} \]
\[ \Gamma^r_{rr} = \Gamma^t_{tr} = -\frac{GM}{r^2} (1 - \frac{2GM}{r})^{-1} \]
\[ \Gamma^r_{rr} = \Gamma^t_{rr} = -\frac{ GM}{r^2} \]
\[ \Gamma^\theta_{r \theta} = \Gamma^\phi_{r \phi} = -\frac{GM}{r^2} \]
\[ \Gamma^\phi_{\theta \theta} = \Gamma^\theta_{\phi \phi} = -\frac{GM}{r^2} \]
\[ \Gamma^\theta_{\phi \phi} = \Gamma^\phi_{\theta \theta} = -\frac{GM}{r^2} \]

Geodesic Eqns:

\[ \frac{d^2 \lambda}{d \lambda^2} + \frac{2GM}{r(r - 2GM)} \frac{dt}{d \lambda} \frac{dr}{d \lambda} = 0 \]
\[ \frac{d^2 r}{d \lambda^2} + \frac{GM}{r^2} (1 - \frac{2GM}{r}) \left( \frac{dt}{d \lambda} \right)^2 - \frac{GM}{r(r - 2GM)} \left( \frac{dr}{d \lambda} \right)^2 \]
\[ -r \left( 1 - \frac{2GM}{r} \right) \left[ \left( \frac{d \theta}{d \lambda} \right)^2 + \sin^2 \theta \left( \frac{d \phi}{d \lambda} \right)^2 \right] = 0 \]
\[ \frac{d^2 \theta}{d \lambda^2} + \frac{2}{r} \frac{d r}{d \lambda} \frac{d \theta}{d \lambda} - \sin \theta \cos \theta \left( \frac{d \phi}{d \lambda} \right)^2 = 0 \]
\[ \frac{d^2 \phi}{d \lambda^2} + \frac{2}{r} \frac{d r}{d \lambda} \frac{d \phi}{d \lambda} + 2 \cot \theta \frac{d \theta}{d \lambda} \frac{d \phi}{d \lambda} = 0 \]
To solve these, use constants of the motion (1st integrals). We have four Killing vectors, three from $so(3)$ symmetry and one timelike Killing vector. For each:

$$K^a \frac{dx^a}{d\lambda} = \text{constant}$$

along the geodesic. Moreover, for massive particles, we can take $\lambda = c$ so that

$$\frac{dx^a}{dt} \frac{dx^a}{dt} g_{\mu \nu} = -1$$

timelike geodesic

and for massless particles

$$\frac{dx^a}{dt} \frac{dx^a}{dt} g_{\mu \nu} = 0$$

null geodesic.

The Killing vectors associated with $so(3)$ are like angular momentum, $L$. Just as in flat space, $L = \text{constant}$ implies motion in a plane orthogonal to $L$ and with the magnitude of $L$. So we can fix the plane of motion choosing

$$\alpha = \frac{\pi}{2}$$

The magnitude of $L$ corresponds to the Killing vector $\partial_\theta$

$$(\partial_\theta)^m = (0, 0, 0, 1)$$

In addition, the timelike Killing vector is

$$(\partial_\alpha)^m = (1, 0, 0, 0)$$

The conserved quantities are

$$E = -g_{\mu \nu} (\partial_\nu)^m \frac{dx^m}{dt} = (1 - 2mR) \frac{dt}{d\lambda}$$

and

$$L = (\partial_\theta)^m \frac{dx^m}{dt} g_{\mu \nu} = r^2 \sin^2 \theta \frac{d\theta}{d\lambda} = r^2 \frac{d\phi}{d\lambda} \quad (\sin^2 \theta = \frac{2}{r}).$$

The constants are named $E$ and $L$, suggestively, but these are just labels. We can discuss $E$ and $L$ and angular momenta in for.
For time-like geodesics we have $(U^0 U_a = -1)$:

$-(1 - \frac{2GM}{r})(\frac{dt}{d\xi})^2 + \left(1 - \frac{2GM}{r}\right)^{-1}(\frac{dr}{d\xi})^2 + r^2(\frac{d\theta}{d\xi})^2 = -1$

or multiply by $(1 - \frac{2GM}{r})$ and using $\xi = t$

$- E^2 + (\frac{dr}{d\xi})^2 + (1 - \frac{2GM}{r})\left(1 + \frac{L^2}{r^2}\right) = 0$

This is like a particle in a central potential

$\frac{1}{2} (\frac{dr}{d\xi})^2 + V(r) = E$

with $V(r) = \frac{1}{2} (1 - \frac{2GM}{r})(1 + \frac{L^2}{r^2}) \quad E = \frac{1}{2} E^2 \quad (1)$

The null geodesic is similar, but the LHS $-1$ is replaced by 0:

$\frac{1}{2} (\frac{dr}{d\lambda})^2 + V_n(r) = E$

$V_n(r) = \frac{1}{2} (1 - \frac{2GM}{r}) \frac{L^2}{r^2} \quad E = \frac{1}{2} E^2 \quad (5)$

(or, together)

$V(r) = \frac{1}{2} (1 - \frac{2GM}{r}) (\kappa + \frac{L^2}{r^2}) \quad \kappa = \frac{2}{r_+} \quad \text{null-like}$

$E_xpanding (5)$:

$V(r) = \frac{1}{2} - \frac{GM}{r} + \frac{L^2}{r^2} - \frac{GM L^2}{r^3}$

for $r < M$ Newttonian $\frac{\delta}{\nu} \rightarrow \frac{GM}{r}$

$\nu(r)$ non-relativistic

$\text{Extrema}$

$V(r)$ non-relativistic (Newtonian) $\frac{GM}{r} \rightarrow \frac{GM}{r_+}$

Schwarzschild
\[
\text{Find extrema } \frac{\partial V}{\partial r} = 0 = GMr^2 - L^2r + 3GML^2
\]
or
\[
R_+ = \frac{1}{2GM} \left[ L^2 \pm \sqrt{L^4 - 12(LGM)^2} \right]
\]

For \( 12GM^2 > L^2 \) \( \Rightarrow \) no extrema of \( V \)

For \( L^2 > 12GM^2 \), \( R_+ \) a minimum, \( R_- \) a maximum as in figure.
\( \Rightarrow \) stable circular orbit at \( r = R_+ \) (unstable at \( r = R_- \)).

Now \( R_+ \) for \( L^2 > 12(GM)^2 \): \( R_+ \approx \frac{L^2}{GM} \) the Newtonian formula.

Note, as we vary \( L^2 \), \( R_+ \) is smallest at \( L^2 = 12(GM)^2 \), while
\[
R_+ = \frac{1}{2GM} \left[ 12(GM)^2 \right] = 6GM
\]
so
\[
R_+ > 6GM
\]
Here is a smallest stable circular orbit.

(Similarly unstable circular orbits are restricted to by \( R_+ < 6GM \).)

\( \text{radii same calculation - and on low end the } L \to 0 \)
\[
R_- \to \frac{1}{2GM} \left[ L^2 - L^2 \left[ 1 - \frac{1}{2} \frac{12(GM)^2}{L^2} + \ldots \right] \right] = 3GM
\]
so \( 3GM < R_- < 6GM \). (Note that this calculation also gives...
Null geodesics:
\[
V_n(r) = \frac{L^2}{2r^2} - \frac{GMm^2}{r^3}
\]

\[
\left(\frac{\partial V_n}{\partial r} = 0 = \frac{L^2}{r^2} - \frac{3GMm^2}{r^3}\right)
\]

Now recall, \(E, L\) are (in arbitrary units) the energy and angular momentum of the particle (photon?), the energy necessary for the particle to go over the potential barrier is the height of the barrier:

\[
\frac{1}{2}E^2 = V_n(3GM) = \frac{L^2}{27(6M)^2} (\frac{1}{2}3GM - GM) = \frac{L^2}{54GM^2}
\]

\[
\Rightarrow \quad \frac{L}{E} = 3\sqrt{3}GM
\]

But \(L/E\) has a simple interpretation. In the asymptotically flat region \((r \gg GM)\) it corresponds to the impact parameter:

\[
b = \frac{L}{E}
\]

\[
L = b\rho \quad \text{and} \quad E = \rho \quad \text{(crossless)}
\]

For \(b < 3\sqrt{3}GM\) the photon is captured
For \(b > 3\sqrt{3}GM\) it is scattered

Capture cross section

\[
\sigma_c = \pi b_{\text{crit}}^2 = 27\pi(6M)^2
\]
Red-Shift

Similar to what we did before:

\[
\begin{align*}
\frac{dt}{d\lambda} & = \sqrt{1 - \frac{2GM}{r}} \\
\frac{dt}{d\lambda} & = \frac{E}{\sqrt{1 - \frac{2GM}{r}}}
\end{align*}
\]

Now, if \( k^\mu = \frac{dx^\mu}{d\lambda} \) on the null geodesic for the photon, then

\( \text{He observers measure} \quad \omega = - u \cdot k = + \sqrt{1 - \frac{2GM}{r}} \frac{dt}{d\lambda} \)

and

\[
\frac{dt}{d\lambda} = \frac{E}{\sqrt{1 - \frac{2GM}{r}}} \Rightarrow \omega = \frac{E}{\sqrt{1 - \frac{2GM}{r}}}
\]

Since \( E \) is constant we have

\[
\omega \sqrt{1 - \frac{2GM}{r_1}} = \omega_0 \sqrt{1 - \frac{2GM}{r_2}}
\]

or

\[
\frac{\omega_2}{\omega_0} = \sqrt{\frac{1 - \frac{2GM}{r_1}}{1 - \frac{2GM}{r_2}}}
\]

Gravitational redshift,

For weak fields,

\[
\frac{\omega_2}{\omega_0} = 1 - \frac{GM}{r_1} + \frac{GM}{r_2} = 1 + \phi_1 - \phi_2 = 1 - \Delta \phi
\]

which is the formula obtained first quantitatively (see Schwarzschild) on general grounds (principle of equivalence) for weak fields,
Kruskal Coordinates and Extension

Coordinate singularities vs real singularities: for models first (warming):

\[ ds^2 = -\frac{1}{t^4} dt^2 + dx^2 \]

Defined for \( x \in (-\infty, \infty) \) and \( t \in (0, \infty) \). Seems singular but defining \( t' = \frac{1}{t} \), we have

\[ ds^2 = -dt'^2 + dx^2 \]

The original spacetime is a portion of Minkowski space with \( t' > 0 \). Note that the original spacetime is not geodesically complete: geodesics approaching \( t \to \infty \) take finite affine parameter to get there (even though approaching \( t \to 0 \) takes infinite affine parameter).

**Check:**

\[ \Gamma_{\alpha \beta \gamma} = \frac{1}{2} g_{\alpha \gamma} \partial_\beta g - g_{\beta \gamma} \partial_\alpha g + g_{\beta \alpha} \partial_\gamma g \]

\[ \Gamma^\alpha_{\beta \gamma} = -\frac{2}{t^4} \]

\[ \frac{d^2 t}{d\lambda^2} + \left[ -\frac{2}{t^2} \left( \frac{dt}{d\lambda} \right)^2 \right] = 0 \]

\[ \frac{dx}{d\lambda} = 0 \Rightarrow \frac{dx}{dt} = v = \frac{c}{s} \]

\[ \left( 1 + \frac{v^2}{c^2} \right) \frac{d^2 t}{d\lambda^2} - \frac{2}{t^2} \left( \frac{dt}{d\lambda} \right)^2 = 0 \]

\[ \left( 1 - \frac{v^2}{c^2} \right) \frac{d^2 x}{d\lambda^2} - \frac{2}{t^2} \left( \frac{dx}{d\lambda} \right)^2 = 0 \]

\[ \frac{dx}{d\lambda} = \frac{1}{2t} \int \frac{dx}{dt} dt = \int \frac{dx}{dt} dt = -\frac{1}{2t} \int dx \]

\[ -\frac{1}{2t} (1 + e^{-2t}) = -\frac{t}{2} \]

\[ \frac{dx}{dt} = -\frac{1}{t} \]

\[ \frac{d\lambda}{dx} = \frac{1}{1 - \frac{v^2}{c^2}} \int \frac{dx}{dt} dt = \sqrt{1 + v^2} \int \frac{dx}{dt} \]

\[ \frac{dx}{dt} = \frac{1}{\sqrt{1 + v^2}} \int \frac{dx}{dt} \]

\[ \int \frac{dx}{dt} = \frac{1}{t} \to t \to \frac{1}{t} \to 0 \to \infty \]

So \( t \to 0 \) as \( x \to \frac{1}{t} \sqrt{1 + t^2} \). However \( t \to 0 \) as \( t \to \infty \).
\[ ds^2 = -x^2 dt^2 + dx^2 \]

\[ t \in (-\infty, \infty) \quad x \in (0, \infty) \]

Singular at \( t = 0 \) and \( x = 0 \)?

Geodesics:
\[
\begin{pmatrix}
\Gamma^x_{tt} = -\frac{1}{2} g_{tt} g_{xx} &=& x \\
\Gamma^x_{tx} = \Gamma^x_{xt} &=& \frac{1}{2} g_{tx} g_{xx} &=& -x
\end{pmatrix}
\]

\( \beta = \alpha + \gamma \Rightarrow g_{tt} \frac{dt}{d\tau} = \beta = \alpha + \gamma \Rightarrow \frac{dt}{d\tau} = -\frac{\beta}{x} \)

\(-1 = -x^2 \left( \frac{dt}{d\tau} \right)^2 + (dx/d\tau)^2 = -\frac{v^2}{x^2} + (dx/d\tau)^2 \)

\[
\frac{dx}{d\tau} = \sqrt{x^2 - 1} \Rightarrow \int \frac{x \, dx}{\sqrt{x^2 - 1}} = \int d\tau
\]

\( \sqrt{v^2 - 1} \Rightarrow \frac{v}{\sqrt{v^2 - 1}} \Rightarrow 2 \int dx = -x + \sqrt{v^2 - 1} \Rightarrow v^2 = x^2 + 1 \)

\( \Rightarrow \quad t = \sqrt{x^2 - 1} \quad x^2 = \frac{v^2 - 1}{1 - v^2} \quad \frac{dt}{d\tau} = \frac{1}{v^2 - 1} \)

Geodesically incomplete. How about curvature?

\[ R^t_{xx} = \partial_x \Gamma^t_{xx} - \partial_x \Gamma^x_{tx} + \Gamma^x_{tx} \Gamma^x_{xx} - \Gamma^t_{xx} \Gamma^x_{tx} = 0 - 2 \partial_x \frac{1}{x} + 0 - \frac{1}{x^2} = 0 \]

This is a portion of Minkowski space, again.

Q: How do you find coordinates that are non-singular starting from this, not using the fact that this is Minkowski?

Use a family of geodesics that head towards the singularity, with affine parameter as one coordinate. Must avoid crossing of geodesics because this would give new coordinate singularities. In 2-DM we can take null going in and outgoing geodesics (they never cross because it null geodesics have same tangent).
null geodesics:
\[ 0 = -x^2 \left( \frac{dt}{d\lambda} \right)^2 + \left( \frac{dx}{d\lambda} \right)^2 \]
\[ \Rightarrow \quad \frac{\pm dx}{x} = \frac{dt}{d\lambda} \]
\[ \Rightarrow \quad t = \pm \ln x + \text{const} \]

Define
\[ U = t - \ln x \quad \Rightarrow \quad t = \frac{1}{2} (U + V) \]
\[ V = t + \ln x \quad \Rightarrow \quad x = e^{\frac{1}{2} (V - U)} \]

So geodesics are \( U = \text{const} \) or \( V = \text{const} \). Then
\[ ds^2 = -x^2 \left( \frac{dx}{d\lambda} \right)^2 + \left( \frac{dx}{d\lambda} \right)^2 = -e^{(V - U)} \frac{1}{2} (dv + du)^2 + e^{(V - U)} \frac{1}{2} (dv - du)^2 \]
\[ \text{or} \quad ds^2 = -e^{U - V} \, dV \, dU \]

We want to analyze the singularity at \( x = 0 \). Can't do this yet since \( U, V \in (-\infty, \infty) \) still has \( x > 0 \). But now we can extend the space beyond \( x = 0 \), i.e., beyond \( U, V \) infinite by introducing new coordinates \( U(v) \) and \( V(u) \). Calculate an affine parameter along null geodesics. Since
\[ \text{Let} \quad \frac{dx}{d\lambda} = -y = \frac{e^{(V - U)}}{x} \]
\[ \Rightarrow \quad \frac{dx}{d\lambda} = \frac{e^{(V - U)}}{x} \]
\[ \Rightarrow \quad x = \frac{1}{2} \int e^{V-U} \, dv = A + \frac{e^{V-U}}{2e} \quad (U = \text{const}) \]

Along outgoing null geodesics \( \lambda_{out} = e^V \) is an affine parameter
while \( \lambda_{in} = -e^U \)

So we \( U = -e^U \) \( V = e^V \) \( ds^2 = -dV \, dU \)
Now \( ds^2 = -dUdV \) for \( U < 0, V > 0 \) but there is no obstruction to extending \( U \) to \( (-\infty, 0) \), and we get Minkowski space by

\[
T = \frac{1}{2} (U + V), \quad U = T - x \\
V = \frac{1}{2} (V - U), \quad V = T + x
\]

\[
ds^2 = -dT^2 + dX^2
\]

The original coordinates are given in terms of these by

\[
\tau = \frac{1}{2} (U + V) = \frac{1}{2} \left( -\ln(-U) + \ln V \right) = \frac{1}{2} \left( -\ln(x-T) + \ln(x+T) \right) = \frac{1}{2} \ln \frac{x+T}{x-T} = \tan^{-1} \left( \frac{x}{T} \right)
\]

\[
x = e^{\frac{1}{2}(v-u)} = \sqrt{-VU} = \sqrt{x^2 - T^2}
\]

The original space is wedge 1 in Minkowski space here, \( (x > 1T) \).
Now do the same for Schwarzschild. We can ignore angular coordinates for most of the discussion. Consider

\[ ds^2 = -(1 - \frac{2GM}{r})dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 \]

Null geodesics:

\[ -\left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dt}{dr}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{dr}\right)^2 = 0 \]

\[ \Rightarrow \left(\frac{dt}{dr}\right)^2 = \left(1 - \frac{2GM}{r}\right)^{-2} \]

\[ t = \pm \Gamma_* + \text{constant} \]

\( \Gamma_* \) is the "Regge-Wheeler tortoise coordinate" given by

\[ \Gamma_* = \int \frac{dr}{r-2GM} = \int dr \left[ r-2GM \frac{1}{r-2GM} \right] = r+2GM \ln \left(\frac{r}{2GM} - 1\right) \]

Then we have null coordinates

\[ U = t - \Gamma_* \]

\[ V = t + \Gamma_* \]

Calculation of metric:

\[ du = dt - d\Gamma_* = dt - (1 - \frac{2GM}{r})dr \]

\[ dv = dt + (1 - \frac{2GM}{r})dr \]

\[ dudv = dt^2 - \left(1 - \frac{2GM}{r}\right)^{-2}dr^2 \]

or

\[ ds^2 = -(1 - \frac{2GM}{r})dudv \]

with \( r \) understood as \( r = r(U,V) \):

\[ \sqrt{\frac{r}{U-V}} = 2r_* = 2r + 4GM \ln \left(\frac{r}{2GM} - 1\right) \]

we have

\[ e^{\frac{V-U}{4GM}} = e^{\frac{r}{2GM}} \left(\frac{r}{2GM} - 1\right) = \frac{2GM}{r} e^{\frac{r}{2GM}} \left(1 - \frac{2GM}{r}\right) \]

so

\[ ds^2 = -\frac{2GM}{r} e^{\frac{r}{2GM}} e^{\frac{V-U}{4GM}} dudv \]
This is useful because the factor \( \frac{2Gm}{r} e^{-r/2Gm} \)

is not singular as \( r \to 2Gm \).

Now, as in Rindler case, we introduce

\[
U = e^{-r/4Gm} \\
V = e^{r/4Gm}
\]

\[
\Rightarrow ds^2 = dU = \frac{1}{4Gm} e^{-r/4Gm} \quad dv = \frac{1}{4Gm} e^{r/4Gm} \quad dv
\]

and

\[
ds^2 = -32(\text{cm})^3 e^{-r/2Gm} \quad dU \quad dv
\]

While this is defined for \( V > 0 \) and \( U < 0 \), we can now extend to

\((-\infty, \infty)

and define \( \tau \) as before

\[
\tau = \frac{1}{2}(U + V) \\
X = \frac{1}{2}(V - U)
\]

The full metric is now

\[
ds^2 = \frac{32(\text{cm})^3}{e^{-r/2Gm}} \quad (-d\tau^2 + dx^2) = \frac{r^2}{r^2(\cos^2 + \sin^2 \theta \sin^2 \phi)}
\]

Noting 6 original coordinates:

\[
X^2 - T^2 = -UV = e^{r/4Gm} = e^{2Gm} (\frac{r}{2Gm} - 1) \quad (\ast)
\]

\[
\tanh^{-1} \frac{T}{X} = \frac{1}{2} \ln \left( \frac{X+T}{X-T} \right) = \frac{1}{2} \ln \frac{V}{U} = \frac{1}{2} \ln e^{\frac{r+U}{4Gm}} = \frac{T}{4Gm}
\]

which would have been hard to guess. Eq. (\ast) also gives

\[
r = r(T, X) \quad \text{for the metric.} \quad \text{Note that} \quad r > 0 \quad \text{in} \quad (\ast) \quad \text{gives the allowed range.}
\]

Thus, \( X, T : X^2 - T^2 > -1 \).
Keep in mind each point is a $S^2$ with radius $r$.

Causal structure: null geodesics are 45° lines.

- Singularities at $r=0$ are spacelike. Two of them:
  - Future of region II
  - Past of region III

NOT a timelike line at origin, as suggested by original coordinates.

- Region I corresponds to original $r>2M$, exterior gravitational field of body.
- Region III has time-reversed properties of $I \Rightarrow \text{white hole}$.
- Region IV has identical properties to $I$, asymptotically flat.

To see what's going on, consider hypersurfaces of $T=\text{constant}$, rotate one angular variable ($\theta$):

$$C \frac{T}{2M} (\frac{r}{2M} - 1) = x^2 - r^2$$

as $x \to 0$, $r \to \infty$ $r$ goes

$r \to $ minimum and $x \to \infty$. 

$$f(r) = r \to 0$$

for $T=1$, $r_{\min} = 0$
There is another space, on the other side of the black hole. Can we communicate with our brothers there? No, as is clear from causality diagram. What happens in this picture is first as an observer near the horizon goes from \( \mathcal{I}^- \) to \( \mathcal{I}^+ \):

\[ \mathcal{I}^- \rightarrow \mathcal{I}^+ \]

the radius of the horizon is shrinking and it necessarily pinches off before the observer makes it to the other side.

**Penrose Diagram**

Recall

\[ ds^2 = -\frac{32 \, \text{c}^4 m^3 \, \text{e}^{-\frac{m}{2\, \text{c} m}} \, dU dV + r^2 d\Omega^2 }{r} \]

Now let

\[ \theta = \arctan \left( \frac{U}{U - \text{c} m} \right) \quad \hat{\theta} = \arctan \left( \frac{V}{V - \text{c} m} \right) \]

so \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \quad \hat{\theta} \in (-\frac{\pi}{2}, \frac{\pi}{2}) \)

(\text{It is quite irrelevant what kinematics is in } \theta, \hat{\theta}, \text{ just note that it is finite range}). Also, since \( -UV = e^{\frac{1}{2} \text{c} m (U - \text{c} m - 1)} \geq -1 \Rightarrow UV < 1 \)

\[ \Rightarrow \tan \theta \tan \hat{\theta} < 1 \Rightarrow \cos (U + V) \times 0 = 0 + V < \frac{\pi}{2} \]

Also \( r = 0 \) is \( UV = 1 \quad U + V = \pm \frac{\pi}{2} \). Now let \( \hat{\Theta} = \frac{\pi}{4} (U + V) \quad \hat{\Theta} = \frac{\pi}{4} (U + V) \)

So \( \hat{\Theta} = 0 \) is \( \hat{\Theta} = \pm \frac{\pi}{2} \) and \( \hat{\Theta} \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), \( \hat{\Theta} \in (-\frac{\pi}{2}, \frac{\pi}{2}) \).
Black holes in nature arise from gravitational collapse of stars. Even this is unlikely to be exactly spherically symmetric, but we can consider the idealized case. Since at some time

The metric is spherically symmetric with some mass/energy density distributed with spherical symmetry. Dirichlet's theorem gives that outside the region of mass the metric is Schwarzschild (well, not quite, that would be true if the solution were static), but let imagine it is slow, so the metric is approximately Schwarzschild. Yet, inside the metric is not Schwarzschild and what is regular at $r=0$.

Moreover at As we let the star collapse (which will happen if the object is dense enough, in fact if the mass of a large object falls within $\frac{R}{2GM}$, it will continue falling inwards towards the singularity. A picture (contour diagram in red line) of the process is

and we see such a space has no white hole nor a

region $\Pi$. 
More General Black Holes

What characterizes black holes?

Recall Schwarzschild

Two important ingredients:

(i) Asymptotically flat, so it looks like Minkowski on the "outside".

(ii) Has an event horizon (future) ($r = 2GM$). Recall we had

So in Schwarzschild all observers that remain in I go to $i^+$ at infinity, and they all share $r = 2GM$ as a future event horizon.

So, more general black hole