Maps of Manifolds

\[ \phi : M \to M' \]
is a map between manifolds

\[ x \downarrow \quad x' \downarrow \]
\[ \mathbb{R}^n \quad \mathbb{R}^m \]

**Notes:**
- In general not one-to-one
- Even if one-to-one may not have an inverse
So it goes one way.

Let \( f : M' \to \mathbb{R} \) a function on \( M' \)
a "scalar field"

\( \nu \circ \phi \) defines a function on \( M \), \( \phi^*f \)

\( \phi^*f : M \to \mathbb{R} \)
defined by

\[ \text{for all } p \in M \text{ for } \phi(p) \in M' \text{ and } f(\phi(p)) \text{ is defined.} \]

(This is a "pull-back" of a \( \phi^*f \) form)
In the other direction

By mapping curves $\lambda(t)$ in $M$ into $M'$, we can get maps of tangent vectors.

If $T_p(M)$ is the tangent space to $M$ at $p$, then

$\phi_* : T_p(M) \rightarrow T_{\phi(p)}(M')$

defined by mapping $(\frac{\partial}{\partial t})_x \rightarrow (\frac{\partial}{\partial t})_{\phi(x)}$

(denote this by $\phi_* \frac{\partial}{\partial t}$)

This is a linear transformation between two vector spaces: if $x^m$ and $y^a$ are coordinates at points of $M \times M'$, then the curve is $x^m(t)$, mapped into $y^a(x^m(t))$ and

$$\frac{dy^a}{dt} \bigg|_0 = \frac{\partial y^a}{\partial x^n} \bigg|_p \frac{dx^n}{dt} \bigg|_0$$

or

$$N^a = \frac{\partial y^a}{\partial x^n} \bigg|_p$$

where $\vec{N} \in T_{\phi(p)}(M')$ and $\vec{M} \in T_p(M)$

so $\phi_*$ is just the matrix $\frac{\partial y^a}{\partial x^n} |_p$, and we write $\vec{N} = \phi_* \vec{M}$
Since a vector \( \vec{M} \) is a directional derivative, we have \( \vec{M}(f) \) defined.

A vector gives a map of any function \( f \) at \( p \) into a number.
If \( \vec{M} = \frac{\partial}{\partial t} \) and \( \vec{M}(f) = \frac{d}{dt} \bigg|_{t=t_0} \bigg|_{p=x(t_0)} \) is the derivative of \( x(t) \).

Explicitly \( \frac{df}{dt}(x(t)) = \frac{d}{dx} \bigg|_{x=x(t)} \), so the action of the vector \( \vec{M} \) with coordinates \( \frac{\partial}{\partial x^i} \) on \( f \) is \( \vec{M}(f) = \frac{d}{dx} \bigg|_{x=x(t)} f(p) \).

\[ T_p(M) \xrightarrow{\phi_*} T_{\phi(p)}(M) \]

\[ M \xrightarrow{\phi} M' \xrightarrow{f} R \]

\[ \phi_* \vec{M}(f) \]

\[ \vec{M}(\phi_* f)|_p = \phi_* \vec{M}(f) \bigg|_{\phi(p)} \]

\[ m^m \frac{\partial}{\partial x^m} \bigg|_p = m^m \frac{\partial f}{\partial x^m} \bigg|_{\phi(p)} = \left( \frac{\partial (\phi \circ f)}{\partial (\phi \circ x^m)} \bigg|_{\phi(p)} \right) \]

Students should always flesh out relations in terms of coordinate patches, to make sure they understand.
Go on to 1 forms: define pull-back

$$\phi^* : T^*_p(M') \to T^*_p(M)$$

by requiring the contraction is mapped properly: $$\phi^* : \omega \to \phi^* \omega$$

\[ (\phi^* : \omega \in T^*_p(m') \quad \omega \mapsto \phi^* \omega \in T^*_p(m) ) \]

with $$\tilde{\omega}(\phi^*_* \tilde{M}) = \phi^* \tilde{\omega}(\tilde{M})$$

$$T_p \xrightarrow{\phi^*} T_{p'(\phi)}$$

$$\tilde{M} \xrightarrow{\phi} \tilde{M}'$$

$$T^*_p \xleftarrow{\phi^*} T^*_{p'(\phi)}$$

Recall $$\tilde{\omega}(\tilde{N})$$ is a number, i.e. $$\tilde{\omega}$$ is a map for $$T_1 \to R$$.

(In components $$\tilde{\omega}(\tilde{N})_b = \omega_a M_a^b$$, the index contraction.

Some texts write $$\langle \tilde{\omega}, \tilde{N} \rangle$$).

So the above gives the action of $$\phi^* \tilde{\omega}$$ on vectors $$\tilde{M}' \in T_{p'(\phi)}$$ in terms of the action of $$\tilde{\omega}$$ on vectors $$\tilde{N} \in T(M')$$, which is

(In components $$\tilde{\omega}(\tilde{M})^a = \omega_a N^a = \omega_a \frac{\partial \tilde{x}^a}{\partial x^m} N^m$$

that is $$\tilde{\omega}(\tilde{N})_b = \omega_a \frac{\partial \tilde{x}^a}{\partial x^m}$$).

In particular $$\phi^* (d\tilde{f}) = d(\phi^* \tilde{f})$$

(In coordinates $$d\tilde{f} = f_a dy^a$$, $$\phi^* (d\tilde{f}) = f_a \frac{\partial \tilde{x}^a}{\partial x^m} dx^m$$

while $$d(\phi^* \tilde{f}) = df(\gamma(x)) = (\phi^* \tilde{f}) (\frac{\partial \tilde{x}^a}{\partial x^m}) dx^m$$).
Clearly this can be extended to tensors of type $T^r_0$ and $T^0_r$.

$\phi^*_x : T^r_0 (p) \rightarrow T^r_0 (x(p))$

Recall $T \in T^r_0 (p)$ acts on $r$ 1-forms $T(\omega^1, \ldots, \omega^r) \epsilon \mathbb{R}$

so $T \rightarrow \phi^*_x T$ by $T(\phi^* \omega^1, \ldots, \phi^* \omega^r) = \phi^*_x T(\omega^1, \ldots, \omega^r)$

And $T \in T^0_r$ acts on $r$ vectors so $\phi^*_x : T^0_r (x(p)) \rightarrow T^0_r (p)$

$\phi^*_x T(\overline{\omega}_1, \ldots, \overline{\omega}_r) = T(\phi^*_x \overline{\omega}_1, \ldots, \phi^*_x \overline{\omega}_r)$

In coordinates: $T_{a\ldots r} = \frac{\partial y^{a_1}}{\partial x^{m_1}} \ldots \frac{\partial y^{a_r}}{\partial x^{m_r}} T_{m_1\ldots m_r}$

$\phi^*_x : T_{a_1\ldots a_r} = \frac{\partial y^{a_1}}{\partial x^{m_1}} \ldots \frac{\partial y^{a_r}}{\partial x^{m_r}} T_{a_1\ldots a_r}$

Let's define $\text{Rank} : \phi \mid M \rightarrow M'$ is rank $k$ (at $p$) if the dimension of the target space at $\phi(p)$ ($\phi_x (T_1 (M))$ is $k$.

Injective $\phi$ above is injective if $\text{Rank} = \text{dimension of } M$

$k = n$. (In this case $n \leq n'$).

Exercise: If $\phi$ is injective then no non-zero vectors in $T_p (M)$ are mapped to zero by $\phi_x$.
Surjective: \( \phi \) is surjective if rank of \( \phi = \text{dimension of } M' \)

\[ k = n' \]

(So that \( n \geq n' \)).

( Immersion: \( \phi \) is an immersion if it has an inverse \( \phi^{-1} \) (with same differentiability as \( \phi \)) such that

for each \( p \in M \) there is \( U \subseteq M \) with \( p \in U \)

\[ \phi^{-1} \circ \phi(p) \rightarrow U \]

(Skip immersion: it depends subtle only when \( C^r \) properties matter)

If \( \phi \) is injective \( \forall p \in M \) we say \( \phi \) is an immersion (actually, defn of immersion is given in terms of existence of differentiable inverse \( \phi^{-1} \), add the equivalence of stuff is proved) \( \Rightarrow \) \( \Phi_x : T_p \rightarrow \phi_x(T_p) \subset T_{\phi(p)} \) is an isomorphism.

Thus \( \phi(M) \subset M' \) is an \( n \)-dimensional immersed submanifold in \( M' \).

This is one-one locally, but may not be so globally.

An embedding is, basically, an immersion that is one-one (actually a homeomorphism onto its image).
Diffeomorphism: one-to-one map $\phi: M \to M'$ with inverse $\phi^{-1}: M' \to M$.

Then $n=n'=k$, $\phi$ is injective and surjective.

Thus, if $\phi_x$ is injective and surjective at $p$ then there is an open $U \subset M$, $p \in U$, and $\phi : U \to \phi(U)$ is a diffeomorphism.

That is, if $\phi_x : T_p \to T_{\phi(p)}$ is an isomorphism then $\phi$ is a local diffeomorphism.

With a diffeomorphism we can $\phi \circ \xi$ with $\phi_x : T_p(M) \to T_{\phi(p)}(M')$ and with $(\phi^{-1})^* : T^*_{\phi(p)}(M') \to T^*_p(M)$.

So, for any tensor $T$,

$$T(\omega^1, \ldots, \omega^s, \overline{m}_1, \ldots, \overline{m}_r) \mid_p = \phi_*(\overline{T}(\phi^{-1})^*\omega^1, \ldots, (\phi^{-1})^*\omega^s, \phi_*\overline{m}_1, \ldots, \phi_*\overline{m}_r) \mid_{\phi(p)}.$$
Di fferentiation without a connection

Two types arise naturally:
- Exterior derivative
- Lie derivative

Exterior derivative \( d : \Omega_s \rightarrow \Omega_{s+1} \)

\( \Omega_s \) : linear space of \( s \)-forms \( \tilde{\alpha} = \alpha_{m_1 \cdots m_s} dx^{m_s} \cdots dx^{m_1} \)

(\( \Omega_s \subset T^* \), is \( \star \) totally antisymmetric \( T^* \) tensors).

Recall if \( \tilde{\alpha} \) and \( \tilde{\beta} \) are \( s \)-forms, \( \tilde{\alpha} \wedge \tilde{\beta} = (-1)^{s^2} \tilde{\beta} \wedge \tilde{\alpha} \).

\[ d \tilde{\alpha} \] by
\[
\frac{\partial}{\partial x^\alpha} dx^\alpha \wedge dx^{m_1} \wedge \cdots \wedge dx^{m_s}
\]

Exercise: show
- This is indeed a \( \Omega_{s+1} \) (tensor) (obvious from first line)
- \( d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^s \alpha \wedge d\beta \) if \( \alpha \) is an \( s \)-form
- \( d(d\tilde{\alpha}) = 0 \)
- \( d(\phi^* \tilde{\alpha}) = \phi^* (d\tilde{\alpha}) \)

Useful integration results (reminder)

If \( \phi \) is a diffeomorphism \( \int_M \tilde{\alpha} = \int_{\phi(M)} \phi^* \tilde{\alpha} \)

and \( \tilde{\alpha} \) is an \( n \)-form (\( n = \dim M \))

If \( \tilde{\beta} \) is an \( n-1 \) form

\[ \int_M \tilde{\beta} = \int_{\phi(M)} d\tilde{\alpha} \]

Stokes' Theorem.
Lie derivative

Let \( \mathbf{X} \) be a vector field on \( M \)

Thm. There exists a unique curve \( \lambda(t) \) through \( p \) with \( \lambda(0) = p \) and \( \mathbf{X} = \frac{d}{dt} \lambda(t) \)

with locally, \( \lambda(t) \) coordinate \( \mathbf{x}_m \), \( \lambda(t) \) is \( \mathbf{x}_m(t) \) and with tangent \( \frac{dx_m}{dt} \); so the theorem above is the statement of uniqueness of solution of:

\[
\frac{dx_m(t)}{dt} = M^m_i(x(t))
\]

\( \lambda(t) \) is the "integral curve of \( \mathbf{X} \)."

Given \( \mathbf{X} \) we can construct a diffeomorphism \( \varphi_t \) of \( M \) into itself (actually for small open neighborhoods \( U \) of \( p \) into \( M \)), that maps \( p \) into the point along the curve a distance \( t \) parameter away

\[
\varphi_t(p)
\]

(Note \( \varphi_t \) for a one parameter local group of diffeomorphism.

\[
\varphi_{t+s} = \varphi_t \circ \varphi_s = \varphi_s \circ \varphi_t \quad \varphi_{-t} = (\varphi_t)^{-1} \quad \varphi_0 = e \text{ identity}
\]

From \( \varphi_t \) construct \( \varphi_t^* : T_S(p) \to T_S(\varphi_t(p)) \)

\[
T_p \to \varphi_t^* T_{\varphi_t(p)}
\]
Since $\varphi_\ast$ is a diffeomorphism, $\varphi_\ast T$ is an isomorphism, we can directly compare $\varphi_\ast T$ with $T$. Here we define at $p$ by
\[ \mathcal{L}_M T = \lim_{t \to 0} \frac{1}{t} [\varphi_\ast T - T] \]

Properties:
(i) If $T \in T_S(p)$, then $\mathcal{L}_M T \in T_S(p)$
(ii) $\mathcal{L}_M$ is linear
(iii) $\mathcal{L}_M$ preserves contraction
(iv) $\mathcal{L}_M (T \circ S) = \mathcal{L}_M T \circ S + T \circ \mathcal{L}_M S$
(v) $\mathcal{L}_M f = \overline{M}(f)$ (if $f$ a scalar $f : M \to \mathbb{R}$)
\[ \frac{L \cdot W}{\partial t} \]  
Start from \[ M \xrightarrow{\alpha_t} \mathcal{M} \xleftarrow{\gamma} R^n \]

with \( \alpha_t \) the integral curve of \( \nabla^\alpha \), \( \frac{d\alpha_t^\alpha}{dt} = V^\alpha(x(t)) \)

For small \( t \), \( x^\alpha(t) = x^\alpha(0) + t V^\alpha(x(0)) \).

Starting from \( x^\alpha \), the coordinate \( y^\alpha \) for \( p \), \( x^\alpha(t) = x^\alpha + t V^\alpha(x) \).

So \( y^\alpha = x^\alpha + t V^\alpha(x) \) (to order \( t \)).

Now for \[ \frac{L \cdot W}{\partial t} \] we need:

\[ M \xrightarrow{\partial_t} \mathcal{M} \]

\[ x^\alpha = x^\alpha - t V^\alpha(x) \]

(0., y, difference is higher order in \( t \)).

So we take the vector field \( \nabla^\alpha \) at \( \phi_t(p) \), \( W^\alpha(x^\alpha + t V^\alpha) \) and push forward by \( \phi_t \):

\( \phi_t : (\phi_t \cdot W)|_p = \frac{\partial x^\alpha}{\partial y^\alpha} \bigg|_{\phi_t(p)} \cdot W^\alpha(x^\alpha + t V^\alpha) \)

with \( y \) the coordinate of \( \phi_t(p) \).

That is, \( \frac{\partial x^\alpha}{\partial y^\alpha} \bigg|_{\phi_t(p)} \cdot W^\alpha(x^\alpha + t V^\alpha) \bigg|_{\phi_t(p)} \)

So in terms of coordinates at \( p \),

\( (\phi_t \cdot W)|_p = \left( \frac{\partial x^\alpha}{\partial y^\alpha} \bigg|_{\phi_t(p)} \right) W^\alpha(x + t V) \)

Now \[ \frac{L \cdot W}{\partial t} = \frac{1}{t \partial \alpha_x} \left[ (\phi_t \cdot W)|_p - W^\alpha \right] = V^\alpha \frac{\partial W^\alpha}{\partial y^\alpha} - \frac{\partial V^\alpha}{\partial y^\alpha} W^\alpha \]

Note, with \( \tilde{V} = V^\alpha \frac{\partial}{\partial y^\alpha} \) and \( W = W^\alpha \frac{\partial}{\partial y^\alpha} \) then

\[ [V^\alpha \frac{\partial}{\partial y^\alpha}, W^\beta \frac{\partial}{\partial y^\beta}] = V^\alpha \frac{\partial W^\beta}{\partial y^\alpha} - W^\beta \frac{\partial V^\alpha}{\partial y^\beta} = (V^\alpha \frac{\partial W^\beta}{\partial y^\alpha} - W^\alpha \frac{\partial V^\beta}{\partial y^\alpha}) \frac{\partial}{\partial y^\beta} \]

\[ \Rightarrow \frac{L \cdot \tilde{W}}{\partial t} = [\tilde{V}, \tilde{W}] \]
Look closely at \( L_v f(p) \)

Recall \( (\Phi^\epsilon) : M \to M' \) \( \frac{\phi}{\Phi^\epsilon} \) \( \phi \mapsto f \mapsto f \circ \phi \)

Also, if \( \phi \) is a diffeomorphism then \( M \xrightarrow{\phi} M' \)

The push forward of the inverse is just the pull-back:

\( (\Phi^\epsilon)^{-1}_* f : M \to \mathbb{R} \) is \( \phi^* f : M \to \mathbb{R} \)

For \( \phi \) to be defined, \( (\Phi^\epsilon)^{-1} \) must be defined for \( \phi \in \text{Diff}(\mathbb{R}) \)

\[ f(\phi(x)) = f(x + \Delta x) = f((x + \Delta x) - f(x)) = f(x + \Delta x) - f(x) = v^* df \]

Finally, \( M \xrightarrow{\phi} M \)

Define \( \phi_* f : \mathbb{R} \)

The idea of \( L_v \):

\[ L_v T = \int_{t_0}^t \frac{1}{\epsilon} [ (\Phi^\epsilon)^{-1}_* T] - V_p \]

Just recall that \( (\Phi^\epsilon)^{-1}_* T \) is a push-forward from \( \text{Rev}(\phi^{-1}) \) to \( \phi \).

Which is the same as a pull-back from \( \phi \) to \( \Phi^\epsilon \).

Above:

Finally, \( L_v (\omega \wedge w^\nu) = V^\nu \omega \wedge (\omega \wedge w^\nu) + \omega \wedge L_v (w^\nu) \)

\[ L_v (\omega \wedge w^\nu) = V^\nu \omega \wedge (\omega \wedge w^\nu) - \omega \wedge (V^\nu \omega \wedge w^\nu - \omega \wedge V^\nu w^\nu) = (V^\nu \omega + 2 V^\nu \omega) \wedge w^\nu \]
Recall \( M \xrightarrow{\phi} M' \)

\[ \frac{d}{dt} W_{\phi(t)} = \phi_\ast \frac{d}{dt} W_{\phi(t)} |_{p(t)} \]

Moreover, for our case \( \phi_\ast = \delta \) is what? Take

\[ M \xrightarrow{\delta} M' \]

\[ \frac{d}{dt} W_{\delta(t)} = \delta_\ast \frac{d}{dt} W_{\delta(t)} |_{p(t)} \]

But \( Y^a(x^m) \) is just the shift in coordinates along to curve:

\( x^a \) are the coordinates of \( p \), \( \delta_{x^a} \) of \( x^m \), \( \delta_{y^a} \) coordinates of \( \phi(y) \).

If the curve is the integral of \( \frac{dx^m}{dt} = M^m \) (\( \vec{M} \) a vector field),

\[ \frac{d}{dt} x^m(t) = x^m(0) + t M^m \]

\[ x^m(t) = x^m(0) - t M^m \]

\[ \frac{\partial Y^a}{\partial x^m} |_{\delta(y)} = \delta^a_m - M^a_m t \]

\[ W^m |_{\delta(y)} \] just \( W^m(x^m(t)) = W^m(x^m(0) + t M^m) = W^m |_{p(t)} + t \left[ M^m W^m |_{p(t)} \right] \),

\[ \phi_\ast W^m |_{p(t)} - W^m |_{p(t)} = (\delta^a_m - M^a_m t) (W^m + t M^m W^m |_{p(t)}) - W^m \]

\[ (\phi_\ast W)^m = M^m W^m |_{p(t)} - W^m M^m |_{p(t)} = [M, W]^m \]
In particular, this shows $L^\lambda_\mu W = - L^\lambda_\mu \omega$

From this, we can obtain the action of $L^\lambda_\mu$ on other tensors:

$$
L^\lambda_\mu (\tilde{\omega} \otimes \tilde{W}) = L^\lambda_\mu \tilde{\omega} \otimes \tilde{W} + \tilde{\omega} \otimes L^\lambda_\mu \tilde{W}
$$

Now, contracting $\tilde{\omega}$:

$$
L^\lambda_\mu (\tilde{\omega} (\tilde{W})) = L^\lambda_\mu \tilde{\omega} (\tilde{W}) + \tilde{\omega} (L^\lambda_\mu (\tilde{W}))
$$

Now if we use $\tilde{W} = \tilde{E}_\mu$, a basis vector we can get $L^\lambda_\mu \tilde{\omega}$.

In particular, if $\tilde{E}_\mu = \frac{\partial}{\partial x^\mu}$, the coordinate basis, then

$$
L^\lambda_\mu (\tilde{\omega} (\tilde{E}_\mu)) = (L^\lambda_\mu \tilde{\omega})_\mu
$$

the components we are looking for.

$$
L^\lambda_\mu (\tilde{\omega} (\tilde{E}_\mu)) = L^\lambda_\mu (\tilde{\omega}) = \tilde{M}_\mu (\omega^\mu) \quad \text{(prop-by}(\tilde{\omega}))
$$

$$
= \frac{\partial \omega^\mu}{\partial x^\nu} M_\nu = \omega_{\nu} M^\nu
$$

and

$$
\frac{\partial X^\nu}{L^\lambda_\mu (E_\mu)^y} = \frac{\partial (E_\mu)^y}{\partial X^\nu} M_\mu - \frac{\partial M_\mu}{\partial X^\nu} (E_\mu)^y = - \frac{\partial M_\mu}{\partial X^\nu}
$$

so

$$
\tilde{\omega} (L^\lambda_\mu (E_\mu))^y = - \omega_{\nu} M^{\nu}_\mu
$$

$$
\Rightarrow \quad (L^\lambda_\mu (\tilde{\omega}))_\mu = \omega_{\nu} M^{\nu}_\mu + M^{\nu}_\mu \omega_{\nu}
$$
Exercise: Show

\[ y \frac{\partial}{\partial \gamma_{\mu}} T^{\alpha_1 \cdots \alpha_k}_{\nu_1 \cdots \nu_l} = M^\alpha \partial_\alpha T^{\alpha_1 \cdots \alpha_k}_{\nu_1 \cdots \nu_l} \]

\[ - (\partial_\alpha M^\alpha) T^{\alpha_1 \cdots \alpha_k}_{\nu_1 \cdots \nu_l} \]

\[ - (\partial_\beta M^\beta) T^{\alpha_1 \cdots \alpha_k}_{\nu_1 \cdots \nu_l} \]

\[ + (\partial_{\nu_1} M^\nu) T^{\alpha_1 \cdots \alpha_k}_{\nu_1 \cdots \nu_l} \]

In particular

\[ y M g_{\mu \nu} = M^\alpha \partial_\alpha g_{\mu \nu} + \partial_\mu M^\nu g_{\nu \nu} + \partial_\nu M^\mu g_{\mu \mu} \]

Since these are tensor equations, we can replace by \( \Box \).

\[ y M g_{\mu \nu} = M^{\alpha \beta} \partial_\alpha \partial_\beta g_{\mu \nu} + M^{\mu \nu} g_{\mu \nu} = M_{\mu \nu} + \Box g_{\mu \nu} \]

0.

\[ y M g_{\mu \nu} = 2 M g_{\mu \nu} \]
This is useful later. We will use it for symmetries later, but here is a simple application. Assume the action for GR breaks down into

\[ S = S_0 (g_{\mu \nu}) + S_m (g_{\mu \nu}, \psi) \]  

where \( \psi = \) matter fields

\( S_0 = \) "Hilbert" action (Grass Einstein's etc. - we'll see this later if time comes).

Consider this theory is "diffeomorphism invariant" i.e. \( (m, g_{\mu \nu}, \psi) \) and \( (M, \alpha g_{\mu \nu}, \alpha \psi) \) represent the same physics. The change in \( S_m \) under a diffeomorphism

\[ \delta S_m = \int d^4 x \frac{\delta S_m}{\delta g_{\mu \nu}} \delta g_{\mu \nu} + \int d^4 x \frac{\delta S_m}{\delta \psi} \delta \psi \]

Since we could have \( \alpha + 1 = 0 \), \( \delta S_0 \) can be considered separately (it is invariant by itself, but is where the separation assumption in (A) come in).

But \( \frac{\delta S_m}{\delta \psi} = 0 \) for any variation. So while here we look only at variations from diffeomorphisms, that term vanishes separately for any variation. Left with first term, we consider diffeomorphisms generated by a vector field \( U^\mu \):

\[ \delta g_{\mu \nu} = \xi g_{\mu \nu} = 2 \xi U^\mu \]

\( \Rightarrow \)

\[ \delta S_m = 0 = \int d^4 x \left[ \frac{\delta S_m}{\delta g_{\mu \nu}} \xi U^\mu \right] + 4 \int d^4 x \frac{\delta S_m}{\delta g_{\mu \nu}} U^\mu U_{\mu \nu} \]

or

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ = 4 \int d^4 x \left( \left[ \frac{\delta S_m}{\delta g_{\mu \nu}} \right] U^\mu \right) - U^\mu \left( \frac{\delta S_m}{\delta g_{\mu \nu}} \right) U_{\mu \nu} \]
Dropping the surface term and multiplying by $\frac{\sqrt{g}}{\sqrt{-g}}$ we have

$$\int d^nu \, U^\mu \nabla_\nu \left[ \frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}} \right] = 0$$

Since this holds for arbitrary $U^\mu$ (diffeomorphisms generated by arbitrary vector fields), it must be that

$$\nabla_\nu \left( \frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}} \right) = 0$$

But

$$T^\mu{}_{\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}}$$

is the energy-momentum tensor.
Symmetries, Isometry, Killing Vectors

\( \phi : M \to M \) a diffeomorphism, \( T \) a tensor.

\( \phi \) is a symmetry of \( T \) if

\[ \phi^* T = T \]

\( T \) symmetric

Some symmetries are discrete. But for continuous symmetries, there is a one parameter set of diffeomorphisms \( \phi \), and

\( T \) is symmetric iff

\[ \partial_u T = 0 \]

(\( U \) generates the curve, \( U = \frac{\partial}{\partial t} \)).

Note that one can choose coordinates locally so that \( t \) itself is one of the coordinates. In such coordinates,

\[ \partial_u T^\mu_{\ \nu \rho} \quad \partial_t \nu^\rho - \nu^\rho = \partial_t T^\mu_{\ \nu \rho} \quad \nu^\rho - \nu^\rho \]

so \( \partial_u T = 0 \) \( \Rightarrow \) all components of \( T \) are independent of \( t \).

(Converse is obviously true!)
An isometry is a symmetry of the metric tensor,\[ g^* g_{uv} = g_{uv} \]

A vector field \( \tilde{K} \) that generates an isometry is called a Killing vector field:\[ \mathcal{L}_K g_{uv} = 0 \]

Then \( \tilde{K} \) satisfies:\[ K(u; v) = 0 \]

One can show the operator: if \( K(u; v) = 0 \) then \( \phi^*_t g_{uv} = g_{uv} \)

where \( \phi^*_t \) is generated by \( \mathcal{L}_K = \frac{\partial}{\partial t} \). This is done by integration (see Hawking & Ellis).

Again, one can choose local coordinates that include \( t \), and then \( g_{uv} \) is independent of \( t \).

Now, in first quarter (Schutz, 7.41) we saw that the geodesic equation can be written in terms of \( \tilde{\rho} = m \tilde{V} \) as:

\[ m \frac{d\tilde{\rho}}{dt} = \frac{1}{2} g_{uv} \tilde{\rho}_v \tilde{\rho}^u \tilde{\rho}^u \]

so if \( g_{uv} \) is independent of one coordinate (say \( t \)), then the corresponding \( \tilde{\rho}_t \) is conserved, \( \frac{d\tilde{\rho}_t}{dt} = 0 \)
This can be done in a covariant language as follows: assume \( p^\mu \) satisfies geodesic equation:

\[
\rho^\mu \rho^\nu = 0 \quad (\nabla p^\mu = 0)
\]

Then

\[
\rho^\mu \nabla_\mu (p^\nu K_{\nu}) = \rho^\mu \rho^\nu \nabla K_\nu + K_\nu \rho^\mu \nabla p^\nu = 0
\]

But LHS is just \( \frac{d}{dt} (p^\nu K_{\nu}) \) so \( [p^\nu K_{\nu}] \) is constant along particle path \( \rho^\mu \) a conserved quantity, as before.

Exercise: If \( K_{\mu \nu} \) is a Killing tensor, i.e., it satisfies

\[
\nabla_\mu (K_{\nu \lambda}) = 0,
\]

show that \( K_{\mu \nu} \rho^\mu \rho^\nu \) is conserved.
We can see this more generally with our Killing field technology:

Let \( P^\mu = T^{\mu\nu} K_\nu \)

Then \( \rho^\mu = T^{\mu\nu} i_\mu K_\nu + T_{\mu\nu} K_{\nu,\mu} \)

\( = T^{\mu\nu} i_\mu K_\nu + \frac{1}{2} T_{\mu\nu} K_{\nu,\mu} = 0 \)

So the vector \( \rho^\mu \) is a "conserved current." By Cartan's theorem.

Example: In flat space (which is highly symmetric):

Killing vectors

\( \vec{\rho}(x) = \frac{\partial}{\partial x^a} \) (a vector for each \( a = 0, 1, 2, 3 \))

\( \vec{M}^{(0)} = x^0 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^0} \) \( \mu = 1, 2, 3 \)

\( \vec{l}_{(ij)} = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i} \) \( i, j = 1, 2, 3 \)

10 isometries generate 10-parameter Lorentz group of isometries of flat space, the "inhomogeneous Lorentz group."

To see how this works, choose obvious coordinates with components out:

\( \rho^{(0)} = (1, 0, 0, 0) \quad \rho^{(1)} = (0, 1, 0, 0) \quad \rho^{(2)} = (0, 0, 1, 0) \)

\( \rho^{(3)} = (0, 0, 0, 1) \)

So \( \rho^{(a)} \) is trivially for all \( a \).

Less trivial:

\( \rho^{(0)} = (x^1, x^2, 0, 0) \rightarrow \rho^{(0)} = (-x^1, x^0, 0, 0) \)

\( \rho^{(0)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \)

Then \( \rho^\mu = T^{\mu\nu} K_\nu \) gives connection of \( E \rho^\mu \), \( \vec{J} \) and \( \vec{F} \) (not \( \rho + \lambda \vec{E} \)).
It is clear from the example that most (if not all) space-time may admit several (or none) Killing vectors. Since these symmetry transformations generally form groups (group multiplication = composition of transformations, i.e. \(g \circ g\)) and these are continuous transformations generated by \(\tilde{F}\)'s, we expect there to be Lie groups of the \(\tilde{F}\)'s to form Lie algebras. This is indeed the case, with the Lie bracket being just the commutator, i.e.

\[
[\mathbf{K}_1, \mathbf{K}_2] = \mathbf{L}_{\mathbf{K}_1, \mathbf{K}_2}
\]
Maximally Symmetric Space

Spaces with high degree of symmetry are easier to analyze.

What is the highest degree of symmetry?

Consider $\mathbb{R}^n$ - Euclidean space. Then we had

$N$ translations

$\frac{1}{2}n(n-1)$ rotations

$= \frac{1}{2}n(n+1)$ symmetries in total

A symmetry under translations at a point $P$ is called "isoclinic" of $\mathbb{R}^n$.

Symmetry under translation is called "homogeneity" of the space.

This is as much as we can have, and we define a

"maximally symmetric space" = one with $\frac{1}{2}n(n+1)$ Killing vector fields.

Let's find them.

At $p \in M$ choose locally inertial coordinates, so that

$\omega$ is given by $\omega^\mu$. Obviously (by construction) this is invariant under local Lorentz transformations. But (so happy means, in this coordinates, at this point $p$, $\omega^\mu$ should also be invariant,

$$\omega^\mu = M^\mu_{\nu} \omega^\nu$$

only tensor with proper symmetries and invariant.

NOTE: A local Lorentz transformation acts on $T_p(M)$, i.e., it is a change of basis vectors $\tilde{E}^\alpha$. It is these vectors that are used to define the co-ordinate at $p$. 

If we write this as

\[ R_{\mu \nu \rho \sigma} = \frac{R}{n(n-1)} (g_{\mu \sigma} g_{\nu \rho} - g_{\mu \rho} g_{\nu \sigma}) \]

since this is a tensor equation it holds in any coordinate system. But then use homogeneity \( \Rightarrow \) it holds everywhere on \( M \) with same constant \( R \).

Covariant indices

\[ R_{\mu \nu \rho \sigma} = \frac{R}{n(n-1)} (g_{\mu \sigma} g_{\nu \rho} - g_{\mu \rho} g_{\nu \sigma}) \]

So, in particular, the Ricci scalar is a constant (should be obvious by homogeneity; same \( R \) everywhere).

A maximally symmetric space is determined by

- dimension
- signature
- \( R \)
- additional topological considerations (global issues).

Warning: \( n=2 \), \( M = (+) \) \((n=2 \text{ almost trivial, since only one aspect of } \mathbb{R}^{2,0})\).

\( R > 0 \): the sphere \( S^2 \)

\[ ds^2 = a^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad R = \frac{a^2}{2} \]

\( R = 0 \): \( \mathbb{R}^2 \)

\[ ds^2 = dx^2 + dy^2 \]

\( R < 0 \): less familiar, the hyperboloid \( H^2 \)

\[ ds^2 = \frac{a^2}{y^2} (dx^2 + dy^2) \quad y > 0 \]

Exercise: For \( H^2 \) show

- \( R = -\frac{2}{a^2} \)
- The distance between \( x, x_2, \text{ any } x = \text{ constant } \) is \( \frac{y_2 - y_1}{\sqrt{a^2}} \)
- Geodesics satisfy \((x-x_0)^2 + y^2 = b^2\) for \( x, b \) constants.
Now do \( n = 4 \) \( \mathbb{M}^{(-+++)} \)

\( R > 0 \) de Sitter space

\( R = 0 \) Minkowski space

\( R < 0 \) anti-de Sitter space

Study these. Study causal structure too.

**Minkowski Space-time:** (Hint: It will help if you understand key concepts for other spaces)

\[
\begin{align*}
\mathcal{d}s^2 &= -(\mathcal{d}x)^2 + (\mathcal{d}t)^2 + (\mathcal{d}y)^2 + (\mathcal{d}z)^2 \\
&= -(\mathcal{d}t)^2 + \mathcal{d}r^2 + r^2 (\mathcal{d}\varphi^2 + \sin^2\varphi \, \mathcal{d}\theta^2)
\end{align*}
\]

Null coordinates:

\[
\begin{align*}
v &= t + r \\
w &= t - r
\end{align*}
\]

\[
\mathcal{d}s^2 = -dw \, dv + \frac{1}{4} (w - w')^2 \left[ (\mathcal{d}\varphi^2 + \sin^2\varphi \, \mathcal{d}\theta^2) \right]
\]

\( v = \text{out} \) and \( w = \text{oast} \) are null hypersurfaces.

Can we change coordinates to have only finite ranges? Let

\[
\begin{align*}
W &= \arctan w \\
V &= \arctan v
\end{align*}
\]

\( v \in [-\frac{\pi}{2}, \frac{\pi}{2}] \)

\[
\mathcal{d}s^2 = \frac{1}{\omega^2} \left[ -4 \mathcal{d}W \mathcal{d}W + \sin^2(V-W) \left( \mathcal{d}\varphi^2 + \sin^2\varphi \, \mathcal{d}\theta^2 \right) \right]
\]

where \( \omega = 2 \cos W \cos V \)

Finally write \( T = W + V \) \( R = V - W \)

\( 0 \leq R \leq \pi \)

\( |T| + R \leq \pi \)

So

\[
\mathcal{d}s^2 = \frac{1}{\omega^2} \left[ -dT^2 + dR^2 + \sin^2 R \, d\varphi^2 \right]
\]

with \( \omega = \cos T \cos R \) (kind of irrelevant for us).

\[
\mathcal{d}s^2 = \frac{1}{\omega^2} \, ds_e^2
\]

where \( ds_e^2 = -dT^2 + dR^2 + \sin^2 R \, d\varphi^2 \) is the line element for Einstein's static universe.
So Minkowski space is conformal to (a part of) the Einstein static universe.

(A conformal transformation is a local change of scales \( \tilde{g}_{\mu\nu} = e^{2\alpha(x)}g_{\mu\nu} \).)

Conformal Diagrams (or Penrose Diagrams)

Space-time diagram for space-time \((M, g)\). It has a "time" coordinate and a "radial" coordinate, with light-cones always at 45°. Also, infinity is a finite coordinate distance (so we can fit in paper).

Conformal transformations leave light-cones invariant (if \( ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 0 \) then \( ds^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = 0 \)).

They are useful in deciphering the causal structure of spacetime.

Drawing a circle for each sphere \( \theta, \phi \).
The constant $t$ surfaces are obtained for

\[ T = V + W = t^{\frac{1}{2}} V + t^{\frac{1}{2}} W = t^{\frac{1}{2}} (t+1) + t^{\frac{1}{2}} (t-r) \]

\[ R = V - W = t^{\frac{1}{2}} (t+1) - t^{\frac{1}{2}} (t-r) \]

More easily: for $t =$ constant, eliminate $r: \Rightarrow W + V = 2t$ fixed

\[ \Rightarrow \tan V + \tan W = 2t \]

\[ \Rightarrow \tan \left( \frac{1}{2} (r+1) \right) + \tan \left( \frac{1}{2} (r-1) \right) = 2t \]

$A^+ = \text{future timelike infinity}$

$A^- = \text{past } A^+$

$A^0 = \text{spatial infinity}$

$I^+ = \text{"scri- plus" future null infinity}$

$J^- = \text{past } I^+$

Features:

(a) light cones $t \rightarrow t^2$

(ii) $A^+$ are points, $A^0$ are surfaces (null) with topology $R \times S^2$

(iii) timelike geodesics from $A^-$ to $\infty$, spacelike from $\infty$ to $A^0$

Null geodesics from $J^-$ to $I^+$

Note: light cones always $\times$ everywhere $(i^0, u^0)$
Some obvious things are now obvious in the recent diagram.
- $\mathcal{V}$ is in the future lightcone of any event
- $\mathcal{I}^-$ is in the past lightcone of any event

i.e., you can reach any point, no matter how far, with a signal if you are willing to wait enough.

- $\mathcal{I}^+$ is in the future lightcone of any event
- You cannot reach space-like infinity with a signal in finite time.

If you are willing to wait an infinite time, a signal can reach spatial infinity, but will not catch up with other signals emitted by you earlier.
Cauchy Surfaces

\[ D^+(S) : \text{"future Cauchy development" of } S : \]

\[ D^+(S) = \{ \mathbf{p} \in M \mid \text{each past directed inextendible, non-spacelike curve through } \mathbf{p} \text{ intersects } S \} \]

Notes:
- Inextendible so that we avoid:
- Non-spacelike, we want the part of space that can be causally affected by \( S \).

Since signals can only travel on non-spacelike curves, if \( \mathbf{p} \in D^+(S) \) for knowing data (value of fields and their derivatives, or particle velocities, etc) on \( S \) is enough to predict for at \( \mathbf{p} \).

Similarly, if we want to evolve back into the past, but have information only on \( S \), we can only refer the state in \( D^+(S) \) (defined by "future" \( \rightarrow \) "past" is def above).
If \( D^+(S) \cup D^-(S) = M \)

\( S \) is called a Cauchy surface

In words, every inextendible non-spacelike curve in \( M \) intersects \( S \). \( S \) is Cauchy.

In Minkowski space \( t^4 = 0 \) is a Cauchy surface. In fact \( t^4 = c = \text{a constant} \) is a collection of Cauchy surfaces that cover the whole of \( M \).

Note every spacelike surface in \( M \) is Cauchy. Clearly extendible surfaces are not.

More interestingly, some inextendible surfaces are not Cauchy. For example, the surface

\[
-\overline{(x^2)}^2 + (x^4)^2 - t^2 + x^2 + y^2 + z^2 = \sigma \in \mathbb{R}
\]

are spacelike if \( \sigma < 0 \). Let \( S_0 \) be the surface with \( \sigma < 0 \).

These \( S_0 \) are not Cauchy, but are inextendible spacelike. The collection fills the past lightcone of its origin.
This is $R > 0$. Note that this means ($\Lambda = 0$).

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{1}{4} R g_{\mu\nu} \quad R = \Lambda + \kappa > 0$$

Comparing with Einstein's equation, (1.24) in Schutz

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

we see that either $\kappa T_{\mu\nu} = -\frac{\kappa}{4} g_{\mu\nu}$ (a very odd fluid??)
or $\Lambda = \frac{\kappa}{4} R > 0$. That is, de Sitter spacetime is a solution to Einstein's equations with a cosmological constant, but no matter.

Since we have observed $\Lambda > 0$, with $\Lambda$ finite, and since $\Lambda$ is constant while $\rho$ is decreasing with the (slow) expansion of the universe, soon $\rho$ will be negligible and the future of the universe will be described by (approximately) de Sitter spacetime.

It is defined by embedding the hyperboloid

$$-U^2 + X^2 + Y^2 + Z^2 + W^2 = \alpha^2$$

defined in $S$-dimension Minkowski space, $ds^2 = -dU^2 + dX^2 + dY^2 + dZ^2 + dW^2$

Let

$$U = \alpha \sinh(\frac{t}{\alpha})$$

$$W = \alpha \cosh(\frac{t}{\alpha}) \cos \chi$$

$$X = \alpha \cosh(\frac{t}{\alpha}) \sin \chi \sin \theta \cos \phi$$

$$Y = \alpha \cosh(\frac{t}{\alpha}) \sin \chi \sin \phi$$

$$Z = \alpha \cosh(\frac{t}{\alpha}) \sin \chi \cos \theta$$
Then

\[ ds^2 = -dt^2 + \alpha^2 \cosh^2 \left( \frac{t}{\alpha} \right) \left[ d\chi^2 + \sin^2 \chi \left( d\vartheta^2 + \sin^2 \vartheta \, d\phi^2 \right) \right] \]

(Exercise: hint for class notes)

(1) \[-u^2 + x^2 + y^2 + z^2 + w^2 = -\alpha^2 \sinh^2 \left( \frac{t}{\alpha} \right) + \cosh^2 \left( \frac{t}{\alpha} \right) \left[ \cos^2 \chi + \sin^2 \chi \left( \cos^2 \vartheta + \sin^2 \vartheta \right) \right] \]

\[ = \alpha^2 \]

(2) \[ g_{\mu \nu} = \frac{\partial x^\mu}{\partial x^\nu} g_{ab} \text{ is a pull back } \alpha^i : g \to \alpha g \]

\[ \phi \text{ given by the above eqs, i.e., } u = \alpha \sinh \left( \frac{t}{\alpha} \right) + \chi \]

So \[ ds^2 = g_{\mu \nu} dx^\mu dx^\nu = \frac{\partial x^a}{\partial x^\mu} \frac{\partial x^b}{\partial x^\nu} g_{ab} dx^a dx^b \]

\[ = -\frac{\partial u}{\partial x^\mu} dx^\mu dx^u + \frac{\partial x^a}{\partial x^\mu} \frac{\partial x^b}{\partial x^\nu} g_{ab} dx^a dx^b + \cdots + \frac{\partial x^a}{\partial x^\mu} \frac{\partial x^b}{\partial x^\nu} g_{ab} dx^a dx^b \]

\[ = -\cosh^2 \frac{t}{\alpha} dt^2 + \sinh^2 \frac{t}{\alpha} \left[ \cos^2 \chi + \sin^2 \chi \left( \cos^2 \vartheta + \sin^2 \vartheta \right) \right] d\chi^2 \]

\[ + \alpha^2 \cosh^2 \frac{t}{\alpha} \left[ \sin^2 \chi + \cos^2 \chi \left( \cos^2 \vartheta + \sin^2 \vartheta \right) \right] d\vartheta^2 \]

\[ + \alpha^2 \sinh^2 \frac{t}{\alpha} \left[ \sin^2 \chi \left( d\vartheta^2 + \sin^2 \vartheta \, d\phi^2 \right) \right] \]

Note \( d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta \, d\phi^2 \) metric on a 2-sphere

Similarly \( dS^2 = d\chi^2 + \sin^2 \chi \, dS^2 \rightarrow \text{metric on a 3-sphere} \)

As Sitter space: spatial 3-sphere that shrinks to a minimum radius \( \alpha \), then re-expands.

Topology: \( \mathbb{R}^1 \times S^3 \)
Another coordinate system that is common is

\[ \hat{t} = \alpha \log \left( \frac{w+u}{\alpha} \right) \quad \hat{x} = \frac{\alpha x}{w+u} \quad \hat{y} = \frac{\alpha y}{w+u} \quad \hat{z} = \frac{\alpha z}{w+u} \]

restricted to the hyperboloid \((w-u)(w+u) + r^2 = \alpha^2 \Rightarrow w-u = \frac{\alpha^2 - r^2 e^{2\lambda}}{\alpha e^{2\lambda}}\) or \(w-u = \alpha e^{2\lambda} - \frac{\alpha^2 (x^2 + y^2 + z^2)}{e^{2\lambda}}\).

Now, proceed with the pullback of \(g^{ab} : ds^2 = d(u+w)\left[(u-w)\right] + dx^2 + dy^2 + dz^2 = \left(\alpha e^{2\lambda}d\hat{t} \right) \left( - e^{2\lambda} - \frac{r^2 e^{2\lambda}}{\alpha^2} \right) d\hat{t} \left( - e^{2\lambda} - \frac{r^2 e^{2\lambda}}{\alpha^2} \right) d\hat{t} + \left( d\hat{x}^2 - \frac{\alpha^2}{\alpha} \frac{d\hat{t}^2}{e^{2\lambda}} \right) e^{2\lambda} \hat{n}

Cross terms cancel!

This coordinates cover only the region

\( w+u > 0 \)

of the hyperboloid
null surface $(t = -\infty)$
boundary of caustic patch

\[ \text{check } w + u = 0 \implies \alpha \sinh \left( \frac{t}{\alpha} \right) + \alpha \cosh \left( \frac{t}{\alpha} \right) \cos \chi = 0 \]

\[ \implies \cos \chi = -\tanh \left( \frac{t}{\alpha} \right) \]

As $t \to \pm \infty$, $\tanh \left( \frac{t}{\alpha} \right) \to \pm 1$, so $\cos \chi \to \pm 1$ or $\chi \to 0 \text{ or } \pi$
The diagram for de Sitter:

Let \( t \to t' \) be the coordinate change by

\[
\tan \left( \frac{1}{2} t' + \frac{\pi}{4} \right) = e^{\frac{t}{2}}
\]

with \( t' \in (-\pi, \pi) \)

Then

\[
dt^2 e^{2\frac{t}{2}} = \left( \frac{1/2}{\cos^2 (t'/2)} \right)^2 dt'^2
\]

or

\[
dt^2 = \frac{1}{4} \frac{1}{\cos^4 (t'/2)} \frac{\cos^2 (dt')}{\sin^2 (dt')} = \frac{d\xi^2}{4 \cos^2 \sin^2 (t'/2)}
\]

and

\[
\cosh \frac{\xi}{2} = \frac{1}{2} \left( \tan + \frac{1}{\tan} \right) = \frac{1}{2} \frac{\sin^2 + \cosh^2}{\sin \cos} = \frac{1}{\sin (t'/2)}
\]

\[
ds^2 = \frac{\alpha^2}{\sin^2 (t'/2)} \, d\xi^2
\]

where \( d\xi^2 = -dt'^2 + d\chi^2 + d\beta^2 = -dt'^2 + d\beta^2 \)

So de Sitter is conformal to the metric \( ds^2 = \text{Einstein-Schwarzschild} \) familiar from Minkowski. Now

... study Schwarzschild...
Exercise:

Steady-state universe of Bondi, Gold, and Hoyle (circa 1948)

Horizons: (Next page)
Horizons

Pero de Sitter future and past infinities are spacelike

(contrast with Minkowski's timelike).

This gives rise to both particle and event horizons.

Particle horizon: defined for an observer at some event $p$.

$\in$ spacelike:

- worldline of observer

\[
\text{worldline of particle that may be observed at } p \quad \text{(crosses past null cone at } p) \quad \text{or}
\]

\[
\text{does not cross past null cone).}
\]

unobservable particles (at $p$)

p - particle horizon

So the particle horizon separates the region of spacetime occupied by particles that may have been seen at $p$ from those that cannot be seen at $p$.

Particle horizons are defined with respect to a congruence of worldlines. Problem is

\[
\text{so we wouldn't be able to separate space into two pieces --- no "horizon".}
\]
Congruence is a set of lines such that each point \( p \) (in some open set \( U \subset M \)) is in exactly one curve.

By definition, curves in a congruence do not cross.

**Examples:**

1. There are no particle horizons in Minkowski space.

   ![Diagram of Minkowski space with a null cone]

   More generally, this is true if \( J^\pm \) is null.

2. De-Sitter does have particle horizons. Consider the congruence of \( \tau \)-constant in the figure diagram.

   ![Diagram of De-Sitter space with a null cone]
Event horizon: While a particle horizon tells us which "particles" may have been seen at $p$, we may ask instead of which particles may influence $p$ at all throughout its whole history. That is, if the space-time is expanding faster than the speed of light then if some observer far away from us, light sent to us will never reach us. We want to characterize this situation with an "event horizon" separating those events that can never influence us from those that can. Clearly, at any event $p$, all events in its past light cone are observable, while those outside are not. The future event horizon is the limiting light cone of an observer as it goes into future infinity, $\mathcal{I}^+$. Similarly, past event horizon is defined to separate each that $O$ will be able to influence in its history from those it won't.
Examples: (1) Minkowski space-time.

If \( O \) is a geodesic (free falling) observer \( \Rightarrow \) no event horizon.

\[ \text{\( A^+ = \) past light cone of } \Omega \text{ at } A^+ \]

\[ \text{\( A^- = \) future light cone of } \Omega \text{ at } A^- \]

(2) Uniformly accelerated observer in Minkowski space-time

\[ \text{picture is } r^2 - t^2 = a^2 \]

his future and past event horizons.

\[ \text{Work it out: recall } ds^2 = \omega^2 d\omega^2, \text{ see above.} \]

and uniformly accelerated \( \Rightarrow r^2 - t^2 = a^2 \) or \( rv = a^2 \)

\( \Rightarrow \) \( tgW \) \( tgV = a^2 \) \( \Rightarrow \) \( tg(\frac{1}{2}(r-p)) \) \( tg(\frac{1}{2}(r+p)) = a^2 \)

Here \( ds^2 = -dT^2 + d\rho^2 + \sin \rho d\Sigma^2 \) \( \quad 0 \leq r \leq \pi, \quad \pi + R < \pi \)

Now \( \sin \rho tgV = -a^2 \) is easy to draw.

\[ \text{Family of different } ds. \]
Consider (in de-Sitter space, or any space with $d^+$ spacelike) an observer $\mathcal{O}$ and a particle worldline $Q$. Suppose $Q$ intersects the past light cone of event $p$ on $\mathcal{O}$:

\[ \mathcal{O} \]

$Q$ is observable to $\mathcal{O}$ at any time after $p$:

\[ \mathcal{O}'s \text{ future event horizon} \]

But note, there is a point $r$ on $Q$ that lies on $\mathcal{O}'s$ future event horizon. Events on $Q$ after $r$ are not observable to $\mathcal{O}$.

Since $r$ is seen at $d^+$, it takes $\infty$ proper time from any event on $\mathcal{O}$ to the observation of $r$ on $\mathcal{O}$.

On $\mathcal{O}$, of course, it takes finite proper time from any past event to $r$.

It takes an infinite time in $\mathcal{O}$ to see a finite part of $Q$'s history (e.g., $\mathcal{O}$ observes infinite redshift of light from $Q$ as it approaches $r$).

Likewise, $\mathcal{O}$ will see infinite history of $\mathcal{O}$ in infinite time.
Even in Minkowski space if we have non-gerodetic observers:

which seems perfectly logical (redshifted light from accelerated light source), light from $\mathcal{I}$ appears as redshifted as $\mathcal{O}$ - $\mathcal{I}$.\)
anti-de Sitter space

(R < 0 case) we now will have \( \Lambda = -\frac{1}{4} R < 0 \).

Consider hyperboloid

\[-U^2 - W^2 + x^2 + y^2 + z^2 = \frac{1}{2} \rho^2 - \alpha^2\]

in flat \( R^5 \) with ---ttt signature

\[ds^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2\]

[compare signs with de Sitter (?), but \( u^2 \) to \( x^2 \) (odd) flipped].

\[\text{let}\]

\[U = \alpha \sin \varepsilon \cosh \rho\]

\[V = \alpha \cos \varepsilon \cosh \rho\]

\[X = \alpha \sinh \rho \sin \Theta \cos \Phi\]

\[Y = \alpha \sinh \rho \sin \Theta \sin \Phi\]

\[Z = \alpha \sinh \rho \cos \Theta\]

\[\text{spherical coordinates in } R^3 \text{ with radius } \alpha \sinh \rho\]

This defines a map from hyperboloid \( H^4 \) to \( R^5 \)

\[\varphi: H^4 \rightarrow R^5\]

with induced metric \( \varphi^* g \) (pullback of \( g \)).

Then

\[ds^2 = \alpha^2 \left[ -\cosh^2 \rho \, dt^2 + dp^2 + \sinh^2 \rho (d\Omega^2 + \sin^2 \delta d\phi^2) \right]\]

Exercise: Check this

\[\frac{1}{\alpha^2} ds^2 = -\, dt^2 \cosh^2 \rho (\cos^2 \varepsilon + \sin^2 \varepsilon) + dp^2 \left[ \sinh^2 \rho (\sin^2 \varepsilon + \cos^2 \varepsilon) + \cos^2 \rho (\cos \Theta + \mu \sin \Theta \cos \varepsilon) \right]

+ \sinh^2 \rho d\Omega^2 \left[ 5 \cosh^2 \varepsilon + \cos^2 \varepsilon (\sin^2 \Theta + \Theta \cosh^2 \varepsilon) \right] + \sin^2 \rho d\phi^2\]

Note that with \( \varepsilon > 0 \) a radius-like coordinate, the
space \( t' = \text{constant} \) sections are \( R^3 \) (topologically).

But for \( \rho, \varepsilon, \Theta \) fixed, \( t' \) lines are periodic \( t' \rightarrow t' + 2\pi \)

\( \rightarrow \) space has closed timelike curves (causal-
no-escape, maybe... see later causality).
Minner coordinate system:

\[ U = \alpha \sin t \]
\[ V = \alpha \cos t \cos \rho \]
\[ x = \alpha \cos t \sin h \rho \sin \theta \cos \phi \]
\[ y = \alpha \cos t \sin h \rho \sin \theta \sin \phi \]
\[ z = \alpha \cos t \sin h \rho \cos \theta \]

Now \( p^a g_{ab} \) is

\[ \int \frac{1}{\alpha^2} \, ds^2 = \left( -\cos^2 t - \sin^2 t (\sin h^2 r - \sin h^2 (\cos^2 t + \sin^2 \theta)) \right) dt^2 \]
\[ + \frac{\beta}{\alpha^2} \cos^2 t (\sin h^2 r + \cos^2 h r (-)) \, dr^2 + \cos^2 t \sin h^2 r \, \sin^2 \theta \, d\phi^2 \]

\[ \frac{1}{\alpha^2} ds^2 = -dt^2 + \cos^2 t \left[ dr^2 + \sin h^2 r \, d\rho^2 \right] \]

As we'll see this system has simple geodesics:

\((r, \theta, \phi) = \text{constant})\). So these lines are orthogonal to \( t = \text{constant surface} \).

But note that at \( t = \pm \frac{1}{2} \pi \) there are singularities. Clearly there are only coordinate singularities, but this frame can only be used for one piece of the space.
So the space described so far is one with topology $S^1 \times \mathbb{R}^3$.

We take de-Sitter space to be the universal covering space of this, meaning, take $t' \in [-\infty, \infty)$ and keep the metric as above (the embedding no longer makes sense).

**Structure at infinity and Penrose diagram:** Let's define (similar to the de-Sitter case)

$$\cos \chi = \frac{1}{\sinh p}$$

So

$$d\tilde{e}^2 = \sinh p \, dp = \frac{\sin \chi}{\cos \chi} \, d\chi$$

$$\Rightarrow \quad (1 + \cosh p) \, dp^2 = \frac{\sin \chi}{\cos \chi} \, d\chi^2$$

$$\Rightarrow \quad dp^2 = \frac{\cosh \chi}{\sin \chi} \, \frac{\sin \chi}{\cos \chi} \, d\chi^2 = \frac{1}{\cos \chi} \, d\chi^2$$

and

$$ds^2 = \alpha^2 \left[ -\frac{1}{\cos^2 \chi} \, dt'^2 + \frac{1}{\cos^2 \chi} \, d\chi^2 + \sec^2 \chi \, d\tilde{e}^2 \right]$$

which has $\chi \in (0, \frac{\pi}{2})$ and

$$ds^2 = \frac{\alpha^2}{\cos^2 \chi} \left[ -dt'^2 + d\chi^2 + \sin^2 \chi \, d\tilde{e}^2 \right] = \frac{\alpha^2}{\cos \chi} \, ds^2$$

recognizing again the metric of Einstein-static universe.

**Note:** that with $t' \in [-\infty, \infty)$ but $\chi \in [0, \frac{\pi}{2}]$ anti-de-Sitter is conformally related to half of the Einstein-static universe (the $\chi \in [\frac{\pi}{2}, \pi]$ is missing).
Geodesics in anti-de Sitter (not for closed)

\[ ds^2 = - \cosh p \, dt^2 + dp^2 + \sinh^2 p \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \]

Find geodesics? \( \text{Stat } g_{\mu \nu} = \frac{1}{2} \left( g_{\mu \nu} + g_{\nu \mu} - g_{\mu \nu} \right) \)

\[ \Gamma_{\mu \nu}^{\rho} = 0 \]

\[ \Gamma_{\rho \nu}^{\mu} = \Gamma_{\rho \mu}^{\nu} = -\frac{1}{2} \left( \cosh p \right)_{\rho} \mu = - \cosh p \sinh p \quad \Rightarrow \quad \Gamma_{\mu \nu}^{\rho} = \Gamma_{\rho \mu}^{\nu} = \frac{\sinh p}{\cosh p} \]

\[ \Gamma_{\rho t}^{t} = \sinh p \sinh p \quad \Rightarrow \quad \Gamma_{\rho t}^{t} = \cosh p \sinh p \]

\[ \Gamma_{\rho \rho}^{\rho} = \Gamma_{\rho \rho} = \frac{1}{2} \left( \sinh^2 p \right) \rho = \cosh p \sinh p \quad \Rightarrow \quad \Gamma_{\rho \rho}^{\rho} = \Gamma_{\rho \rho} = \frac{\cosh p}{\sinh p} \]

\[ \Gamma_{\rho \rho}^{\rho} = - \cosh p \sinh p \quad \Rightarrow \quad \Gamma_{\rho \rho}^{\rho} = - \cosh p \sinh p \]

Ignore \( \rho \); always look at \( \rho = \cos \theta \) plane (could have done \( \rho = \sin \theta \) as well)

Then

\[ \frac{d^2 x^m}{dt^2} + \Gamma_{nu}^{m} \frac{dx^n}{dt} \frac{dx^u}{dt} = 0 \]

To be sure, let's keep \( \theta \);

\[ \Gamma_{\rho \rho}^{\rho} = \Gamma_{\rho \rho} = \frac{1}{2} \sin^2 \theta \quad 2 \sinh^2 p \cos \theta = \sin^2 \theta \sinh p \cosh p \quad \Rightarrow \quad \Gamma_{\rho \rho}^{\rho} = \Gamma_{\rho \rho} = \frac{\cosh p}{\sinh p} \]

\[ \Gamma_{\rho \rho}^{\rho} = - \sin^2 \theta \sinh p \cosh p \]

\[ \Gamma_{\rho \rho}^{\rho} = \frac{\cosh p}{\sinh p} \]

\[ \Gamma_{\rho \rho} = - \sin^2 \theta \cos \theta \sinh p \]

Curved quantities

\[ g_{\mu \nu} \frac{dx^m}{dt} = - \cosh p \, dt \frac{dx^m}{dt} = 0 \]

\[ g_{\mu \nu} \frac{dx^m}{dt} = - \sinh^2 p \frac{d\theta}{dt} = 0 \]

\[ g_{\mu \nu} \frac{dx^m}{dt} = \sin^2 \theta \sinh p \frac{d\varphi}{dt} = 0 \]

\[ \frac{d^2 x^{\rho}}{dt^2} + \sinh p \sinh p \left[ \left( \frac{dt}{dt} \right)^2 - \left( \frac{d\theta}{dt} \right)^2 - \sin^2 \theta \left( \frac{d\varphi}{dt} \right)^2 \right] = 0 \]

\[ \frac{d^2 x^T}{dt^2} + \frac{\sinh p}{\cosh^2 p} \left( - \frac{\cosh p \cos^2 \theta - \cosh p \sin^2 \theta}{\sinh^2 p} \right) = 0 \]
This equation has a 1st integral that is easy to find. But, even easier, use \( t = \cosh \rho \) time to \( \rho \)

or
\[
\frac{(d\rho)^2}{d\tau^2} - \frac{T^2}{\cosh^2 \rho} + \frac{\Theta^2}{\sinh \rho} + \frac{\Phi^2}{\sinh \rho \sin \Theta} = 1
\]

Look for solutions with \( \Theta = \Phi = 0 \). Then
\[
\frac{d\rho}{d\tau} = \sqrt{\frac{T^2}{\cosh^2 \rho} - 1}
\]

This is like motion in a potential \( -\frac{\Theta T^2}{2 \cosh \rho} \) with total energy \( -\frac{1}{2} \).

And clearly there are "bound state" solutions, with turning point at \( \cosh^2 \rho = T^2 \) or \( \rho_0 = \text{arc} \cosh T \). Now, it is easy to integrate (\( \Theta \))

\[
\int \frac{d\rho}{\sqrt{T^2 - \cosh^2 \rho}} = \int \frac{\cosh \rho d\rho}{\sqrt{T^2 - \cosh^2 \rho}} = \int \frac{d\sinh \rho}{\sqrt{T^2 - 1 - \sinh^2 \rho}}
\]

Let \( \sinh \rho = \sqrt{T^2 - 1} \sin \theta \) \( \Rightarrow \) \( \int \frac{d\theta}{\sqrt{1 - \sin^2 \theta}} = \int \frac{\cos \theta d\theta}{\cos \theta} = \theta = \arcsin \rho = \text{arc} \sin \frac{\sinh \rho}{\sqrt{T^2 - 1}} \)

or \( \arctan \left( \frac{\sinh \rho}{\sqrt{T^2 - 1 - \sinh^2 \rho}} \right) = \arctan \left( \frac{\sinh \rho}{\sqrt{T^2 - 1}} \right) \).
Then $t(\tau)$ is obtained from

$$\frac{dt}{d\tau} = -\frac{T}{\cosh^2 \rho}$$

For this we need

$$5 \sin \gamma \rho = \frac{\sin \gamma \rho \sqrt{T^2 - 1}}{\sqrt{T^2 - 1}}$$

or

$$5 \sin^2 \gamma \rho = 5 \sin^2 \gamma \rho = \cosh^2 \rho - 1$$

So

$$\frac{dt}{d\tau} = -\frac{T}{1 + (T^2 - 1) \sin^2 \gamma \rho}$$

We need

$$\int \frac{d\tau}{1 + k^2 \sin^2 \gamma \rho} = \frac{\tan^{-1} \left[ \frac{\sqrt{1 + k^2} \gamma \rho}{\sqrt{1 + k^2}} \right]}{\sqrt{1 + k^2}}$$

(mol.)

So

$$\frac{t + T_0}{T} = \frac{1}{\sqrt{1 + (T^2 - 1) \sin^2 \gamma \rho}} \tan^{-1} \left[ T + \gamma \rho \right]$$

or

$$-\frac{\tan^{-1} (T + \gamma \rho)}{T} = T + \gamma \rho$$

(The sign is because $\gamma$ is taken distinct, but positive.)

We can also obtain the trajectory. Since $T = \frac{1}{T} \frac{\tan (t_0 - t)}{\sqrt{\gamma^2 (t_0 - t) + T^2}}$

So

$$\frac{1}{\sqrt{1 + T^2 \gamma^2 (t_0 - t)}} = \frac{\sin \gamma \rho}{\sqrt{T^2 - 1}}$$

In all these it's worth remarking $T = -\cosh \rho$.
Check the piece (recall $\Phi=\exp$) so we were with

from \( \Phi = \exp \) constant

Now

\[
\frac{d^2 \Phi}{dt^2} + 2 \frac{\cosh \theta}{\sinh \theta} \frac{d \theta}{dt} - \frac{\sin \theta}{\sinh \theta} \left( \frac{d \theta}{dt} \right)^2 = 0
\]

But if \( \Phi = \text{constant} \) (\( \Phi=0 \)) we have

\[
\frac{d}{dt} \left( \frac{d \Phi}{dt} \right) + 2 \frac{\cosh \theta}{\sinh \theta} \frac{d \theta}{dt} = 0
\]

Now, check: \( \frac{d \Phi}{dt} = \frac{\partial \Phi}{\partial \theta} \frac{d \theta}{dt} \) gives \( \frac{d}{dt} \left( \frac{d \Phi}{dt} \right) = -2 \frac{\cosh \theta}{\sinh \theta} \frac{d \theta}{dt} \frac{d \theta}{dt} \)

while the right is \( 2 \frac{\cosh \theta}{\sinh \theta} \frac{d \theta}{dt} \frac{d \theta}{dt} \)

so they cancel $\blacksquare$
Geodesics are $r=0, \theta = \text{constant}$

\[ \sin \theta \sinh \rho = \sin \tau \]
\[ \cosh \rho = \cos \theta \cos \tau \]
\[ \sinh \rho = \cos \theta \sin \tau \]
\[ \tan \tau = \frac{\cosh \rho}{\sin \theta} \]

\( (0, 0) \text{ remains fixed) } \)

\[ \sinh \rho_0 \sin \tau = \cos \theta \sin \tau \]

\[ \Rightarrow \tau = \frac{\pi}{2} + 2 \rightarrow +1 \]

\[ \tan \left( \frac{\tau}{2} - \frac{\pi}{4} \right) = \frac{\cosh \rho_0 \tan \left( \frac{\pi}{2} + 2 \frac{\pi}{4} \right)}{2 \sin \tau} = \frac{\cosh \rho_0}{2 \sin \tau} \]

\[ \Rightarrow \rho_0 = \frac{\pi}{2} \Rightarrow \text{it works} \]
The lines \( t = \frac{\pi}{2} \) are easy to understand. Since

\[ \sin t = \sin t' \cosh \rho \]

we have

\[ \pm 1 = 3 \sin t' \cosh \rho = \sin t' + \cos \chi \]

or

\[ \cos \chi = \pm 1 \]

Hence we introduced the variable \( \chi \) for the conformal mapping.

Note that the apparent singularity in \( t \), \( \rho \), and \( \theta \) is related to the converse of geodesics.
Causal structure of anti-de Sitter space:

\[ \text{NO CAUCULY SURFACE} \]

\[ t' = \frac{\sqrt{s}}{2} \]

\[ t = 0 \]

\[ \chi = 0 \]
\[ \chi = \frac{\pi}{2} \]

Evident \( \Rightarrow \) information flows in/out from boundary at \( \infty \).
null (go to o) from p

timelike geodesics from p are confined to infinite sequence of diamonds

At the are timelike curves (non-geodesic) that can reach any point future of the null-one from p.

Also

every point in D(t,s) can be reached by a unique geodesic from S, and to S.
Maximally symmetric hypersurfaces: isotropic cosmology.

Since this is a separate course in cosmology, we won't study cosmology here. But we do set up the stage by analyzing the spacetime one obtains from the requirements of homogeneity and isotropy.

Why? In cosmology (the study of the history, dynamics, evolution of the universe on a large scale) our observations are limited because:

(i) can only be done from one specific point, Earth, at one specific time, now (cosmologically the fact that we have been doing astronomy for 1000 years is still basically ad instantaneous observation event).

(ii) can only see part of the universe; observations are limited by

- dust and other intervening stuff
- physics, i.e. the universe is opaque before recombination
- luminosity
- only see few bandwidths of light.

Hence, to make progress it is always the case that assumptions are made in building mathematical models that describe the universe.
Generally, two assumptions are made: approximate.

1. Homogeneity: that there is no preferred point in space.
2. Isotropy: that there is no preferred direction.

These are often referred to as the Copernican Principle (since we would not occupy a preferred place in the Universe, just like Copernicus moved the center from Earth to Sun, how we are disposed with a center anywhere).

Notice that we are talking about approximate conditions. There is matter in the universe and it is not uniformly spread (there are galaxies, planets, asteroids...).

But, on average, smoothing over distance scales of several intergalactic lengths, the universe appears fairly smooth. So this is a starting approximation that has to be improved to account for very interesting irregularities — while course on cosmology.

We will restrict our attention to determining the properties that are homogeneous and isotropic, and discuss briefly what Einstein's equations imply for them.
Technical def.: of

Homogeneity. For spacetime to be homogeneous:

Need a foliation of spacetime by a 1-parameter family of spacelike surfaces $\Sigma_t$, and

for any $\Sigma_t$ (any "time"), for any two points $p, q \in \Sigma_t$ there is an isometry taking $p \rightarrow q$.

(Recall an isometry $\phi$ is $\phi : M \rightarrow M$ + $\phi^* g = g$).

In other words, there is some definition of time for which at each $t = \text{constant}$ hypersurface, the metric is the same at all points.
Isotropy: First define isotropy for an observer. We want to say that an observer sees no shift in any direction. So a space-time is spatially isotropic if

\[ \mathbf{u} = (\text{timelike}) \text{ tangent to world line at } p, \]
\[ \mathbf{v} = (\text{spacelike}) \text{ tangent vectors at } p, \text{ unit magnitude,} \]

(ie, \( \mathbf{v} \) are \( \perp \mathbf{u} \)).

Timelike curve

Isotropy of \( p \): an isometry leaving \( p \) and \( \mathbf{u} \) fixed but taking \( \mathbf{v}_1 \rightarrow \mathbf{v}_2 \) for any pair of \( \mathbf{v}_1, \mathbf{v}_2 \).

(ie, \( \mathbf{u} \) is no preferred direction, ie, no preferred spatial vector \( \perp \mathbf{u} \)).

Isotropic space: if there is a congruence of timelike curves such that every point \( p \) and the space he has isotropy at every point on his curves is congruent.

Note:
- For the 2 definitions (having a isometry) require a preferred collection of subspaces.
- If space-time is homogeneous and isotropic, then \( \mathbb{Z}_2 \) act on \( \mathbf{u} \). For if \( \mathbf{u} \) had a component along \( \mathbb{Z}_2 \), say \( \mathbf{v} \), then that would be a spatial vector in a preferred direction (we can project out that part that is not orthogonal to \( \mathbf{u} \), to construct \( \mathbf{v} \), a preferred vector with \( \perp \mathbf{u} \)).
Actually, one must be careful; the previous note is true when \( \mathbb{E}^+ \) and the isotropic observers are unique. If not unique, one can choose \( \Sigma \) to \( \mathbb{E}^+ \). (E.g., in flat Minkowski.)

Now, we \( \mathbb{E}^+ \rightarrow M \) (embedding) to define \( h = \delta g \). (This is the same as \( h = g \) restricted to act on vectors tangent to \( \mathbb{E}^+ \).)

So consider the space \( \mathbb{E}^+ \) with metric \( h_{ij} \), inverse \( h^{ij} \).

We expect isotropy \( \Rightarrow \) homogeneity \( \Rightarrow \mathbb{E}^+ \) is a 3-dim, maximally symmetric space.

In fact, isotropy is enough to show this (and so isotropy \( \Rightarrow \) homogeneity). Consider the 3-d curvature tensor \( \mathcal{R}^{ij}_{\ k} \).

With indices raised as shown, this is a map on \( \Omega^2 \) to \( \Omega^2 \),

\[
\mathcal{R}^{ij}_{\ k} = \langle \omega, \nabla_k \omega \rangle = \langle \omega, \mathcal{R}^{ij}_{\ k} \rangle, \quad \omega = a_{ij} \, d\xi^i \wedge d\xi^j
\]

Now, defining the inner product on \( \Omega^2 \) by

\[
\langle \xi, \omega \rangle = \langle \omega, \xi \rangle = a_{ij} \xi_k \omega^k
\]

then \( \mathcal{L} \) is self-adjoint; \( \langle \xi, \mathcal{L} \omega \rangle = \langle \mathcal{L} \xi, \omega \rangle \), which follows from \( \mathcal{R}^{ij}_{\ k} = \mathcal{R}^{kj}_{\ i} \). Hence a basis of orthonormal eigenvectors of \( \mathcal{L} \) has all the same eigenvalues. By isotropy the eigenvalues must be all the same (else, special direction), so \( \mathcal{L} \mathbb{E}^+ \) is isomorphic to \( \mathbb{E}^+ \).
or
\[
\overline{R}_{\text{ijk}e} = k (\delta^i_k \delta^j_e - \delta^j_k \delta^i_e) \\
\Rightarrow \overline{R}_{\text{ijkl}} = k (h_{ik} h_{j}e - h_{ie} h_{jk})
\]

Now, this is the statement that $h_{ij}$ is maximally symmetric if $k$ is constant (same everywhere on $\Sigma_t$). This follows from homogeneity, but it also follows from the Bianchi identity

\[
\overline{R}_{\text{ijkl}} + \overline{R}_{\text{ijkm}} + \overline{R}_{\text{ijnj}} = 0 \\
\Rightarrow k_{nm} (h_{im} h_{jn} - h_{im} h_{jn}) + k_{ln} (h_{ln} h_{jm} - h_{ln} h_{jm}) + k_{jk} (\cdots) = 0
\]

Now contract with $h_{im} h_{jn}$:

\[
k_{nm} (3 \cdot 3) + k_{ln} (1 \cdot 3) + (1 \cdot 3) k_{ln} = 0 \\
\Rightarrow k_{nm} = 0
\]

So, if $h_{ij}$ is homogeneous AND maximally symmetric.

Also, $k = \frac{k}{6}$ a constant for each $\Sigma_t$

Finally, we have the

So we have a space which admits a congruence of iso- metric observers with a corresponding foliation by space-like surfaces $\Sigma_t$ orthogonal to $\mathcal{O}$, which are Reimannian 2D-in maximally symmetric spaces with metric $h$. The null space time has metric $g$, and if $\Sigma_t \leftrightarrow \Sigma_t$, then $g(\delta^i_j, \delta^i_j) = h(3, 3)$.
Let \( \tilde{\mathcal{U}}(t) = g(t, \mathbf{x}) \). Then clearly, if we define \( h(t, \mathbf{x}) = 0 \)

\[
g = h + \lambda \tilde{\mathcal{U}} \times \tilde{\mathcal{U}}
\]

however, since \( g(0, \mathbf{0}) = -1, \quad \mathcal{U}(0) = -1 \) and

\[
-1 = 0 + \lambda \quad \Rightarrow \lambda = -1
\]

\[
g = -\tilde{\mathcal{U}} \times \tilde{\mathcal{U}} + h
\]

In components, \( g_{\mu\nu} = -\gamma_{\mu\nu} + h_{\mu\nu} \)

Useful coordinates:

1. Obvious choice on each \( \mathbb{E}_t \), i.e., spherical coordinates if \( R > 0 \)

2. Assign a fixed spatial coordinate label to each isotropic observer ("comoving coordinates")

3. Homogeneity \( \Rightarrow \) all isotropic observers agree on proper time of \( \mathbb{E}_t \), so label \( \mathbb{E}_t \) by proper time of isotropic observer.

\[
ds^2 = -dt^2 + a^2(t) \left\{ \begin{array}{ll}
dx^2 + \sin^2 x \, d\rho^2 & R > 0 \\
dx^2 + \chi^2 \, d\rho^2 & R = 0 \\
dx^2 + \sinh^2 x \, d\rho^2 & R < 0 \end{array} \right. 
\]

Robertson-Walker metric.

Note, this is a preferred set of observers: the isotropic observers. In comoving coordinates the distance between fixed points fixes on the hypersurface \( \mathbb{E}_t \) evolves with \( t \) as \( a(t) \).
Einstein's Equations

We have \( R_{ij} = 2k \delta_{ij} \).

Need to apply Einstein's tensor. It is fairly standard to introduce radial coordinates by

\[
dx = \frac{dr}{\sqrt{1-kr^2}}
\]

with \( k = 1, 0, -1 \) for \( R > 0, = 0, < 0 \). Then

\( r = \sin \chi, \chi, \sin \chi \) in each case. Then

\[
ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right]
\]

As usual, \( g_{\mu\nu} = \frac{1}{2} \left( g_{\mu\nu} + g_{\nu\mu} - g_{\mu\nu} \right) \) and \( R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\beta\mu\nu} \)

so

\[
\Gamma_{00} = -\frac{1}{2} \frac{2a^2}{1-kr^2} = -\Gamma_{00} = -\Gamma_{0i}
\]

\[
\Gamma_{00} = -\frac{1}{2} \frac{2a^2 r^2}{1-kr^2} = -\Gamma_{00} = -\Gamma_{00} = -\Gamma_{00} = -\Gamma_{0i}
\]

\[
\Gamma_{00} = -a^2 r^2 \sin \chi \theta = -\Gamma_{00} = -\Gamma_{00} = -\Gamma_{0i}
\]

\[
\Gamma_{11} = \frac{\partial}{\partial t} \frac{2kr}{(1-kr^2)^2}
\]

\[
\Gamma_{10} = \Gamma_{01} = -\Gamma_{10} = -\Gamma_{01} = -\frac{a^2 r}{1-kr^2}
\]

\[
\Gamma_{10} = -\Gamma_{10} = -\Gamma_{01} = -\Gamma_{01} = -\frac{a^2 r}{1-kr^2}
\]

\[
\Gamma_{10} = -\Gamma_{10} = -\Gamma_{01} = -\Gamma_{01} = -\frac{a^2 r}{1-kr^2}
\]

\[
\Gamma_{\theta\theta} = -r(1-kr^2) \quad \Gamma_{\phi\theta} = \frac{1}{r} 
\]

\[
\Gamma_{\phi\phi} = -r(1-kr^2) \sin \theta \cos \theta \quad \Gamma_{\theta\phi} = \frac{1}{r} 
\]

\[
\Gamma_{\phi\phi} = -r(1-kr^2) \sin \theta \cos \theta \quad \Gamma_{\theta\phi} = \frac{1}{r} 
\]

and

\[
R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = \partial_{\rho} \Gamma^{\rho}_{\mu\nu} - \partial_{\nu} \Gamma^{\rho}_{\mu\rho} + \Gamma^{\rho}_{\mu\lambda} \Gamma_{\lambda\nu} - \Gamma^{\rho}_{\nu\lambda} \Gamma_{\lambda\mu}
\]

so we have
\[ R_{\alpha\alpha} = -3 \frac{\ddot{a}}{a} (\frac{\dot{a}}{a})^2 - 3 \left( \frac{\ddot{a}}{a} \right)^2 = -3 \frac{\dddot{a}}{a} \]

\[ R_{\dot{\alpha}\dot{\alpha}} = \dot{a} \left( \frac{\ddot{a}}{a} \right) + \dot{a} \left( \frac{\ddot{a}}{a} \right) + \dot{a} \left( \frac{\dddot{a}}{a} \right) - \frac{\dddot{a}}{a} \]

\[ R_{\beta\beta} = \frac{\dddot{a} + \dot{a}^2 + 3 \dot{a}^2 + 2 \dddot{a}}{1-k_1^2} \]

\[ R_{\dot{\beta}\dot{\beta}} = \dot{a} \left( \frac{\ddot{a}}{a} \right) + \dot{a} \left( \frac{\ddot{a}}{a} \right) + \dot{a} \left( \frac{\ddot{a}}{a} \right) - \frac{\dddot{a}}{a} \]

\[ R_{\gamma\gamma} = \dot{a} \left( \frac{\ddot{a}}{a} \right) + \dot{a} \left( \frac{\ddot{a}}{a} \right) + \dot{a} \left( \frac{\ddot{a}}{a} \right) - \frac{\dddot{a}}{a} \]

\[ R_{\dot{\gamma}\dot{\gamma}} = \dot{a} \left( \frac{\ddot{a}}{a} \right) + \dot{a} \left( \frac{\ddot{a}}{a} \right) + \dot{a} \left( \frac{\ddot{a}}{a} \right) - \frac{\dddot{a}}{a} \]
\[ R = g^{\mu \nu} \mathcal{R}_{\mu \nu} = 3 \frac{a'}{a} + \frac{1}{a^2} \left[ (a'' a + 2 a'^2 + 2k) - 3 \right] \]
\[ = 6 \left[ \frac{a''}{a} + (\frac{a'}{a})^2 + \frac{k}{a^2} \right] \]

Now Einstein's equations are \( R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = 8\pi G T_{\mu \nu} \)
or, the \(- R = 8\pi G T \), \( R_{\mu \nu} = 8\pi G (T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T) \)

Model energy a matter in the universe by a perfect fluid.

For consistency with the isotropy and homogeneity of the relic
we must choose the fluid to be homogeneous and isotropic, that is
the fluid is at rest in comoving coordinates:

\[ U^\mu = (1, 0, 0, 0) \]

\[ T_{\mu \nu} = (\rho + p) U^\mu U_\nu + p g_{\mu \nu} \quad T^\mu_{\nu} = (\rho + p) U^\mu U_\nu + p g^\mu_{\nu} \]

Note that \( T^\mu_{\nu,\mu} = 0 \) and for \( \nu = 0 \)

\[ \Rightarrow \quad T^{\mu}_{\nu,\mu} = T^{\mu}_{\nu,\mu} + \Gamma^{\mu}_{\nu,\mu} T^\mu_\rho - \Gamma^{\mu}_{\nu,\mu} T^\mu_\rho \]
\[ \Rightarrow \quad T^{\mu}_{\nu,\mu} = T^{\mu}_{\nu,\mu} + \Gamma^{\mu}_{\nu,\mu} T^\mu_\rho - \Gamma^{\mu}_{\nu,\mu} T^\mu_\rho \]

and so

\[ T^{\mu}_{\nu,\mu} = -(\rho + p) \frac{\dot{a}}{a} + p_0 + (\rho)(3 \frac{a'}{a}) - (3 \frac{a'}{a})(\rho) \]

so

\[ \rho + 3 \frac{a'}{a} (\rho + p) = 0 \]

This equation could be obtained from Einstein's, but this is
simpler.
Now \( T = g^{\mu \nu} T_{\mu \nu} = -(p+p) + 4\rho = 3\rho - \rho \)

\[ R_{00} = 8\pi G (T_{00} - \frac{1}{2} g_{00} T) \]

\[-3 \frac{a^3}{a} = 8\pi G (\rho + \frac{1}{2} (\rho - p)) = 8\pi G (\frac{1}{2} \rho + \frac{3}{2} \rho) \]

So

\[ \frac{\dot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3\rho) \]

and \( R_{\mu \nu} = 8\pi G (T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T) \)

\[ \frac{\dot{a}^2 + 2a^2 + 2\dot{a}}{1-kR^2} = 8\pi G \left[ \frac{a^2}{1-kr^2} \rho - \frac{1}{2} \frac{a^2}{1-kr^2} (\rho - p) \right] \]

\[ = \frac{1}{1-kr^2} 8\pi G a^2 (\rho - \frac{1}{2} (3\rho - p)) \]

\[ \Rightarrow \frac{\dot{a}^2}{a} + 2(\dot{a})^2 + 2\frac{\ddot{a}}{a^2} = 4\pi G (p - p) \]

Eliminate \( \frac{\dot{a}}{a} \) using above, \( \frac{\dot{a}}{a} \) and \[ \frac{4\pi G}{3} (\rho + 3\rho + 3p - 3\rho) = \frac{4\pi G}{3} \rho \]

\[ \Rightarrow (\dot{a})^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho \]

These are Friedmann equations. Metrics that obey these are called FRW metrics (Friedmann-Robertson-Walker).

We have two equations for three unknowns, \( a, \rho, p \).

But \( \rho \) and \( p \) are not independent if we know what constitutes the matter/energy in the universe. For example, a collisionless fluid (dark) has \( p = 0 \), while radiation has \( p = \frac{1}{3} \rho \). So we use an "equation of state" \( p = \omega \rho \).
We will take $w$ to be a fixed number, and are particularly interested in the cases:

$$w = \begin{cases} 
0 & \text{dust (or "matter")} \\
\frac{1}{3} & \text{radiation} \\
-1 & \text{cosmological constant.}
\end{cases}$$

The last one is just the statement that if we add modify Einstein's equations by adding Einstein's cosmological constant $\Lambda$:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Then we can rewrite:

$$G_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu})$$

and think of $-\frac{\Lambda}{8\pi G} g_{\mu\nu}$ as a contribution to $T^\Lambda_{\mu\nu} = T_{\mu\nu} + T^\Lambda_{\mu\nu}$.

Then, $T^\Lambda_{\mu\nu}$ is of the form of a fluid with $\rho = \frac{\Lambda}{8\pi G}$ and $p = -\rho$ (so $w = -1$).

Then $T^\Lambda_{0\mu} = 0 \Rightarrow \rho + 3 \frac{a^2}{a} (\rho + wp) = 0$ or

$$\frac{\dot{\rho}}{\rho} = -3(1 + w) \frac{a}{a} \Rightarrow \frac{a}{a} \ln \rho = -3(1 + w) \frac{a}{a} \ln a$$

$$\rho = \rho_0 (\frac{a}{a_0})^{-3(1 + w)}$$

Note, $w = 0 \Rightarrow \rho \sim \frac{1}{a^3}$ makes sense, $\rho \sim \frac{1}{a}$ value

$w = \frac{1}{3} \Rightarrow \rho \sim \frac{1}{a^2}$

$w = -1 \Rightarrow \rho = \text{constant, i.e.} \quad \rho \sim \frac{\Lambda}{8\pi G}$.
One can then solve Einstein equations &

\[
\left( \frac{d^2}{dt^2} \right) + \frac{k}{a^2} = \frac{8\pi G}{3} \rho = \frac{8\pi G \rho_0}{3a^2} \left( t+\omega \right)
\]

or

\[
\frac{d^2 a}{dt^2} - \frac{8\pi G \rho_0}{3} \frac{1}{a^{1+3\omega}} = -k
\]

This is like the equation

\[
E = \frac{1}{2} m \dot{r}^2 + V(r)
\]

multiplied by \( \frac{2}{3} \), so if a particle the solution has same time dependence as a particle in a potential \( V \sim \frac{C}{r^{1+3\omega}} \) with

\[ E \sim R \]

For \( 1+3\omega > 0 \) \( V \to -\infty \) as \( r \to 0 \) and \( V \to 0 \) as \( r \to \infty \). So

\[ \text{Now if } E < 0 \text{ the motion has a turning point at some maximum } r \text{ and then eventually } r \to 0. \]

For \( E > 0 \) the motion is unbounded provided \( r > 0 \) initially. So for \( \omega > \frac{1}{3} \) we have

\[ \text{Note } k + \omega \text{ for } k < 0 \]

\[ \dot{a} \to 0 \text{ as } t \to \infty \]

while for \( k < 0 \) \( \dot{a} > 0 \) for \( t \to \infty \).\]
Clearly it is of great (political) interest to know if the universe will expand forever (and if so whether it will do so by slowing down to $a=0$ asymptotically) or if it will collapse into a "big crunch". Need $\Omega > 0$.

(Note, however, that if we start with $a>0$ at some point, running the clock back in any case gives $a\to 0$, so it looks like the universe grew out of a singularity ($a=0$) condition, or better, started small at some to and quietly grew. This is called the big bang. However, it is not an explosion. Recall, comoving observers are separated by fixed comoving separation. It is just that the distance (space between any two of them) is $\to 0$ as $t\to \text{big bang}$.

To figure out whether $k>0$, $k=0$ or $k<0$ in our present universe we can measure each term in the left of

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G \rho_0}{3} a_0^3 H_0^2 = -\frac{k}{a^2}$$

First, if this is evaluated today, then $a=a_0$ and such

$$\left(\frac{\dot{a}}{a}\right)_0^2 = \frac{8\pi G \rho_0}{3} = -\frac{k}{a_0^2}.$$

Need $H_0 = \frac{\dot{a}}{a_0}$ the Hubble parameter (should be called Hubble velocity since $H \equiv \frac{\dot{a}}{a}$ is not constant)

and $\rho_0 = \text{energy density}$. 


H$_0$ can be measured from redshift vs luminosity of standard candles. $C$ are below) while $f_0$ can be "counted".

Actually, we should be more careful to include all types of matter (different equations of state) in the analysis --- we have assumed one dominant type.

Write

\[
\left(\frac{a}{a_0}\right)^2 = \frac{8\pi G}{3} \sum_i \frac{\rho_i}{a_0^3(1+z)} - \frac{1}{a_0^2}
\]

You will often see this written as

\[
H^2 = \frac{8\pi G}{3} \sum_i \rho_i
\]

where one term, $i=k$, curvature has

\[
\rho_k = \frac{3}{8\pi G} \left( -\frac{k}{a_0^2} \right)
\]

(set aside $\rho_c$ for critical density).

Also divide this by $H^2$,

\[
1 = \sum_i \Omega_i
\]

where

\[
\Omega_i = \frac{8\pi G}{3H^2} \rho_i = \frac{\rho_i}{f_c}
\]

where $f_c = \frac{3H^2}{8\pi G}$ is a quantity depending only on the geometry, through

which gives the critical value for which $k$ changes sign: if we define

\[
\Omega = \sum_{i\neq k} \Omega_i
\]

then we have

\[
\Omega_k = 1 - \Omega
\]

and $\Omega_k > 0$, $\Omega_c \geq 0$, $\Omega_k < 0$ (if $k < 0$, $\Omega_k > 0$).
So we need to measure all components of \( p \) and compare with \( \Phi \), obtained from measuring \( H \).

Note that the different components scale differently:

\[
\Omega_m \sim 9^0, \quad \Omega_k \sim \frac{1}{a^2}, \quad \Omega_{\Lambda} \sim \frac{1}{a^4}, \quad \Omega_{\Lambda} \sim \frac{1}{a^4}
\]

If they were all small today, then in the past, \( \Omega_m \approx \Omega_k \approx 0 \), and should be dominant.

In fact, today it's \( \Omega_m \sim \frac{1}{2} \Omega_k \), and with \( \Omega_\Lambda \approx 0.7 \).

Moreover, the evolution of \( a(t) \) is still given, as before, by

\[
a^2 - \frac{8\pi G}{3} \sum \rho_i \frac{a^3(t_i)}{a^{1+3w_i}} = -k
\]

At small \( \Lambda \), the largest \( w \) dominates; at large \( a \), the smallest \( w \) dominates. With matter, radiation, and cosmological constant, we have \( w_{\text{max}} = \frac{1}{3} \), \( w_{\text{min}} = -1 \), so the "potential"

\[V(a) \propto -\frac{8\pi G}{3} \rho_m \frac{a^3}{a^2} \propto a^4 \quad \text{as} \quad a \to 0 \quad \text{and}
\]

\[V(a) \propto -\frac{8\pi G}{3} \rho_\Lambda \frac{a^6}{a^2} \propto a^2 \quad \text{as} \quad a \to \infty.
\]

So, if \( \Lambda < 0 \),

\[
V(a) = \frac{1}{3} \Lambda a^2
\]
while, if $\Lambda > 0$

\[ V(a) \]

\[ a \]

Let's look at this in more detail. For $k < 0$ ("E" > 0) or $k = 0$ ("E" = 0"), the "particle" motion is unbounded, describing an ever-expanding universe. But for $k > 0$ ("E" < 0) there is a critical value of parameters beyond which the universe collapses. This occurs if the maximum of the potential $V(\sigma)$ is above $k$ for every $E$.

Recollapse condition

\[ \max_a \left[ -\frac{8\pi G}{3} \sum \rho_i \frac{a_i^3(1+w_i)}{a^{1+3w_i}} \right] > -k \]

or multiply by $-a$ - sign and using $\rho_{oi} = \frac{8\pi G}{3H_0^2} \rho_i$

\[ \min_a \left[ \frac{H_0}{a} \sum \frac{\rho_{oi} a_i^3(1+w_i)}{a^{1+3w_i}} \right] < k = -H_0^2 a_0^2 \rho_k \]

or simply

\[ \min_a \left[ \frac{\rho_{rad} a_0^4}{a^2} + \frac{\rho_{om} a_0^3}{a} + \rho_{om} a^2 \right] < -a_0^2 \rho_k \]

To simplify matters, let's ignore radiation, since it is already negligible today. Then, to get one derivative:
\[
\frac{d}{da} \left[ \frac{\rho_0 a_0^3}{a} + \rho_0 a^2 \right] = 0
\]

\[\Rightarrow - \frac{\rho_0 a_0^3}{a^2} + 2 \rho_0 a = 0\]

\[\Rightarrow a = \left( \frac{\rho_0 a_0^3}{2 \rho_0} \right)^{\frac{1}{3}} = a_0 \left( \frac{\rho_0}{2 \rho_0} \right)^{\frac{1}{3}}\]

Now plug back into potential to find minimum

\[\text{minimum} = \frac{\rho_0 a_0^3}{a_0 (2 \rho_0)^{\frac{1}{3}}} + \rho_0 a_0^2 \left( \frac{\rho_0}{2 \rho_0} \right)^{\frac{2}{3}}\]

\[= a_0^2 \rho_0^{\frac{2}{3}} (2 \rho_0)^{\frac{1}{3}} (1 + \hat{c})\]

and the condition for recollapse is

\[\frac{3}{2} \rho_0 (2 \rho_0)^{\frac{1}{3}} < -\rho_0\]

Moreover, recall that \(\rho_k = 1 - \rho_0 - \rho_0\), so the condition is an \(\rho_0 vs \rho_0\) :

\[\frac{3}{2} \rho_0 (2 \rho_0)^{\frac{1}{3}} < \rho_0 + \rho_0 - 1\]

And keep in mind that we are doing the \(k > 0\) case, so \(\rho_k < 0\) (although the treatment has been general).

\text{CAUTION! The solution to the inequality must be dealt with great care because of the cube root. There are two large (one positive and one negative) roots of the cubic (set \(x^3 = \rho_0\)), and a small, positive root. Only the last is physical.}
Let's put together our result into one graph: the $\Omega_m, \Omega_k$ parameter space.

Expansion vs. Recollapse

If $\Lambda > 0$ (at with $k < 0$), then $\Lambda < 0$.

$(\Lambda < 0 \Rightarrow \text{re-collapse})$

Note that there is an unstable solution to $a^2 + V(a) = -k$

with $\dot{a} = 0$ and $V(a) = -k$ at the "top" of the hill.

That is Einstein's static universe.

Sci Am March 2005 p96 has "misprint in subcosmology"
Redshift and Distance (a la Carroll).

FRW has no timelike Killing vector (the metric depends explicitly on t). But there is a Killing tensor, let $U^m = (1, 0)$, that is, $U$ is the 4-vector tangent to isotropic observers in comoving coordinates (i.e., their 4-velocity). Then let

$$K_{mn} = a^2 (g_{mn} + U_n U_m)$$

where $g_{mn}$ is the FRW metric with scale factor $a$. Then $\nabla_m K_{mn} = 0$ (see next page for check of this).

Now, take $V^m$ to be a tangent to a particle trajectory $V^m = \frac{dx^m}{d\lambda}$. This is the 4-velocity for a massive particle, or the wave 4-vector for a massless particle.

Along the geodesic

$$K^2 = K_{mn} V^m V^n$$

is constant. Then, for a massive particle $V_\mu V^\mu = -1$

$$\frac{K^2}{a^2} = V^m V_m + (U_m V^m)^2$$

$$= -1 + (V^0)^2$$

But $V_\mu V^\mu = -1 \Rightarrow (V^0)^2 - g_{ij} V^i V^j = 1 \Rightarrow$

$$V^2 = g_{ij} V^i V^j = \frac{K^2}{a^2}$$

For massless particles $V_\mu V^\mu = 0$ and $V_\mu V^\mu = -\infty$

so

$$\frac{K^2}{a^2} = \omega^2 \quad \text{or} \quad \omega = \frac{K}{a}$$
\[
K_{\nu;\sigma} = K_{\nu;\sigma} - \Gamma_{\nu\sigma}^\lambda K_{\lambda\mu} - \Gamma_{\nu\sigma}^\mu K_{\mu\lambda}
\]

Check
\[
K_{\nu;\nu} = K_{\nu;\nu} - 2\Gamma_{\nu0}^\lambda K_{\lambda0} = 0
\]
\[
K_{\nu0;\nu} = K_{\nu0;\nu} - 2\Gamma_{\nu0}^\lambda K_{\lambda0} = 0 \quad (K_{\nu0} = 0 = K_{\nu0})
\]
\[
K_{0i;\nu} = K_{0i;\nu} - \Gamma_{0i}^\lambda K_{\lambda0} - \Gamma_{0i}^0 K_{i\lambda} = 0
\]
\[
K_{ij;\nu} = K_{ij;\nu} - \Gamma_{ij}^\lambda K_{\lambda0} - \Gamma_{ij}^0 K_{i\lambda} = 0
\]

Here \( K_{ij} = a^2 g_{ij} = a^4 h_{ij} \)

where \( h_{ij} \) is the metric on the hypersurface of constant \( k \).

so \( K_{\nu0} = 4\left(\frac{\dot{a}}{a}\right) K_{ij} \)

Also \( \Gamma_{0i}^\lambda K_{0\lambda j} = \Gamma_{0i}^\lambda K_{\lambda0} = \frac{\dot{a}}{a} K_{ij} \)

so \( K_{0i;\nu} = 2\left(\frac{\dot{a}}{a}\right) K_{ij} \)

so \( K_{0i;\nu} = \Gamma_{0i}^\lambda K_{\lambda0} - \Gamma_{ij}^\lambda K_{\lambda0} - \Gamma_{ij}^0 K_{i\lambda} = -2\left(\frac{\dot{a}}{a}\right) K_{ij} \)

so \( K_{0i;\nu} = (2-1-1)\left(\frac{\dot{a}}{a}\right) K_{ij} = 0 \)

Finally
\[
K_{ij;\nu} = K_{ij;\nu} - \Gamma_{ij}^\lambda K_{\lambda\mu} - \Gamma_{ij}^\mu K_{\mu\lambda}
\]
\[
K_{ij;\nu} = a^4 h_{ij;\nu}
\]

Recall \( \Gamma_{\mu\nu}^{\lambda} = \frac{1}{2} g^{\mu\rho} (g_{\lambda\rho,\nu} + g_{\nu\rho,\lambda} - g_{\lambda\nu,\rho}) \)

so \( \Gamma_{ij}^\mu K_{ij;\nu} = \frac{1}{2} a^4 h_{ij;\nu} \Gamma_{ij}^\mu = \frac{1}{2} a^4 (h_{ij,\nu} + h_{\nu,i,j} - h_{ij} \delta_{\nu,j}) \)

so \( K_{ij;\nu} = a^4 [h_{ij,\nu} - \frac{1}{2} (h_{ij,\nu} + h_{\nu,i,j} - h_{ij,j}) - \frac{1}{2} (h_{ij,\nu} + h_{\nu,i,j} - h_{ij,j})] = 0 \) even before symmetrizing.
Consider two comoving observers (that have \( \mathbf{v} \) as target vector):

[Diagram showing emission and observation of a photon]

Then, since \( k = \text{constant} \),

\[
\omega_{\text{em}} a_{\text{em}} = \omega_{\text{obs}} a_{\text{obs}}
\]

or, since \( \omega_{\text{em}} = \frac{1}{\lambda_{\text{em}}} \)

![\frac{\lambda_{\text{em}}}{a_{\text{em}}} = \frac{\lambda_{\text{obs}}}{a_{\text{obs}}}
]

That is \( \lambda_{\text{obs}} = \frac{a_{\text{obs}}}{a_{\text{em}}} \lambda_{\text{em}} \)

and since \( \lambda \) is increasing \( \lambda_{\text{obs}} > \lambda_{\text{em}} \Rightarrow \text{redshift} \).

Define the redshift as

\[
z \equiv \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}} = \frac{a_{\text{obs}}}{a_{\text{em}}} - 1
\]

or

\[
\frac{a_{\text{em}}}{a_{\text{obs}}} = \frac{1}{1 + z}
\]

\( \Rightarrow \) Measuring \( z \) gives the factor by which the universe has grown since emission as \( 1 + z \).
The instantaneous physical distance \( d_p(t) \) between isotropic observers is the distance between two on a common \( t = \text{constant} \) surface. Recall

\[
ds^2 = -dt^2 + a^2(t) \left[ dx^2 + S_x^2(x) \, dx^2 \right]
\]

where \( S_{+1} = \sin x \), \( S_0 = x \), \( S_{-1} = \sinh x \). Then, the distance between an isotropic observer at \( x = 0 \) and one at \( x \) is

\[
d_p(t) = a(t) x
\]

Taking \( \frac{\partial}{\partial t} \), we have

\[
d_p = \dot{a} x = \ddot{a} (dx) = (\frac{\partial}{\partial a}) dp
\]

So, interpreting \( dp = \frac{\partial}{\partial t} \), for "velocity of separation" of the isotropic observers, we have

\[
V_p = \frac{\partial}{\partial t} dp
\]

which is Hubble's law. (If we evaluate that today we have

\[
V_p = H_0 \, dp_0
\]

The problem, of course, is that \( H_0 \), which is of cosmological interest, cannot be directly determined from the above because we have no way of measuring \( dp \) or \( V_p \) directly. The problem (beyond other accidental issues, like the fact that galaxies are not necessarily isotropic observers) is that

(i) we have no ruler to measure \( dp_0 \), we have to infer it from other observations, like luminosity (see below)

(ii) we cannot observe \( V_p \), its velocity today of an observer far away, because light was emitted in the past. This is a small effect if the time \( T \) of light travel is much smaller than \( \frac{1}{H_0^{-1}} \).
In flat space, the luminosity $L$ (defined as energy/time emitted) of a source, and the flux $F$ (defined as energy/increase received) are related by

$$L = 4\pi d^2 F$$

So we define a luminosity distance, $d_L$, by

$$d_L^2 = \frac{L}{4\pi F}$$

This is useful if we can identify objects in the sky as "standard candles", i.e., objects that have the same intrinsic luminosity. Then measuring the flux at Earth, we can directly infer the relative distance to Earth.

In a FRW background, photons from a source at $x=0$ get redshifted by $(1+z)$. Moreover, since we are looking at energy/time emitted vs received, the energy emitted over an interval of time $\Delta t$ at $x=0$ is received over a time interval $(1+z)\Delta t$. So

$$\frac{F}{L} = \frac{1}{(1+z)^2 A}$$

where $A$ is the area of a sphere centered at $x=0$ with varying radius $x$. Now, for $ds^2$ we have

$$A = 4\pi a^2 S_k(x)$$

So

$$d_L = \sqrt{\frac{L}{4\pi F}} = (1+z)a_0 S_k(x)$$
(Note: check on $\delta t$ argument. Emit two photons at $t=0$ and $t=\delta t$. They follow null geodesics for $x=0 \to x^<$.)

$$ds^2 = 0 = -dt^2 + a^2 dx^2$$

$$\Rightarrow \quad \frac{dX}{dt} = a^{-1}$$

$$\Rightarrow \quad X = \int_0^{\delta t} a(t')dt' = \int_0^{\delta t} a(t')dt'$$

and we want $\delta T$. But then, from the equality

$$\int_0^{\delta t} a(t')dt' = \int_0^{\delta T} a(t')dt'$$

and if $dt$ is infinitesimal

$$a(0)\delta t = a(t')\delta T \quad \text{or} \quad \delta T = \left(\frac{a(t')}{a(0)}\right)^{-1} \delta t = \left(\frac{a(t')}{a(0)}\right)^{-1} \delta t$$

Now, the expression for $dX$ is not very useful since it depends on $X$ explicitly, not as an observable. However, as in the note above,

$$X = \int_0^{\delta t} a(t')dt' = \int dt' \frac{da}{a} = \int \frac{da}{a^2} \quad \text{a_{obs}}$$

Now using

$$\frac{a_{en}}{a_{obs}} = \frac{a}{a_0} = \frac{1}{1+z}$$

and $a_{en} = a$, the scale factor at emission corresponding to redshift $z$, we can choose variables from a to $z$. Using $a = \delta t a$, we have

$$X = \int_0^{\delta t} \left[ \frac{a_0}{(1+z)^2} \left( \frac{1}{a^2H} \right) \right] \frac{dt'}{dt_0} = \int_0^z \frac{dz'}{H(z')}$$

Note added: At this point a solution of Friedmann equations gives $a(t)$, the integral could be done if we insert $t = t(a)$, and then express the result in terms of the redshift. We instead write the integral as an integral over $z$.\end{quote}
To perform the integral we need a solution to Friedmann equations, which give $H(z)$. Of course,

$$\dot{H}^2 = \frac{8\pi G}{3} \sum p_i$$

and we know $p_i = \frac{\dot{R}_i}{R_i} \left( \frac{a_0}{a} \right)^{3(1+w_i)} = R_i (1+z)^{3(1+w_i)}$

Moreover, recall that evaluating this today and divide by $H_0^2$ we get

$$1 = \frac{1}{a_0^2} \sum R_i$$

So

$$\frac{\dot{H}^2}{H_0^2} = \frac{8\pi G}{3} \sum R_i \frac{a_0}{a} (1+z)^{3(1+w_i)} = \frac{1}{a_0^2} \sum R_i (1+z)^{3(1+w_i)}$$

Let $E(z) = H(z)/H_0$. Then

$$X = \frac{1}{a_0} \int_0^z \frac{dz'}{E(z')}$$

with $E(z) = \sqrt{\frac{1}{a_0^2} \sum R_i (1+z)^{3(1+w_i)}}$

and this can be plugged into $dL = (1+z)^{a_0} S_k (X)$

to get $dL$ in terms of $z$, $a_0$ and $H_0$. Not for integration has to be done numerically.

Note that now we need $a_0$ in addition to $H_0$ and $z$. But if we know $\Omega_k$ we can get $a_0$ (since $\Omega_k = \frac{3}{8\pi G a_0^2}$) except for the case $k=0$. However, for $k=0$, $S_k (X) = X^2$ and $a_0$ drops out of $dL$. For $k \neq 0$, we can use $\Omega_k = 1 - \Omega_0$ to infer $\Omega_0$ using it above. So, starting with $S_k = 1 - \Omega_0$ to infer

$$\Omega_k = \frac{8\pi G}{3H_0^2} p_k = -\frac{k}{H_0^2 a_0^2} \Rightarrow$$

Thus

$$a_0^2 = \frac{k}{\Omega_k H_0^2} \quad \text{or} \quad a_0 = \frac{1}{H_0 \sqrt{\Omega_k}} = \frac{1}{H_0 \sqrt{1 - \Omega_0}}$$

(provided $k \neq 0$).

So, finally

$$dL = \frac{(1+z)}{H_0 \sqrt{1 - \Omega_0}} S_k \left[ \int_0^z \frac{dz'}{E(z')} \right]$$
Exercise: do the integral $\int_0^1 \frac{d^2}{\partial l^2}$ numerically (Elliptic integral... and numerics don't go together anyway).

for the case that we have only $\Lambda$ and nother (and the three cases $k=0, \pm 1$).

$E(z) = \Omega_m (1+z)^3 + \Omega_{\Lambda} + \Omega_k (1+z)^2$

where $\Omega_k = 1 - \Omega_m - \Omega_{\Lambda}$.

\[\begin{align*}
\Omega_m &= 0.3 \\
\Omega_{\Lambda} &= 0.5 \\
\Omega_k &= 0.2 & k &= -1
\end{align*}\]

Note the maximum from $S_k(x) = \sin x$ eventually has a zero.
These are other measures of distance:

1) Proper notion distance, \( d_M \).

In flat space
\[
\delta s = \frac{dt}{\Theta}
\]
so, \( d_M = \delta s \).

So define:
\[
d_M = \frac{r}{\Theta}
\]

ii) Angular diameter distance, \( d_A \):

In flat space, \( d_A = \Theta \), so \( d_A = \frac{\Theta}{\Theta} \).

Exercise: Show \( d_A = (1+\ell)^{-2}d_L \) and \( d_M = (1+\ell)^{-1}d_L \).

Ans: For \( d_A \), let the observer be at \( x = 0 \) and the light emitted from \( x \), with \( \Theta \) varying from 0 to \( \Theta \).

\[
\int_0^\Theta \frac{d\Theta}{\Theta} = \ln \frac{\Theta}{\Theta} = 0
\]

This makes \( x = \frac{1}{\ell} \).

But doing this way is problematic since changing the target vectors at the observer (the origin) is bad (coordinate singularity).

Avoiding coordinate singularity is messy.

Example: with \( \Theta = 2\pi \), \( d_A = \frac{\Theta}{\Theta} \). But now

\[
D = 2\pi a_M S_F(x)
\]

so
\[
d_A = \frac{2\pi a_M S_F(x)}{2\pi} = a_M S_F(x) = \frac{1}{1+\ell} a_0 S_F(x) = \frac{d_L}{(1+\ell)^2}
\]

Similarly
\[
d_M = \frac{\delta s}{\delta \Theta} = \frac{\delta s}{\delta \Theta} \cdot \frac{\delta \Theta}{\delta \ell} = \frac{\delta s}{\delta \ell}
\]

Not
\[
\frac{\delta s}{\delta \Theta} = d_A = \frac{1}{1+\ell} a_0 S_F(x) \quad \text{and} \quad \frac{\delta s}{\delta \ell} = (1+\ell) \Rightarrow d_M = a_0 S_F(x) = \frac{d_L}{1+\ell}
\]
Lookback Time:

If today's time is to find the time when a photon was emitted by a comoving observer (or at an event coinciding with a comoving observer) with coordinate $x$ is $t_{em}$, then

$$
\Delta t = t_0 - t_{em} = \int_{t_{em}}^{t_0} dt = \int_{a_{em}}^{a_0} \frac{da}{a} = \int_{a_{em}}^{a_0} \frac{da}{aH}.
$$

Using $H = H_0 E(z)$ and $a = \frac{a_0}{1+z}$ we have

$$
\Delta t = \frac{1}{H_0} \int_{0}^{z} \frac{dz}{(1+z)E(z)}.
$$

The integral is dimensionless; its units are set by $H_0 \approx 10^{6}$ yrs.

In particular, as $z \to \infty$ the integral goes to a fixed finite number (that depends on the details of $E(z)$), of order 1. So we are tempted to say

$$
T = \text{age of universe} = \frac{1}{H_0} \int_{0}^{z} \frac{dz}{(1+z)E(z)} \approx \frac{1}{H_0}.
$$

In fact, we get (from mathematics) that for $\Omega_m = 0.3$, $\Omega_r = 0.7$,

So, for fixed $\Omega_m$, $T$ increases with $\Omega_m$ (albeit slowly). This is not the whole story because there is also radiation! But adding $\rho_r = 10^{-5} \rho_m$ changes the result a negligible amount.
Black Holes

Start with Schwarzschild:
As seen briefly in 1st quarter

\[ ds^2 = - \left( 1 - \frac{2GM}{r} \right) dt^2 + \left( 1 - \frac{2GM}{r} \right)^{-1} dr^2 + r^2 d\Omega^2 \]

is a solution of Einstein's equations in empty space

\[ R_{\mu\nu} = 0 \]

that has spherical symmetry and it is static.

Birkhoff's theorem asserts that the Schwarzschild metric is the unique static, spherically symmetric solution of Einstein's equations in empty space.

We won't go over the proof here, but the ingredients are:
(a) Define spherical symmetry as having corresponding symmetries; there are three Killing vectors that generate the symmetries of the sphere. These are the generators of the Lie Algebra of the group of rotations, $SO(3)$, that leave the sphere ($S^2$) invariant.
(b) You are familiar with the algebra, it is just the same as angular momentum in Q.M.:

\[ [L_i, L_j] = i \epsilon_{ijk} L_k \]

or, since we use anti-hermitian generators, let \( X_i = i L_i \Rightarrow \)

\[ [X_i, X_j] = \epsilon_{ijk} X_k \]

Exercise: transforming to spherical coordinates \( X_i = r \partial_r - \theta \partial_\theta, X_\theta = \theta \partial_\theta - r \partial_r \)

\( X_\phi = x \partial_\phi - y \partial_x \)

show that these Killing vectors are:

\[ R = \theta_\phi \quad S = \cos \theta_\phi e_r - \sin \theta_\phi \sin \theta e_\theta \quad T = -\sin \theta_\phi e_\phi - \cos \theta_\phi e_r \]
(ii) Frobenius theorem then allows one to show the space is foliated by 2-spheres. Basically the theorem says that if you have a set of vector fields that closes under commutation, \[ [X_i, X_j] = \text{lin combination of } X_a \text{'s}, \] then the integral curves form a submanifold of the manifold on which they are defined.

(iii) Put spherical coordinates \( \theta, \phi \) on one sphere. Extend to other neighboring spheres using orthogonal geodesics

![Diagram](both points same(\theta, \phi))

and characterize the other spheres by two coordinates, say \( p, q \).

The space of orthogonal geodesics through a point on a sphere is \( 4-2 = 2 \) dimensional. Then by suitable one has

\[
ds^2 = g_{pp}(p,q) \, dp^2 + 2g_{pq}(p,q) \, dp \, dq + g_{qq}(p,q) \, dq^2 + r^2(p,q) \, d\Sigma^2
\]

and by changing variables one can write

\[
ds^2 = T(\theta, r) \, d\Sigma^2 + R(\theta, r) \, dr^2 + r^2 d\Sigma^2
\]

(iv) Plug this into Einstein's equations and solve. Impose the condition that the metric is static. This too has to be defined with some care. A metric is stationary if it has a timelike Killing vector near infinity, and a stationary metric is static if in addition the timelike Killing vector is orthogonal to a family of hypersurfaces.
It is difficult to define in general what is meant by a singularity. One common means of determining whether there is a singularity is to look for infinities in geometric quantities (coordinate independent), such as $R$, $R^\mu_\nu R^\nu_\rho$, $R^\lambda_\mu R^\mu_\lambda$, etc.

In the case at hand, the metric

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2$$

is singular at $r = 2GM$ and at $r = 0$. But are these real singularities or artifacts of the metric?

In this case $R = 0$ and $R^\mu_\nu = 0$. But $R^\mu_\nu R^\nu_\lambda = 0$ and computing explicitly one finds

$$R^\mu_\nu R^\nu_\lambda = \frac{48G^2M^2}{r^6}$$

Therefore, there is no singularity at $r = 2GM$ as far as this invariant can show, but there certainly is one at $r = 0$.

In fact we will introduce coordinates that have a perfectly regular metric at $r = 2GM$.

Another way of defining singularities is by finding inextendible geodesics that terminate at finite affine parameter. Let's study geodesics.
Geodesics

\[ ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 + \sin^2 \theta d\phi^2 \]

\[ \Gamma^\mu_{\nu\lambda} = g^\mu\rho \Gamma_{\rho\nu\lambda} \quad \Gamma^\mu_{\mu\lambda} = \frac{1}{2} \left( g^\mu\lambda + g^\lambda\nu - g^\nu\lambda \right) \]

\[ \Gamma^r_{tt} = -\frac{1}{2} g^{rr, r} = \frac{GM}{r^2} \quad \Gamma^r_{rr} = \Gamma^{r}_{tr} = \frac{GM}{r^2} \left( 1 - \frac{2GM}{r} \right)^{-1} \]

\[ \Gamma^r_{rrr} = \frac{1}{2} g^{rr, r} = - \left(1 - \frac{2GM}{r}\right)^{-2} \frac{GM}{r^2} \quad \Gamma^r_{rr} = - \left(1 - \frac{2GM}{r}\right)^{-1} \frac{GM}{r^2} \]

\[ \Gamma^r_{\theta\theta} = -\frac{1}{2} g^{\theta\theta, r} = -r \quad \Gamma^r_{\phi\phi} = - r \sin^2 \theta \left(1 - \frac{2GM}{r}\right) \]

\[ \Gamma^r_{\phi\theta} = - r \sin^2 \theta \quad \Gamma^r_{\phi\phi} = - r \sin^2 \theta \sin \theta \cos \theta \]

\[ \Gamma^r_{\theta\phi} = - r^2 \sin \theta \cos \theta \quad \Gamma^r_{\phi\theta} = - r^2 \sin \theta \sin \theta \]

Geodesic Eqn:

\[ \frac{d^2 t}{d\lambda^2} + \frac{2GM}{r \left( r - 2GM \right)} \frac{dt}{d\lambda} \frac{dr}{d\lambda} = 0 \]

\[ \frac{d^2 r}{d\lambda^2} + \frac{GM}{r^2} \left( 1 - \frac{2GM}{r} \right) \left( \frac{dt}{d\lambda} \right)^2 - \frac{GM}{r \left( r - 2GM \right)} \left( \frac{dr}{d\lambda} \right)^2 - r \left( 1 - \frac{2GM}{r} \right) \left[ \left( \frac{d\theta}{d\lambda} \right)^2 + \sin^2 \theta \left( \frac{d\phi}{d\lambda} \right)^2 \right] = 0 \]

\[ \frac{d^2 \theta}{d\lambda^2} + \frac{2}{r} \frac{dr}{d\lambda} \frac{d\theta}{d\lambda} - \sin \theta \cos \theta \left( \frac{d\phi}{d\lambda} \right)^2 = 0 \]

\[ \frac{d^2 \phi}{d\lambda^2} + \frac{2}{r} \frac{dr}{d\lambda} \frac{d\phi}{d\lambda} + 2 \cot \theta \frac{d\theta}{d\lambda} \frac{d\phi}{d\lambda} = 0 \]
To solve these, use constants of the motion (1st integrals). We have four Killing vectors, three from $so(3)$ symmetry and one timelike Killing vector. For each
\[ K_\mu \frac{dx^\mu}{d\lambda} = \text{constant} \]
along the geodesic. Moreover, for massive particles, we can take $\lambda = \tau$ so that
\[ \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} g_{\mu\nu} = -1 \]
timelike geodesic
and for massless particles
\[ \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} g_{\mu\nu} = 0 \]
null geodesic.

The Killing vectors associated with $so(3)$ are like angular momentum, $L$. Just as in flat space, $E = 0$ must imply motion in a plane orthogonal to $L$ and with the magnitude of $L$. So we can fix the plane of motion choosing
\[ \theta = \frac{\pi}{2} \]

The magnitude of $L$ corresponds to the Killing vector $\partial_\theta$
\[ (\partial_\theta)^\mu = (0, 0, 0, 1) \]
In addition, the timelike Killing vector is
\[ (\partial_\tau)^\mu = (1, 0, 0, 0) \]
The conserved quantities are
\[ E = - g_{\mu\nu} (\partial_\tau)^\mu \frac{dx^\nu}{d\tau} = (1 - 2mc^2) \frac{dt}{d\lambda} \]
and
\[ L = (\partial_\theta)^\mu \frac{dx^\mu}{d\lambda} g_{\theta\mu} = r^2 \sin^2 \theta \frac{dd}{d\lambda} = r^2 \frac{dd}{d\lambda} \quad (\sin \theta = \frac{2}{3}). \]
The constants are named $E$ and $L$, respectively, but these are just labels. We can discuss energy and angular momentum later.
For time-like geodesics we have \( U^\mu U_\mu = -1 \):
\[ -(1 - \frac{2GM}{r})(dt/d\xi)^2 + (1 - \frac{2GM}{r})^{-1}(dr/d\xi)^2 + r^2(d\phi/d\xi)^2 = -1 \]
or multiply by \((1 - \frac{2GM}{r})\) and using \(E = L\)
\[ -E^2 + (dr/d\xi)^2 + (1 - \frac{2GM}{r})(1 + \frac{L^2}{r^2}) = 0 \]

This is like a particle in a central potential
\[ \frac{1}{2} (dr/d\xi)^2 + V(r) = E \]

with \(V(r) = \frac{1}{2}(1 - \frac{2GM}{r})(1 + \frac{L^2}{r^2})\) \(E = \frac{1}{2} E^2\) (I)

The null geodesic is similar, but for LHS \(-1\) is replaced by \(0\):
\[ \frac{1}{2} (dr/d\xi)^2 + V_n(r) = E \]
\[ V_n(r) = \frac{1}{2}(1 - \frac{2GM}{r})\frac{L^2}{r^2} \]

\(E = \frac{1}{2} E^2\) (II)

(or, together)
\[ V(r) = \frac{1}{2}(1 - \frac{2GM}{r})(\kappa + \frac{L^2}{r^2}) \]
\(\kappa = \frac{1}{2} E^2 \) null-like

\( \kappa = \frac{1}{2} r^2 \) time-like

Expanding (II):
\[ V(r) = \frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GM L^2}{r^3} \]

The curve Newtonian solid curve new radially non-relativistic (Newtonian) Schwarzschild
and extrema \( \frac{dv}{dr} = 0 = GMr^2 - L^2 r + 3GM^2 \)

or \( R_\pm = \frac{1}{2GM} \left[ L^2 \pm \sqrt{L^4 - 12(GM)^2} \right] \)

For \( 12GM^2 > L^2 \) \( \rightarrow \) no extrema of \( V \)

For \( L^2 > 12GM^2 \) \( R_+ \) a minimum, \( R_- \) a maximum as in figure.
\( \Rightarrow \) stable circular orbit at \( r = R_+ \) (unstable at \( r = R_- \)).
Now put for \( L^2 > 12GM^2 \) \( R_+ = \frac{L^2}{2GM} \) be Newtonian formula.

Now, as we vary \( L^2 \), \( R_+ \) is smallest at \( L^2 = 12(GM)^2 \), where
\( R_+ = \frac{1}{2GM} \left( \frac{L^2}{2GM} \right) = 6GM \) so

\( R_+ > 6GM \)

Here is a smallest stable circular orbit \( \delta \)

(Similarly, unstable circular orbits are restricted to \( R_- < 6GM \))

-same calculation—and on low end take \( L = 0 \)
\( R_- \rightarrow \frac{1}{2GM} \left[ L^2 - L^2 \left( 1 - \frac{1}{2} \frac{12(GM)^2}{L^2} + \ldots \right) \right] = 3GM \)
so \( 3GM < R_- < 6GM \).

(Note that this calculation also gives)
(Note: at this point comparison of \( c_0 \) (two small perturbations about circular orbit, \( d^2u = \frac{d^2V}{dr^2} \)) and \( \omega - \phi \) gives precession of perihelion \( \rightarrow \) mercury \( \rightarrow \) classical test. This must have been covered?).

Null geodesics:

\[
V_n(r) = \frac{L^2}{2r^2} - \frac{GMm^2}{r^3}
\]

\[
\left( \frac{\partial V}{\partial r} = 0 \rightarrow L^2r - 3GMm^2 \right)
\]

Now recall \( E \times L \) are (in arbitrary units) the energy and angular moment of the particle (photon?), and the energy necessary for the particle to go over the potential barrier is the height of the barrier:

\[
\frac{1}{2}E^2 = V_n(3GM) = \frac{L^2}{27(6M)^3}(\frac{1}{2}3GM - GM) = \frac{L^2}{54G^2M^2}
\]

\[
\Rightarrow \quad \frac{L}{E} = 3\sqrt{3}GM
\]

But \( L/E \) has a simple interpretation. In the asymptotically flat region \( (r \gg GM) \) it corresponds to the impact parameter

\[
b = \frac{L}{E} \quad \rightarrow \quad L = bE \quad \text{and} \quad E = p \quad \text{(crossless)}
\]

For \( b < 3\sqrt{3}GM \) the photon is captured

For \( b > 3\sqrt{3}GM \) it is scattered

Capture cross section

\[
\sigma_c = \pi b_{\text{crit}}^2 = \frac{3\pi}{27}G^2M^2
\]
Red-Shift

Similar to what we did before:

\[ u \cdot u = -1 \Rightarrow u^0 = \sqrt{-g^{tt}} = \frac{1}{\sqrt{1 - \frac{2GM}{r}}} \]

Now, if \( \frac{dx^\mu}{d\lambda} \) are null geodesics, for the photon, then the observer measures

\( \omega = -u \cdot k = \pm \sqrt{1 - \frac{2GM}{r}} \frac{dt}{d\lambda} \)

and

\[ \frac{dt}{d\lambda} = \frac{E}{\sqrt{1 - \frac{2GM}{r}}} \Rightarrow \omega = \frac{E}{\sqrt{1 - \frac{2GM}{r}}} \]

Since \( E \) is constant, we have

\[ \omega_1 \sqrt{1 - \frac{2GM}{r_1}} = \omega_2 \sqrt{1 - \frac{2GM}{r_2}} \]

or

\[ \frac{\omega_2}{\omega_1} = \sqrt{\frac{1 - \frac{2GM}{r_1}}{1 - \frac{2GM}{r_2}}} \]

Gravitational redshift.

For weak fields,

\[ \frac{\omega_2}{\omega_1} \approx 1 - \frac{GM}{r_1} + \frac{GM}{r_2} = 1 + \phi_1 - \phi_2 = 1 - \Delta \phi \]

which is the formula obtained first qualitatively (see Schwarzschild) on general grounds (principle of equivalence) for weak fields.
Kruskal Coordinates and extension

Coordinate singularity vs real singularity: for models first (warm-up):

\[ ds^2 = -\frac{1}{t^4} dt^2 + dx^2 \]

defined for \( x \in (-\infty, \infty) \) and \( t \in (0, \infty) \). Seems singular but defining \( t' = \frac{t}{e} \), we have

\[ ds^2 = -dt'^2 + dx^2 \]

\( \Rightarrow \) original spacetime is a portion of Minkowski space with \( t' > 0 \).

Note that the original spacetime is not geodesically complete: geodesics approaching \( t = \infty \) take finite affine parameter to get there (even though approaching \( t = 0 \) take infinite affine parameter).

Check:

\[ R_{\mu \nu} = \frac{1}{2} g_{\mu \nu} \left( \frac{2}{t^5} \right) \]

\[ R_{\mu \nu} = -\frac{2}{t^6} \]

\[ \frac{d^2 t}{d \lambda^2} + \left( -\frac{2}{t^2} \right) \left( \frac{dt}{d \lambda} \right)^2 = 0 \]

\[ \frac{dx}{d \lambda} = 0 \Rightarrow \frac{dx}{dt} = \frac{d x}{d \lambda} = 0 \]

\[ \lambda + \mu = \frac{dt}{d \lambda} \rightarrow \frac{d \lambda}{d \mu} = \frac{1}{t^2} \int d\lambda = \frac{2 \ln \lambda}{t^2} = \frac{1}{2 \ln t} \]

\[ \sigma x = \frac{dt}{d \sigma} = \frac{1}{-2 \ln t} \int d\lambda \ln \lambda = -\frac{1}{2} \int dx \]

\[ -6 \ln \left( t + \frac{1}{2} \right) \]

\[ R_{\mu \nu} = \frac{1}{2} g_{\mu \nu} \left( \frac{2}{t^5} \right) \]

\[ \frac{d^2 t}{d \lambda^2} + \left( -\frac{2}{t^2} \right) \left( \frac{dt}{d \lambda} \right)^2 = -1 \]

\[ \frac{d t}{d \lambda} = \frac{t^2 \sqrt{1 + \nu^2}}{d \lambda} \]

\[ \int \frac{d t}{t^2} = \sqrt{1 + \nu^2} \int d \lambda = \frac{1}{t} \ln t \equiv -\frac{1}{2} \ln \left( t^2 \right) \]

So \( t \to \infty \) as \( \zeta \to \frac{1}{2} \ln (1 + \nu^2) \). However \( t \to 0 \) as \( \zeta \to \infty \).
2nd example: Rindler spacetime

\[ ds^2 = -x^2 dt^2 + dx^2 \]

\( t \in (-\infty, \infty) \quad x \in (0, \infty) \)

Singularity at \( t = 0 \)?

Geodesics:

\[
\begin{pmatrix}
\Gamma^x_{tt} &=& -\frac{1}{2} g_{tt} x &=& 0
\end{pmatrix}
\begin{pmatrix}
\Gamma^x_{tx} &=& \frac{1}{2} g_{tx} x &=& 0
\end{pmatrix}
\begin{pmatrix}
\Gamma^t_{xx} &=& -\frac{1}{2} x^2 x &=& 0
\end{pmatrix}
\begin{pmatrix}
\Gamma^t_{tx} &=& -\frac{1}{2} x^2 x &=& 0
\end{pmatrix}
\]

\[ \gamma_c = \alpha t \quad \Rightarrow \quad \gamma_{tt} \frac{dt}{d\tau} = \dot{x} = \alpha \]

\[ \dot{t} = \frac{x}{x_t}, \quad \dot{x} = \frac{x_t}{x} \]

\[ -1 = -x^2 \left( \frac{\dot{x}}{x} \right)^2 + \left( \frac{dt}{d\tau} \right)^2 = -\frac{v^2}{x^2} + \left( \frac{dt}{d\tau} \right)^2 \]

\[ \frac{dx}{d\tau} = \sqrt{\frac{v^2}{x^2} - 1} \quad \int \frac{x \, dx}{\sqrt{v^2 - x^2}} = \int d\tau \]

\[ \left( x^2 + \dot{x}^2 \right) = x^2 \quad \Rightarrow x^2 + \dot{x}^2 = 2x \dot{x} \quad \int \frac{ds}{\sqrt{1 - \frac{x^2}{v^2}}} = \frac{1}{\sqrt{1 - \frac{x^2}{v^2}}} = \frac{1}{\sqrt{1 - \frac{x^2}{v^2}}} \]

\[ c = \sqrt{\frac{v^2}{x^2} - 1} \quad x^2 = \frac{v^2}{c^2 - 1} \quad \frac{dt}{d\tau} = -\frac{v}{c^2 - 1} \]

\[ \frac{dx}{d\tau} = \frac{\dot{x}}{x} \quad \frac{1}{x^2} + \frac{1}{x_t^2} = \frac{2v^2}{c^2 - 1} \quad \Rightarrow t = \frac{1}{2} \ln \frac{c}{x} \quad \Rightarrow x \to 0 \quad \text{near } t \to \infty \]

Geodesically incomplete. How about curves? 

\[ R^t_{xtx} = \partial_x \Gamma^t_{xx} - \partial_x \Gamma^t_{tx} + \Gamma^t_{tx} \Gamma^x_{xx} - \Gamma^t_{tx} \Gamma^x_{tx} \]

\[ = 0 \quad \partial_x \frac{1}{x} + 0 - \frac{1}{x^2} = 0 \]

So this is a portion of Minkowski space, again.

Q: How to find coordinates that are non-singular starting from this, without the fact that this is Minkowski?

A: Use a family of geodesics that head towards the singularity, with affine parameter as one coordinate. Must avoid crossing of geodesics because this would give new coordinate singularities. In 2-DM we can take null ingoing and outgoing geodesics (they never cross, because it null geodesics have same tangent they agree everywhere)
null geodesics:

\[ 0 = -x^2 \left( \frac{dt}{d\lambda} \right)^2 + \left( \frac{dx}{d\lambda} \right)^2 \]

\[ \Rightarrow \quad x \frac{dt}{d\lambda} = \pm \frac{dx}{d\lambda} \]

\[ \Rightarrow \quad \pm \frac{dx}{x} = dt \]

\[ \Rightarrow \quad t = \pm \ln x + C \text{ or } t = \mp \ln x + C \]

Define

\[ U = t - \ln x \quad \Rightarrow \quad t = \frac{1}{2} (U + V) \]

\[ V = t + \ln x \quad \Rightarrow \quad x = e^{\frac{1}{2} (V - U)} \]

So geodesics are \( U = c \text{ for } t \) or \( V = \text{constant} \). Then

\[ ds^2 = -x^2 dt^2 + dx^2 = -e^{(V-U)} \frac{1}{2} (dv+dv)^2 + e^{(V-U)} \frac{1}{4} (dv-dv)^2 \]

or

\[ ds^2 = -e^{V-U} dv^2 \]

We want to analyze the singularity at \( x = 0 \). Can't do that yet since \( U, V \in (-\infty, \infty) \) still has \( x > 0 \). But now we can extend the space beyond \( x = 0 \), i.e., beyond \( U, V \) finite by introducing new coordinates \( U(v) \) and \( V(u) \). Calculate affine parameter along null geodesics. Since

\[ \frac{dt}{d\lambda} = -\frac{V}{x} = \text{const} \quad \Rightarrow \quad dt = -\frac{V}{x} \frac{dx}{d\lambda} \]

\[ \Rightarrow \quad \lambda = \frac{1}{2} e \int e^{V-U} dv = A + \frac{e^{V-U}}{2e} \quad (U = \text{const, } t) \]

Along outgoing null geodesics \( \lambda_{out} = e^v \) is an affine parameter

while \( \lambda_{in} = -e^{-v} \)

So we \( U = -e^{-v} \quad V = e^v \quad ds^2 = -dv^2 \)
Now $ds^2 = -dUdV$ when $U < 0, V > 0$.

But there is no obstruction to extending $(t, \rho)$ to $(-\rho, \rho)$, and we get Minkowski space above.

$$T = \frac{1}{2} (U + V) \quad \Rightarrow \quad U = T - x$$
$$x = \frac{1}{2} (V - U) \quad \Rightarrow \quad V = T + x$$

$$ds^2 = -dT^2 + dx^2$$

The original coordinates are given, in terms of these, by

$$\tau = \frac{1}{2} (U + V) = \frac{1}{2} (-\ln(-U) + \ln V)$$
$$= \frac{1}{2} (-\ln(x-t) + \ln(x+t))$$
$$= \frac{1}{2} \ln \frac{x+t}{x-t} = \tanh^{-1} \left( \frac{t}{x} \right)$$

$$x = e^{\frac{1}{2}(V-U)} = \sqrt{-VU} = \sqrt{x^2 - T^2}$$

The original space is wedge II in Minkowski space here $(x > 1)$.
Now do the same for Schwarzschild. We can ignore angular coordinate for most of the discussion. Consider

\[ ds^2 = -(1 - \frac{2GM}{r}) dt^2 + (1 - \frac{2GM}{r})^{-1} dr^2 \]

Null geodesics:

\[ -(1 - \frac{2GM}{r}) \left( \frac{dt}{dr} \right)^2 + (1 - \frac{2GM}{r})^{-1} \left( \frac{dr}{dr} \right)^2 = 0 \]

\[ \Rightarrow \left( \frac{dt}{dr} \right)^2 = (1 - \frac{2GM}{r})^{-2} \]

\[ t = \pm r_* + \text{constant} \]

\[ r_* \text{ is the "Regge-Wheeler tortoise coordinate" given by} \]

\[ r_* = \int \frac{dr}{1 - \frac{2GM}{r}} = \int dr \left[ \frac{r - 2GM + 2GM}{r - 2GM} \right] = r + 2GM \ln \left( \frac{r}{2GM} - 1 \right) \]

They define null coordinates

\[ U = t - r_* \]

\[ V = t + r_* \]

Calculate metric:

\[ du = dt - dr_* = dt - (1 - \frac{2GM}{r}) dr \]

\[ dv = dt + (1 - \frac{2GM}{r})^{-1} dr \]

\[ du dv = dt^2 - (1 - \frac{2GM}{r})^{-2} dr^2 \]

\[ \Rightarrow \quad ds^2 = -(1 - \frac{2GM}{r}) du dv \]

with \( r \) understood as \( r = r(U, V) \),

\[ \text{with } V - U = 2r_* = 2r + 4GM \ln \left( \frac{r}{2GM} - 1 \right) \]

we have

\[ e^{\frac{V - U}{2}} = e^{\frac{r}{2GM}} \left( \frac{r}{2GM} - 1 \right) = e^{\frac{r}{2GM}} (1 - \frac{2GM}{r}) \frac{r}{2GM} \]

\[ \Rightarrow \quad ds^2 = -2GM e^{\frac{r}{2GM}} e^{\frac{V - U}{2}} du dv \]
This is useful because the factor \( \frac{2\varepsilon M}{r} e^{-\varepsilon r/2M} \) is not singular as \( r \to 2\varepsilon M \).

Now, as in Rindler case, we introduce

\[
U = - e^{-\varepsilon r/4M} \\
V = e^{\varepsilon r/4M}
\]

\[
\Rightarrow ds^2 = dU = \frac{1}{4M} e^{-\varepsilon r/4M} \, dU \quad dV = \frac{1}{4M} e^{\varepsilon r/4M} \, dV
\]

and

\[
ds^2 = -\frac{32(GM)^3}{r} e^{-r/2GM} \, dU \, dV
\]

While this is defined for \( V > 0 \) and \( U < 0 \), we can now extend to \((\varepsilon, \infty)\) and define \( \Phi \) as before

\[
\Theta = \frac{1}{2}(U + V) \\
X = \frac{1}{2}(V - U)
\]

The full metric is now

\[
ds^2 = \frac{32(GM)^3}{r} e^{-r/2GM} (-d\Theta^2 + dX^2) + r^2 (d\Theta^2 + \sin^2 \Theta \, d\Omega^2)
\]

In terms of original coordinates:

\[
X^2 - \Theta^2 = -UV = e^{\frac{r}{4GM}} = e^{\frac{r}{4GM}} \left( \frac{r}{2GM} - 1 \right) \tag{14}
\]

\[
\tanh \frac{1}{X} = \frac{1}{2} \ln \left( \frac{2GM}{X} \right) = \frac{1}{2} \ln \frac{V}{-U} = \frac{1}{2} \ln e^{\frac{r}{4GM}} = \frac{r}{4GM}
\]

which would have been hard to guess. Eq \((14)\) also gives

\[
r = r(X, T) \text{ for the metric. Note that } r > 0 \text{ in } \Theta \text{ gives the allowed region.}
\]

\[
X, T: \quad X^2 - \Theta^2 > -1
\]
Keep in mind each point is a $S^2$ with radius $r$.

Causal Structure; null geodesics are 45° lines.

• Singularities at $r=0$ are spacelike. Two of them:
  - Future of region II
  - Past of region III

• NOT a timelike line at origin, as suggested by original coordinates.

• Region I corresponds to original $r>2M$, exterior gravitational field of body, spherical body. Radially falling observer that crosses $x=T$ can never escape back to region I AND will eventually hit singularity ergo “black hole”.

• Region III has future-reversed properties of I ⇒ “white hole”.

• Region IV has identical properties to I, asymptotically flat.

To see what’s going on, consider hypersurfaces of $T=constant$, restore one angular variable ($\theta$):

\[
\frac{r}{\sqrt{2\cos(\theta)}} = x^2 + t^2
\]

as $x \to \infty$, $\theta \to \infty$ or $r \to 0$.

$r = \infty$ is a minimum and $t \to 0$.

For $T=1$:

$r_{\text{min}} = 0$
There is another space, on the other side of the black hole. Can we communicate with our brothers there? No, as is clear from causality diagram. What happens in this picture is that as an observer may go from 1 to UV.

The radius of the horizon is shrinking and it necessarily pinches off before the observer makes it to the other side.

**Penrose Diagram**

Recall

\[ ds^2 = -\frac{32 (cm)^2}{r} e^{-\frac{r}{2cm}} dUdV + r^2 d\Omega^2 \]

Now let

\[
\theta = \arctan \left( \frac{U}{V} \right) \quad \hat{\nu} = \arctan \left( \frac{V}{U} \right)
\]

so \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) \( \hat{\nu} \in (-\frac{\pi}{2}, \frac{\pi}{2}) \)

(\[\text{It is quite irrelevant what \( k \) and \( h \) is in \( \theta, \hat{\nu} \), just write \( k, h \) at \( \pm \infty \) respectively.} \]

Also, since \(-UV = e^{\frac{r}{2cm}} (cm - 1) > -1 \Rightarrow UV < 1\)

\[ \Rightarrow \tan \theta \tan \hat{\nu} < 1 \Rightarrow \cos (\theta + \hat{\nu}) < 0 \Rightarrow (\theta + \hat{\nu}) < \frac{\pi}{2} \]

Also \( \tau = 0 \) if \( UV = 1 \) \( \Upsilon + \hat{\Upsilon} = \mp \frac{\pi}{2} \). Now let \( q = \frac{1}{2}(\Upsilon + \hat{\Upsilon}) \) \( \hat{q} = \frac{1}{2}(\Upsilon - \hat{\Upsilon}) \)

So \( \tau = 0 \) if \( q = \pm \frac{\pi}{4} \) and \( \hat{q} \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), \( \hat{q} \in (-\pi, \pi) \)

\[ r = 0 \]

\[ t = \pm \frac{\pi}{2} \] and \( \hat{t} \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), \( t \in (-\pi, \pi) \)
Black holes in nature arise from gravitational collapse of stars. Even this is unlikely to be exactly spherically symmetric, but we can consider the idealized case. Since at some time:

The metric is spherically symmetric with some mass-energy density distributed with spherical symmetry. Ditchfield's theorem gives that outside the region of mass the metric is Schwarzschild (well, not quite, that would be true if the solution were static), but let imagine it is slow, so the metric is approximately Schwarzschild. Yet, inside the metric is not Schwarzschild and what is regular at $r=0$.

Moreover, as we let the star collapse (which will happen if the object is dense enough, a fact if the mass $GM < \frac{\gamma t^2}{x}$), once the core falls within $R=2GM$, it will continue falling irreversibly towards the singularity. A picture (coordinate diagram in riccii) of this process is and we see such a spacetime has no white hole nor a region II.
More General Black Holes

What characterizes black holes?

Recall Schwarzschild

\[ r = 2GM \]

\[ i^+ \]

\[ j^0 \]

\[ j^- \]

\[ \text{Plot Minkowski} \]

Two important ingredients:

(i) Asymptotically flat, so it looks like Minkowski on "both sides."

(ii) Has an event horizon (future) \((a + r = 2GM)\). Recall we had

\[ \text{future horizon of } \partial \]

So in Schwarzschild all observers that remain in \( I^+ \) go to \( i^+ \) at infinity, and they all share \( r = 2GM \) as a future event horizon.

\[ \text{event horizon observers} \]

So, more general black hole
Reissner-Nordstrom: Charged Black Hole.

Look for spherically symmetric and static (or pseudo-static?)

\[ ds^2 = -T(r, t) dt^2 + R(r, t) dr^2 + r^2 d\Omega^2 \]

solution to Einstein's Equations

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu} \]

with matter given by electromagnetic field.

Recall \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \)

(For contact with conventional non-relativistic notation in Minkowski space-time

\[ E_i = -\frac{\partial A^0}{\partial x^i} - \nabla^0 \phi \quad (\phi \equiv A^0) \text{ and } \nabla^0 = \partial_0 \]

with low indices,

\[ E_i = -\partial_0 A_i + \partial_i A_0 = -F_{0i} \]

Similarly \( \bar{B} = (\nabla \times \bar{A}) \) or \( B_i = \epsilon_{ijk} \partial_j A_k = \frac{1}{2} \epsilon_{ijk} F_{jk} \)

Note that \( \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (E^2 - \bar{B}^2) \) as it should.

Then, as we saw earlier,

\[ T_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial g_{\mu\nu}} \mathcal{L} \]

\[ \Rightarrow T_{\mu\nu} = \int d^4x \left[ \frac{1}{\sqrt{-g}} \frac{\partial}{\partial g_{\mu\nu}} \mathcal{L} + \frac{\partial g_{\mu\nu}}{\partial g_{\mu\nu}} \right] \]

The first term requires \( \delta (\det A) = \delta \prod A_{ij} = \det A \sum_{\text{perm}} \epsilon_{ijkl} \epsilon_{mnop} \frac{\partial \ln x}{\partial g_{ij}} x \frac{\partial x}{\partial A_{ij}} \)

or \( \delta g = g^{-1} \delta g_{\mu\nu} \) so \( \sqrt{-g} \delta g = \frac{1}{2} g^{\mu\nu} \delta g_{\mu\nu} \)
For the second term use
\[ L = -\frac{1}{4} \, F_{\mu \nu} F^{\mu \nu} g^{\rho \sigma} \]
so that there is no implicit dependence on \( g_{\nu \mu} \) in \( F \), and then
\[ \frac{\delta g_{\nu \mu}}{\delta g_{\rho \sigma}} = -g^{\rho \nu} g^{\mu \sigma} - g^{\mu \nu} g^{\rho \sigma} \, , \quad \text{so} \]
\[ \frac{\delta}{\delta g_{\rho \sigma}} \left( F_{\lambda} \right) = F^{\mu}_{\lambda} F^{\nu}_{\rho} g^{\lambda \mu} \]
\[ \text{or} \quad T^{\mu \nu} = F^{\mu}_{\lambda} F^{\nu}_{\rho} g^{\lambda \mu} - \frac{1}{4} \, g^{\mu \nu} F^{\rho \sigma} F_{\rho \sigma} \]
\[ \text{or} \quad \begin{bmatrix} T_{\nu} \vspace{1em} \end{bmatrix} = F_{\lambda} F^{\lambda}_{\nu} - \frac{1}{4} \, g_{\nu \lambda} F^{\rho \sigma} F_{\rho \sigma} \]

For spherical symmetry need radial \( E^r \) (and possibly \( B^r \)),
so in radial coordinates we have \( E_{\theta} = E_{\phi} = 0 \) and
\[ F_{tr} = -F_{rt} = f(t, r) \]

For \( B \) to be radial we need to generalize \( B_{t} = \frac{1}{2} \epsilon_{ijk} F^{ik} \):
\[ B^r = \frac{1}{2} \epsilon_{ijk} F^{jk} \] and use \( \epsilon^{\mu \nu \rho \sigma} = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} \). Or, more directly, but pedestrianly, go back to cartesian \( x, y, z \). Then
\[ \bar{E}_{\phi} = E_{ij} \frac{\partial x^j}{\partial \phi} \frac{\partial x^i}{\partial \phi} \quad \text{and} \quad \bar{E}_r = \epsilon_{ijk} \bar{B}^k \propto \epsilon_{ijk} x^k \text{ for radial} \]

But then \( F_{\phi \phi} = g(r) \epsilon_{ijk} \frac{\partial x^i}{\partial \phi} \frac{\partial x^j}{\partial \phi} x^k = \sin \theta g(r) \).

(The factor \( \epsilon_{ijk} \frac{\partial x^i}{\partial \phi} \frac{\partial x^j}{\partial \phi} x^k \) is just the determinant
\[ \begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} & x \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} & y \\ \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} & z \end{vmatrix} \]
which is the measure for the volume integral at \( r = 1 \), \( \sin \theta \).)
So we take
\[ ds^2 = -T(x, r) dt^2 + R(r) dr^2 + r^2 d\Omega^2 \]
\[ F_{tr} = f(t) \]
\[ F_{\theta \phi} = g(t) \sin \theta \]
and plug into Einstein's.

Quick calculation
\[ \Gamma^\mu_{\nu \lambda} = \frac{1}{2} (\partial^\mu \gamma_{\nu \lambda} + \partial^\nu \gamma_{\mu \lambda} - \partial^\lambda \gamma_{\mu \nu}) \]

\[ \Gamma^{t}_{tt} = \frac{1}{2} \frac{T''}{T} \quad \Gamma^{r}_{t r} = \frac{1}{2} \frac{T'}{T} \quad \Gamma^{\theta}_{t r} = \frac{1}{2} \frac{T'}{T} \]

\[ \Gamma^{r}_{rr} = \frac{1}{2} \frac{R''}{R} \quad \Gamma^{\theta}_{r r} = \frac{1}{2} \frac{R'}{R} \]

\[ \Gamma^{\phi}_{\theta \theta} = -\frac{r \sin \theta \cos \theta}{T} \quad \Gamma^{\phi}_{\theta \phi} = -\frac{r \sin \theta \sin \phi}{T} \]

\[ \Gamma^{\phi}_{\phi \phi} = \frac{1}{T} \]

\[ \Gamma^{\phi}_{\phi \theta} = -\frac{r \sin \theta \cos \theta}{T} \quad \Gamma^{\phi}_{\phi \phi} = -\frac{r \sin \theta \sin \phi}{T} \]

\[ \Gamma^{\phi}_{\phi \phi} = \frac{1}{T} \]

\[ R_{\mu \nu} = \partial_\mu \Gamma^\rho_{\nu \lambda} - \partial_\lambda \Gamma^\rho_{\mu \nu} + \Gamma^\rho_{\mu \beta} \Gamma^\beta_{\nu \lambda} - \Gamma^\rho_{\nu \beta} \Gamma^\beta_{\mu \lambda} \]

\[ R_{tt} = \frac{1}{2} \frac{T''}{R} - \frac{1}{2} \frac{T'' T^2}{R^3} + \left( \frac{1}{2} \frac{T'}{T} + \frac{1}{2} \frac{R''}{R} + \frac{1}{T} \right) \left( \frac{1}{2} \frac{T'}{T} - \frac{1}{2} \frac{R'}{R} \right) \]

\[ = \frac{1}{2} \frac{T''}{R} - \frac{1}{4} \frac{T'' T^2}{R^3} - \frac{1}{4} \frac{T'^2}{R^2} + \frac{1}{2} \frac{T'}{T} \]

\[ R_{rr} = \frac{1}{2} \frac{R''}{R} - \frac{1}{2} \frac{R'^2}{R^2} - \partial^\lambda \frac{1}{2} \left( \frac{1}{2} \frac{T'}{T} + \frac{1}{2} \frac{R'}{R} + \frac{1}{T} \right)^2 \]

\[ \left( \frac{1}{2} \frac{T'}{T} + \frac{1}{2} \frac{R'}{R} + \frac{1}{T} \right)^2 - \frac{1}{4} \frac{R''}{R^2} - \frac{1}{4} \frac{R'^2}{R^2} \]

\[ = -\frac{1}{2} \frac{T''}{T} + \frac{1}{2} \frac{T'^2}{R^2} + \frac{1}{4} \frac{T'^4}{R^4} + \frac{R''}{R^4} \]
\[ R_{tt} = R_{tr} = \frac{2\dot{r} + \dot{\theta}^2}{r^2} + \dot{\theta}^2 = 0 \]

\[ R_{\theta \theta} = \frac{\partial}{\partial \theta} \left( -\frac{r}{R} \right) + \left( \frac{1}{2} \frac{\ddot{r}}{R} + \frac{1}{2} \frac{\dot{\theta}^2}{R} \right) \left( -\frac{r}{R} \right) - 2 \left( \frac{\ddot{r}}{R} \right) \left( \frac{1}{2} \right) - \left( \frac{\dot{\theta}^2}{R} \right) \left( \frac{3}{2} \right) \]

\[ = \frac{1}{2} \frac{\ddot{r} R}{R} - \frac{1}{R} - \frac{1}{2} \frac{\dot{\theta}^2}{R} + \frac{1}{2} \frac{\dot{r}^2}{R} \]

\[ R_{\phi \phi} = \frac{\partial}{\partial \phi} \left( -\frac{r \sin^2 \theta}{R} \right) + \frac{\dot{\phi}}{R} \left( -\frac{r \sin^2 \theta}{R} \right) + \left( \frac{\cos \phi}{\sin \theta} \right) \left( -\sin \theta \cos \theta \right) \]

\[ = \sin^2 \theta \left[ \frac{\ddot{r} R}{2 R^2} - \frac{1}{R} - \frac{1}{2} \frac{\dot{\theta}^2}{R} + \frac{1}{2} \frac{\dot{r}^2}{R} \right] = \sin^2 \theta R_{\theta \theta} \]

**Ricci scalar**

\[ R = g^\mu \nabla \nu R_{\mu \nu} = \ldots \text{better use trace of } T \]

So we have \( R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = 8 \pi G T_{\mu \nu} \)

\[ \rightarrow \quad -R = 8 \pi G T \]

\[ = \quad R_{\mu \nu} = 8 \pi G T_{\mu \nu} + \frac{1}{2} g_{\mu \nu} \left( -8 \pi G T \right) = 8 \pi G \left( T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T \right) \]

**Now compute** \( T_{\mu \nu} :**

\[ T_{tt} = F_{t \lambda} F^{\lambda t} - \frac{1}{4} g_{tt} F_{\rho \sigma} F^{\rho \sigma} = \frac{1}{R} \frac{\dot{f}^2}{R^2} + \frac{1}{2} \frac{\dot{T}}{R} \left( -\frac{1}{R^2} \right) + \frac{2}{R^4} \]

\[ = \frac{1}{2} \dot{f}^2 + \frac{1}{2} \frac{\dot{T}}{R} \frac{g^2}{R^4} \]

\[ T_{rr} = F_{r \lambda} F^{\lambda r} - \frac{1}{4} g_{rr} F_{\rho \sigma} F^{\rho \sigma} = -\frac{1}{2} - \frac{1}{2} \frac{\dot{R}}{R} \left( -\frac{1}{R^2} \right) \]

\[ = -\frac{1}{2} \frac{\dot{f}^2}{R} - \frac{g^2}{R^2} \frac{R}{R^4} \]

\[ T_{\theta \theta} = F_{\theta \lambda} F^{\lambda \theta} - \frac{1}{4} g_{\theta \theta} F_{\rho \sigma} F^{\rho \sigma} = \frac{1}{R^2 \sin^2 \theta} \sin^2 \theta \left( -\frac{1}{R^2} \right) + \frac{2}{R^4} \]

\[ = \frac{1}{2} \frac{g^2}{R^2} + \frac{1}{2} \frac{\dot{f}^2}{R} \]

\[ T_{\phi \phi} = \sin^2 \theta T_{\theta \theta} \]
\[ T = g^{\mu\nu} T_{\mu\nu} = -\frac{1}{4} \left( \frac{1}{R^2} \right) \left( \frac{1}{r^2} + \frac{1}{2} \frac{T g^2}{r u} \right) + \frac{1}{R} \left( -\frac{1}{2} \frac{f^2}{r^2} - \frac{g^2}{2 r u^2} \right) \]

\[ = 0 \]

And we have

\[ 6 \theta : \begin{cases} \frac{1}{2} \frac{T^R}{R^2} - \frac{1}{4} \frac{T^R}{R^2} - \frac{1}{4} \frac{T^R}{R^2} + \frac{1}{R} \frac{T^R}{R^2} = \left( \frac{1}{2} \frac{g^2}{r^2} + \frac{1}{2} \frac{T g^2}{r u} \right) 8\pi \theta \text{ Same eqn.} \\ \frac{1}{2} \frac{r^2}{R^2} - \frac{1}{R} \frac{r^2}{R^2} + \frac{1}{4} \frac{T^R}{R^2} = \left( -\frac{1}{2} \frac{g^2}{r^2} - \frac{g^2}{2 r u^2} \right) 8\pi \theta \end{cases} \]

\[ \theta_0 : \frac{1}{2} \frac{r^2}{R^2} - \frac{1}{R} \frac{r^2}{R^2} + \frac{1}{4} \frac{T^R}{R^2} + 1 = 8\pi \theta \left( \frac{1}{2} \frac{g^2}{r^2} + \frac{1}{2} \frac{T g^2}{r u} \right) \]

Reg's 4 Unknowns: and more on Maxwell's Equations

\[ g^{\mu\nu} \nabla_{\mu} F_{\nu\kappa} = 0 \quad \text{and} \quad \nabla_{\kappa} F_{\mu\nu} = 0 \]

Recall

\[ \nabla_{\mu} F_{\nu\kappa} = \partial_{\mu} F_{\nu\kappa} - \Gamma_{\mu\rho\sigma} F_{\rho\nu} - \Gamma_{\mu\nu} F_{\sigma\kappa} \]

So, in components

\[ g^{\mu\nu} \nabla_{\mu} F_{\nu\kappa} = -\frac{1}{r} \left[ \frac{1}{2} \frac{T^R}{r^2} (\pm 1) \right] + \frac{1}{R} \left( -\frac{1}{2} \frac{r^2}{R^2} (\pm 1) - \frac{1}{2} \frac{T^R}{r^2} (\pm 1) \right) \]

\[ + \frac{1}{r^2} [\pm \frac{1}{2} \frac{T^R}{r^2} (\pm 1) + \frac{1}{2} \frac{f^2}{r^2} (\pm 1) - \frac{1}{r^2} \frac{g^2}{r^2} (\pm 1) ] \]

\[ = -\frac{1}{r} \frac{1}{R^2} + \frac{1}{2} \frac{r^2}{R^2} + \frac{1}{2} \frac{T^R}{r^2} - \frac{2}{r} \frac{T^R}{R^2} \quad \text{equ}\]

\[ g^{\mu\nu} \nabla_{\mu} F_{R} = -\frac{1}{r} \left[ 0 \right] + \frac{1}{R} \left( 0 \right) + \frac{1}{r} \left( 0 \right) = 0 \quad \text{autonomous} \]

\[ g^{\mu\nu} \nabla_{\mu} F_{\Theta} = -\frac{1}{r} \left( 0 \right) + \frac{1}{R} \left( 0 \right) + \frac{1}{r} \left( 0 \right) + \frac{1}{r^2} \left[ 0 \right] + \frac{1}{r^2} \left[ 0 \right] = 0 \]

\[ g^{\mu\nu} \nabla_{\mu} F_{\phi} = -\frac{1}{r} \left( 0 \right) + \frac{1}{R} \left( 0 \right) + \frac{1}{r^2} \left[ 0 \right] + \frac{1}{r^2} \left[ \cos \Theta \left( \Theta - \sin \Theta \sin \phi \right) \right] + \frac{1}{r^2} \left( 0 \right) = 0 \]
\[ \nabla_r F_{xy} + \nabla_\theta F_{x\theta} + \nabla_\phi F_{x\phi} = (\sin \theta g' - \frac{2}{r} \sin \phi g) + (-\frac{1}{r} (-g \sin \theta)) + (-\frac{1}{r} (-g \sin \phi)) \]

\[ = \sin \theta g' \]

So \( \nabla_r F_{xy} = 0 \Rightarrow \sin \theta g' = 0 \Rightarrow g = \text{constant}. \]

And Maxwell's equation gives:

\[ \frac{f'}{f} = \left[ \frac{1}{2} \frac{R'}{R} + \frac{1}{2} \frac{T'}{T} - \frac{2}{rR} \right] R \]

Next:

\[ \frac{T'}{T} = -\frac{2}{r} \]

\[ \frac{d}{d\tau} = -2 \frac{d}{d\tau} \quad f = \frac{k}{r^2} \]

and:

\[ \frac{R^2}{R^2} - \frac{1}{R} + 1 = 4\pi G \left( \frac{g^2 + k^2}{r^2} \right) \quad (a) \]

and:

\[ -\frac{1}{2} \frac{T''}{T} - \frac{T'}{T} = -4\pi G \left( \frac{k^2 + g^2}{r^4} \right) \]

or:

\[ T'' + \frac{2}{r} T' = 8\pi G \left( \frac{k^2 + g^2}{r^4} \right) \]

\[ \Rightarrow \frac{1}{2r}(r^2 T')' = 8\pi G \left( \frac{k^2 + g^2}{r^4} \right) \Rightarrow r^2 T' = -8\pi G \left( \frac{k^2 + g^2}{r} \right) + 8\pi GM \]

\[ T = 1 - \frac{2GM}{r} + \frac{4\pi G}{r^2}(g^2 + k^2) \]

Check (a):

\[ -\frac{T'}{TR} = T + 1 = -T' + T + 1 = \left[ \frac{8\pi G}{r^2} \left( \frac{k^2 + g^2}{r^4} \right) - \frac{2GM}{r} \right] + \frac{2GM}{r^2} \]

\[ = \frac{4\pi G}{r^2}(k^2 + g^2) \quad (c) \]
The solution is \( ds^2 = -T dt^2 + R dr^2 + \ldots \delta \delta ^2 \)

\[ T = \frac{1}{R} = 1 - \frac{2GM}{r} + \frac{4\pi G (\rho^2 + q^2)}{r^2} \]

and \( E_r = \frac{\rho}{r^2}, \quad F_{\theta \phi} = \rho \sin \theta \)

(Note that \( E_r = E_r = \frac{\rho}{r^2}, \quad B_r = B_r = \frac{\rho}{r^2} \sin \theta \))

so \( q, p \) are electric magnetic charges, and the notation \( (q, p) \) is standard for "dyons".

Singularity at \( r = 0 \). Horizons sometimes (see below) are coordinate effects.

**Event horizons?** In a static space-time (Killing vector \( \delta_t \)) asymptotically flat-like, \( \delta_t \partial_t = 0 \) choose coordinates \( (r, \theta, \phi) \) so metric looks Minkowski as \( r \to \infty \).

Hyper-surface \( r = \text{const} \) : timelike "cylinder" (topology \( S^1 \times \mathbb{R} \)) as \( r \to 0 \)

Now decrease \( r \) from infinity to some \( r_H \) where the surface becomes null \( \Rightarrow \) an event horizon

\[ \text{timelike path crossing \( r_H \), unable to escape for } r > r_H. \]

How to determine \( r_H \)? \( \partial_t \) is a 1-form normal to \( r = \text{const} \) hyper-surface, with norm

\[ g^{\mu \nu} (\partial_t, r)(\partial_t, r) = g^{rr} \]

We want this to vanish, so \( g^{rr}(r_H) = 0 \)

This method is very restrictive (to spaces that are static and with coordinates \( (t, \theta, \phi) \) as above, found).
Method applies for RN metric. So
\[ g^{rr} = 1 - \frac{2GM}{r} + \frac{4\pi\rho}{r^2} = 0 \]

There are no solutions (at most):
\[ r_+ = \frac{2GM \pm \sqrt{(2GM)^2 - 4\pi\rho (p^2 + q^2)}}{2} \]
\[ = GM \pm \sqrt{(GM)^2 - 4\pi\rho (p^2 + q^2)} \]

Cases:
\( (1) \ 4\pi(p^2 + q^2) > GM^2 \)

No solutions \( \Rightarrow \) no event horizon,

"Naked Singularity"

\[ \text{Penrose:} \]

Cosmic Censorship Conjecture (Penrose): Nature abhors a naked singularity, or

Naked singularities cannot form in gravitational collapse from generic, initially nonsingular states in an asymptotically flat spacetime obeying the dominant energy condition:

For all timelike vectors \( \ell^a, \ T^a \ell^a \ell^a > 0 \) (so for "weak energy adition")

And \( T^a_\nu \ell_\nu \) is a non-space-like vector
\( (T^a_\nu, \ T^a_\nu \ell^a \ell^b \leq 0) \) (Basically, \( p > 1 \)).
\[4\pi (\rho + q) > 6M^2\]

Two distinct solutions, with \(r < r^*_+\)

In \(r < r < r^*_+\), \(dr/dt\) is timelike and \(dt\) is spacelike.

But both for \(r > r^*_+\) and \(r < r^*_+\), \(dr/dt\) is spacelike and \(dt\) is timelike.

If you fall into this black hole with a spaceship full of gas, once you get to \(r = r^*_+\) you must continue falling towards lesser \(r\), but once you come out to \(r < r^*_+\) you can turn on your thrust engines, turn around before you hit \(r = 0\), go back to \(r = r^*_+\). Then you must continue, until you come out to \(r = r^*_+\). You can then decide to continue out to \(r = 0\) or turn around and "re-enter" the black hole?

The \(r = 0\) singularity is timelike (recall, for Schwarzschild, spacelike).
Conformal diagram

MTW has a step by step on how to derive this.
(iii) \[ GM^2 = 4\pi (q^2 + p^2) \]

"Extreme" RN-solution.

In this case \( r_+ = r_0 = GM \), and

\[ g^{rr} = \left(1 - \frac{GM}{r}\right)^2 \]

In fact

\[ ds^2 = -\left(1 - \frac{GM}{r}\right)^2 dt^2 + \left(1 - \frac{GM}{r}\right)^{-2} dr^2 + r^2 d\Omega^2 \]

So, there is a horizon at \( r = GM \), but \( r \) is never timelike. The singularity is at \( r = 0 \) and it is timelike.

Penrose diagram
Solutions with many extreme RN black holes: remarkably, we can produce metrics which are exact solutions of Einstein's equations in empty space with as many RN black holes as we want.

\[ ds^2 = -(1 - \frac{2\rho M}{r}) dt^2 + (1 - \frac{2\rho M}{r})^{-1} dr^2 + r^2 d\Omega^2 \]

Let \( \rho = r - GM \)

so \( r^2 = (\rho + GM)^2 = \rho^2 H^2(\rho) \)

where \( H(\rho) = 1 + \frac{GM}{\rho} \)

Also \( 1 - \frac{2\rho M}{r} = 1 - \frac{2\rho M}{\rho + GM} = \frac{\rho}{\rho + GM} = \frac{1}{1 + \frac{GM}{\rho}} = H^{-1} \)

so \( ds^2 = -H^{-2}(\rho) dt^2 + H^2(\rho) \left[ d\rho^2 + \rho^2 d\Omega^2 \right] \)

Now, the term \( \sqrt{\gamma} \) is just the metric of flat Euclidean 3-space in spherical coordinates, so we can write

\[ ds^2 = -H^{-2}(\rho) dt^2 + H^2(\rho) \left[ d\rho^2 + \rho^2 d\Omega^2 \right] \]

where \( 1 \times 1^2 = x^2 + y^2 + z^2 \)

If we take the metric (3) as an ansatz and plug it into Einstein's equation, we find it is a solution provided \( \nabla^2 H = 0 \). To be precise, we need an EM field too. To make it, we have to find the static extremal RN solution

\[ F_{rr} = \frac{\rho}{r^2} = -\partial_r A_\rho + \partial_\rho A_r \]

so \( A_r = \frac{q}{r} = \frac{\sqrt{GM}}{r} \frac{1}{\rho + GM} \) but \( 1 - H^{-1} = \frac{\rho}{\rho + GM} = \frac{1}{\sqrt{\rho + GM}} \)

Thus, the metric is

\[ ds^2 = -H^{-2}(\rho) dt^2 + H^2(\rho) \left[ d\rho^2 + \rho^2 d\Omega^2 \right] \]

with \( H = H(\rho) \).
\[ A_t = \sqrt{\frac{GM^2}{4\pi}} \frac{1 - H^{-1}}{GM} = \sqrt{4\pi a} (1 - H^{-1}) \] (24)

So one looks for solutions with (3) and (24). Then \( H \) will satisfy
\[ \nabla^2 H = 0 \quad \text{where} \quad \nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 \]

The most general solution with \( H \to 1 \) as \( |x| \to \infty \) is
\[ H = 1 + \sum_{k=1}^{N} \frac{GM_k}{|x - R_k|} \]

(Actually, I guess
\[ H = 1 + \int d^3x \frac{G\rho(x)}{|x - R|} \]
works too, provided \( \rho \) has support in a finite region, \( |R| < R \).
But \( \nabla^2 H = 4\pi G\rho(x) \to 0 \) ... works outside that region.

I think, however, these solutions are inconsistent with the electric forces between the charges! I don't know what the proper resolution of this is!

Note that the electric repulsion between holes cancels the gravitational attraction:
\[ F_{12} = -\frac{GM_1 M_2}{r^2} + \frac{q_1 q_2}{r^2} = 0 \quad \text{if} \quad q_1 q_2 = GM_1 M_2 \]

or \( q_1 = \sqrt{GM_1} \), \( q_2 = \sqrt{GM_2} \)
and we are off by \( \sqrt{2 GM} \).
\[ ds^2 = -H^{-2} dt^2 + H^2 (dx^2 + dy^2 + dz^2) \]

\[ \Gamma^i_{tt} = \frac{1}{2} g_{tt} \delta^i_t = \frac{1}{2} (-1) (-2H^{-3}) \partial_t H = -H^{-3} \partial_t H \]

\[ \Gamma^i_{tt} = -H^{-3} \partial_t H \]

\[ \Gamma^t_{tt} = -H^{-3} \partial_t H \]

\[ \Gamma^{ij}_{jk} = \frac{1}{2} \left( \left( H^2 \delta_{ij} \right)_{jk} + \left( H^2 \delta_{ij} \right)_{jk} - \left( H^2 \delta_{jk} \right)_{ij} \right) = H \left( \delta_{ij} H_{jk} + \delta_{jk} H_{ij} - \delta_{jk} H_{ij} \right) \]

\[ \Gamma^{i}_{jk} = H^{-1} \left( \delta_{ij} H_{jk} + \delta_{jk} H_{ij} - \delta_{jk} H_{ij} \right) \]

\[ R_{\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\rho} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\rho} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\rho} \]

\[ R_{tt} = \partial_t \left( -H^{-5} \partial_t H \right) + \left( -H^{-1} \partial_t H + 3H^{-2} \partial_t H \right) \left( -H^{-5} \partial_t H \right) - 2 \left( H^{-2} \partial_t H \right) \left( H^{-5} \partial_t H \right) \]

\[ = 5H^{-6} \frac{\partial^2 H}{\partial t^2} - H^{-5} \frac{\partial^2 H}{\partial t^2} - 4H^{-2} \frac{\partial^2 H}{\partial t^2} = H^{-6} \frac{\partial^2 H}{\partial t^2} - H^{-5} \frac{\partial^2 H}{\partial t^2} \]

\[ R^{ij} = \partial_k \left[ H^{-1} \left( \delta_{ij} H_{jk} + \delta_{jk} H_{ij} - \delta_{jk} H_{ij} \right) \right] - \frac{1}{2} \left[ 2H^{-1} \partial_t H \right] \]

\[ + \left[ 2H^{-1} \partial_t H \right] \left[ H^{-1} \left( \delta_{ik} H_{jk} + \delta_{jk} H_{ij} - \delta_{jk} H_{ij} \right) \right] \]

\[ - \left( H^{-1} \partial_t H \right) \left( H^{-1} \partial_t H \right) - H^{-2} \left( \delta_{ik} H_{jk} + \delta_{jk} H_{ij} - \delta_{jk} H_{ij} \right) \]

\[ \left( \delta_{ik} H_{jk} + \delta_{jk} H_{ij} - \delta_{jk} H_{ij} \right) \]

\[ = -H^{-2} \left( \delta_{ik} H_{jk} - \delta_{jk} H_{ik} \right) + H^{-1} \left( \partial_t H_{ij} - \partial_t H_{ij} \right) + 2H^{-2} \left( \partial_t H_{ij} - \partial_t H_{ij} \right) \]

\[ + 2H^{-2} \left( \partial_t H_{ij} - \partial_t H_{ij} \right) - H^{-2} \left( \partial_t H_{ij} - \partial_t H_{ij} \right) \]

\[ + 2 \left( \partial_t H_{ij} - \partial_t H_{ij} \right) \]

\[ = -\delta_{ij} \frac{\partial^2 H}{\partial t^2} - H^{-2} \frac{\partial^2 H}{\partial t^2} - 2H^{-2} \frac{\partial^2 H}{\partial t^2} \]

\[ \sum_{i<\mu<\nu} \Gamma^i_{\mu\nu} = F_{t\bar{t}} = -\partial_t A_t = \sqrt{\frac{\mu_0 c}{\lambda}} \left( -H^{-2} H_{ij} \right) \]

\[ F_{t\bar{t}} = \partial_t A_t \]

\[ T_{\mu\nu} = F_{\mu\alpha} F^{\alpha\nu} - \frac{1}{2} g_{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma} \]

\[ T_{tt} = F_{\bar{t}t} F^{\bar{t}t} - \frac{1}{4} g_{tt} F_{\lambda\sigma} F^{\lambda\sigma} = H^{-2} \left( -H^{-2} H_{ij} \right) \left( -H^{-2} H_{ij} \right) \]

\[ = \frac{1}{806} H^{-6} H_{ij} \]

\[ R_{tt} = \gamma_{tt} T_{tt} \Rightarrow H^{-6} H_{ij}^2 - H^{-5} \frac{\partial^2 H}{\partial t^2} = \gamma_{tt} c \left( \delta_{ij} H_{ij}^2 \right) \]

\[ (\text{contr} j_{\mu} T_{\mu\mu} g^{\mu\nu} = 0). \]

\[ \Rightarrow \frac{\partial^2 H}{\partial t^2} = 0 \]
Kerr Metric: Rotating Black Holes

No hair "theorem": all stationary, asymptotically flat solutions to Einstein's + Maxwell's are fully characterized by $M$, $Q$, ($a \neq 0$) and $J=am$

$$ds^2 = -(1 - \frac{2GMr}{\rho^2})dt^2 - \frac{2GMr \sin^2 \theta}{\rho^2} (dt d\phi + d\phi dt)$$

$$+ \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} \left[(l^2 + a^2) - a^2 \Delta \sin^2 \theta \right] d\phi^2$$

where $\Delta (r) = r^2 - 2GMr + a^2$

$$\rho^2 (r, \theta) = r^2 + a^2 \cos^2 \theta$$

Note: Include charge by changing $2GMr \rightarrow 2GMr - 4\pi \epsilon (Q^2 + \rho^2)$.

And $E_x = 2M - \omega \cdot A_0 \text{ w.r.t.}$

$$A_t = \frac{Qr - Pa \cos \theta}{\rho^2} \quad A_\phi = \frac{-Qa r \sin \theta + P(l^2 + a^2) \cos \theta}{\rho^2}$$

(Kerr-Newman)

The novel feature is $J=ma \neq 0$, so let's simply set $a=1=0$ and study Kerr's solution.

Note that for $M=0$ we have flat space but in weird coordinates (called Boyer-Lindquist coordinates): 

$$ds^2 = -dt^2 + \frac{r^2 + a^2 \sin^2 \theta}{r^2 + a^2} dr^2 + (l^2 + a^2 \sin^2 \theta) d\theta^2 + (l^2 + a^2) \sin^2 \theta d\phi^2$$

ellipsoidal coordinates

$$x = \sqrt{r^2 + a^2} \sin \theta \cos \phi$$

$$y = \sqrt{r^2 + a^2} \sin \theta \sin \phi$$

$$z = r \cos \theta$$
Killing vectors: $\theta^t$ and $\theta^\phi$

Also a killing tensor $X_{\mu\nu} = R_{\mu\nu}^2 (\delta_{\mu\nu} + \omega_{\mu\nu}) + \Gamma^2 g_{\mu\nu}$

with $\omega^\mu = \frac{1}{\Lambda} (r^{1+\alpha}, \Delta, \omega_\alpha)$ and $\omega^\mu = \frac{1}{2\Lambda} (r^{1+\alpha}, -\Delta, \omega_\alpha)$

(key have $\rho^2 = n^2 = 0 \quad e \cdot n = -1$).

So geodesics are easy to find: three constants (not just per unit mass).

\[ \frac{d\xi}{m} = - (2\rho^2 m^2 + \rho^4) g_{\mu\nu} \frac{d\xi}{m} = \frac{1}{2\lambda} \frac{d\xi}{m} \]

plus \[ \frac{d\xi}{m} \quad d\xi = -1 \text{ or } 0 \quad \text{for timelike or null} \]

Note that

\[ \dot{\xi} = (1 - 2GM) \frac{dt}{\rho^2} + \frac{2GMsin^2 \theta}{\rho^2} \frac{d\phi}{d\lambda} \]

\[ \dot{\lambda} = g_{\xi\xi} \frac{dt}{d\lambda} + g_{\xi\phi} \frac{d\phi}{d\lambda} \]

and \[ \dot{\phi} = \frac{g_{t \phi}}{\dot{\xi}} \frac{dt}{d\lambda} + \frac{g_{\phi \phi}}{\dot{\xi}} \frac{d\phi}{d\lambda} \]

so if $\omega = d\phi/dt$ we have

\[ g_{\xi \xi} + c_\omega g_{\phi \phi} + \frac{1}{\epsilon} (g_{t \phi} + g_{\phi \phi} \omega) = 0 \]

\[ \omega = -\frac{c_\omega}{g_{\phi \phi} + \frac{1}{\epsilon} g_{t \phi}} \]

so in particular, even if $L=0$ we can have $c_\omega \neq 0$ ($\omega = -g_{t \phi}/g_{\phi \phi}$) or with $\omega = 0$ we can have $L \neq 0$. 
Horizon: $g^{rr} = 0 \iff \frac{A}{r^2} = 0 \iff p^2 > 0$ \text{Hence} $\Delta = 0 \text{ or } r^2 - 2GMr + a^2 = 0$

or $r = r_+ = GM \pm \sqrt{(GM)^2 - a^2}$ \text{ (if } GM > 1a \text{)}.

Stationary Limit Surface: by definition, this is a surface where $\partial_t$ becomes null:

$g_{\nu \lambda} (\partial_t)^\nu (\partial_t)^\lambda = 0 \iff 1 - \frac{2GM}{r^2} = 0$

or

$r^2 + a^2 \cos^2 \theta - 2GMr = 0 \iff \theta = \Delta(t) = 0$

\begin{align*}
\theta \text{ is spacelike outside the inner horizon, i.e. the \text{"ergosphere".} } \\
\text{Moreover, at } r = r_+ \quad g_{\nu \lambda} (\partial_t)^\nu (\partial_t)^\lambda = \frac{\partial_t^2 - a^2 \sin^2 \theta - 2GM}{r^2 + a^2 \cos^2 \theta} = \frac{a^2 \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \geq 0
\end{align*}

and the equality holds only at $a = 0$ where the stationary limit surface and $r = r_+$ coincide.

The ergosphere is quite peculiar. Consider for simplicity the sector $0 < \theta < \frac{\pi}{2}$.

Null lines have (with $r_0$ constant) look at tangential emission.

$0 = \frac{\partial \psi}{\partial t} dt^2 + 2\frac{\partial \psi}{\partial \theta} d\theta d\theta + g_{\theta \theta} d\theta^2$

or

$\omega = \frac{\partial \psi}{\partial t} = -\frac{\partial \psi}{\partial \theta} \sqrt{\left(\frac{\partial \psi}{\partial \theta}\right)^2 - \left(\frac{\partial \psi}{\partial \theta}\right)^2}

\text{Ergosphere: } \frac{\partial \psi}{\partial \theta} > 0 \iff \sqrt{\left(\frac{\partial \psi}{\partial \theta}\right)^2 - \left(\frac{\partial \psi}{\partial \theta}\right)^2} \neq \left|\frac{\partial \psi}{\partial \theta}\right| \text{, so both solutions } \omega \text{ have same sign.}

and at stationary limit surface one solution has $\omega = 0$
In fact \( \frac{g_{t t}}{g_{\theta \theta}} = \frac{2GMa \sin^2 \theta}{\sin^2 \theta \left[ (r^2+a^2)^2 - a^2 \Delta \sin^2 \theta \right]} = \infty \)

Her k\text{r} sign determined by \( a = J/m \).

\& photons emitted tangentially (with \( i = 0 \) and \( \theta = 0 \)) from the ergosphere move in same direction as rotation of black hole.
Null geodesics in more detail:

We had

\[ E = -g_{tt} \frac{dt}{d\lambda} - g_{t\phi} \frac{d\phi}{d\lambda} \quad L = g_{tt} \frac{dt}{d\lambda} + g_{\phi\phi} \frac{d\phi}{d\lambda} \]  

(1)

Instead of doing next general stuff, we limit ourselves to \( \theta > \frac{\pi}{2} \) trajectories. Then

\[ g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \Rightarrow \]

\[ g_{tt} \left( \frac{dt}{d\lambda} \right)^2 + 2g_{t\phi} \frac{dt}{d\lambda} \frac{d\phi}{d\lambda} + g_{\phi\phi} \left( \frac{d\phi}{d\lambda} \right)^2 + g_{rr} \left( \frac{dr}{d\lambda} \right)^2 = 0 \]

Solve (1) above for \( \frac{dt}{d\lambda} \) and \( \frac{d\phi}{d\lambda} \), write

\[ M \left( \frac{dt}{d\lambda} , \frac{d\phi}{d\lambda} \right) = \begin{pmatrix} E \\ L \end{pmatrix} \quad \text{where} \quad M = \begin{pmatrix} g_{tt} + g_{t\phi} \\ g_{\phi\phi} \end{pmatrix} \]

we need \( M^{-1} \) which is just the inverse of the matrix:

\[ M^{-1} = \frac{1}{g_{tt} g_{\phi\phi} - g_{t\phi} g_{\phi t}} \begin{pmatrix} g_{\phi\phi} & -g_{t\phi} \\ -g_{t\phi} & g_{tt} \end{pmatrix} \]

\[ \frac{1}{g_{tt} g_{\phi\phi} - g_{t\phi} g_{\phi t}} \left( \begin{pmatrix} E \\ L \end{pmatrix} \right) = \begin{pmatrix} -g_{\phi\phi} E - g_{t\phi} L \\ g_{tt} E + g_{t\phi} L \end{pmatrix} \]

So

\[ g_{tt} \left( g_{tt} E + g_{t\phi} L \right)^2 = \frac{2g_{t\phi}}{D^2} \left( g_{tt} E + g_{t\phi} L \right) \left( g_{tt} E + g_{t\phi} L \right) \]

\[ + \frac{1}{D^2} g_{\phi\phi} \left( g_{tt} E + g_{t\phi} L \right)^2 = + g_{rr} \left( \frac{dr}{d\lambda} \right)^2 = 0 \]

This is of the form

\[ \left( \frac{dr}{d\lambda} \right)^2 + V_{eff} = 0 \]
\[ \text{Compute } (r + \Theta = \frac{\pi}{2}) \quad (p^2 = r^2) \quad (\Delta = r^2 + a^2 - 2CMr = p^2 - 2CMr) \]

\[ D = g_{tt} g_{tt} - g_{ee} = - \left( 1 - \frac{2CMr}{p^2} \right) \left( \frac{r^2 + a^2}{p^2} \right) - \left( \frac{2CMr}{p^2} \right)^2 \]

\[ = - \frac{A}{p^4} (p^2 - a^2) - \left( \frac{2CMr}{p^2} \right)^2 \]

Deeper yet, since \( \omega = -\frac{g_{ee}}{g_{tt}} \quad D = g_{tt} g_{tt} - \omega^2 g_{ee} = g_{tt} (g_{tt} - \omega^2 g_{ee}) \]

\[ V_{ee} \frac{D^2 g_{ee}}{g_{ee}} = g_{tt} \left( g_{ee} E^2 + 2EL g_{tt} g_{tt} + L^2 g_{ee} \right) \]

\[ = g_{tt} \left( g_{ee} g_{ee} E^2 + EL \left( g_{tt} \frac{\Delta}{g_{tt} g_{tt} + g_{ee} g_{ee}} \right) + L^2 g_{ee} \right) \]

\[ + g_{ee} \left( g_{tt} E^2 + 2 g_{ee} g_{tt} EL + g_{tt} L^2 \right) \]

\[ = E^2 \left( g_{ee} g_{ee} - 2 g_{ee} g_{tt} + g_{ee} g_{ee} \right) \]

\[ + 2EL \left( g_{ee} g_{tt} g_{tt} - g_{tt} g_{tt} g_{tt} g_{tt} - g_{ee} + g_{ee} g_{ee} g_{tt} \right) \]

\[ + L^2 \left( g_{tt} g_{tt} - 2 g_{tt} g_{tt} + g_{ee} g_{ee} \right) \]

\[ = E^2 \left( g_{tt} g_{tt} - \omega^2 g_{ee} \right) + 2EL \left( \omega^3 g_{tt} - \omega^2 g_{ee} \right) \]

\[ + L^2 \left( g_{tt} g_{tt} - \omega^2 g_{ee} \right) \]

\[ V_{ee} = \frac{g_{ee}}{g_{tt}} \left[ -2EL \omega + L^2 \frac{g_{tt}}{g_{ee}} \right] \]

\[ A \frac{1}{p^2} \rho^2 \left[ (r^2 + a^2 - \omega^2) \right] \left[ \frac{E^2 - 2EL \omega + L^2 \frac{g_{tt}}{g_{ee}}}{p^2 (r^2 + a^2 - \omega^2)} \right] \]

\[ = \frac{1}{r^2 (r^2 + a^2 - \omega^2)} \left( \frac{g_{tt} - 2CMg_{ee}}{g_{ee}} \right) \left[ \frac{E^2 - 2EL \omega + L^2 \frac{g_{tt}}{g_{ee}}}{g_{ee}} \right] \]
Since \((\frac{\partial r}{\partial x})^2 > 0\), solutions only exist for \((E - V_+)(E - V_-) > 0\), that is both \(V_+ > E\) or both \(V_- < E\).

So study \(V_+\). As \(r \to \infty\), \(V_+ \to \pm \frac{a}{r}\) (if \(L > 0\)), but we should also consider \(L < 0\), since presumably the relative sign of \(L\) and \(a: \frac{\gamma}{M}\) matters (and we are assuming \(a > 0\)).

Clearly \(V_+ = V_-\) at \(\Delta = 0\) \(\rightarrow\) event horizon \(r = R_+\).

Therefore
\[
V_+ = V_- = \frac{2GM_{\star}aL}{(a^2 + \alpha^2)^2} = \frac{aL}{2GM_{\star}}
\]

Note that \(V_+\) has no zeroes, while \(V_-\) has a zero at
\[
2GM_{\star}a = r_0^2 \sqrt{\Delta(r_0)}
\]
\[
(2GM_{\star})^2 \gamma^2 = r_0^2 (r_0^2 + a^2 - 2GM_{\star}r_0)
\]
In principle four zeroes, but note that \(\Delta = 0\) three zeroes are at \(r_0 = 0\) while one is at \(r_0 = 2GM_{\star}\). So we suspect only one zero is in the region \(r > R_+\).

---

\[
\frac{aL}{2GM_{\star}}
\]

\[
L > 0
\]

\[
\frac{aL}{2GM_{\star}}
\]

\[
L < 0
\]

---

**Diagram:**
- **Region:** The diagram illustrates the relationship between \(V_+\) and \(V_-\) with respect to \(r\), showing the forbidden region of \(E\) and the separation of \(V_+\) and \(V_-\) based on the sign of \(L\).
Now, writing \(-g_{tt} = 1 - \frac{2GMr}{\rho^2} = \frac{A - a^2s_1}{\rho^2}\)
we have \(\Delta = \frac{a^2}{(1 + a^2)^2} - a^2\Delta\)

\[
\frac{g_{tt}}{g_{\phi\phi}} = -\frac{\Delta - a^2}{(1 + a^2)^2 - a^2\Delta}
\]

\[
\omega = \frac{2GMra}{(1 + a^2)^2 - a^2\Delta}
\]

\[
\frac{a^2 - g_{\phi\phi}}{g_{\phi\phi}} = \frac{(2GMra)^2 + (\Delta - a^2) [Cr^2 + a^2]}{[(Cr^2 + a^2)^2 - a^2\Delta]^2}
\]

\[
\text{numerator} = (2GMra)^2 + (r^2 - 2GMr) [(Cr^2 + a^2) - a^2(Cr^2 + a^2) + 2GMra^2]
\]
\[
= r^2 [(Cr^2 + a^2)^2 - 2GMra^2] - 2GMr[(Cr^2 + a^2) + 2GMra^2]
\]
\[
= (Cr^2 + a^2)r^4 - 2GMr^5
\]
\[
= r^4 (1 + a^2 - 2GMr) = r^4 \Delta
\]

so

\[
V_\pm = L \left[ \frac{2GMra \pm r^2 \sqrt{\Delta}}{(1 + a^2)^2 - a^2\Delta} \right]
\]

and we have

\[
\left(\frac{dr}{d\lambda}\right)^2 = \frac{(E - V_+)(E - V_-)}{g_{rr} (g_{tt} - \omega^2 g_{\phi\phi})}
\]

\[
\Delta = -g_{rr} g_{\phi\phi} (\omega^2 - \frac{g_{tt}}{g_{\phi\phi}})
\]
\[
= \frac{\rho^2}{\Delta} [\frac{1}{(1 + a^2)^2 - a^2\Delta}] \frac{r^4 \Delta}{[(Cr^2 + a^2)^2 - a^2\Delta]^2} = -\frac{r^4}{[(Cr^2 + a^2)^2 - a^2\Delta]^2}
\]

and

\[
\left(\frac{dr}{d\lambda}\right)^2 = \frac{(1 + a^2)^2 - a^2\Delta}{r^4} (E - V_+)(E - V_-)
\]
Renrose process

\[ p_{\text{out}} = p_{(i)}^{\text{out}} + p_{(o)}^{\text{out}} \]

\[ E^{(o)} = E^{(i)} + E^{(r)} \]

Clearly \( E^{(o)} > 0 \), but if you push \( E^{(r)} \) hard enough you can arrange \( E^{(o)} < 0 \) so \( E^{(i)} < E^{(r)} \)

\( \Rightarrow \) come out of ergosphere with more than the original total energy.

Energy comes from black hole \( \Rightarrow \) reduce bh's angular momentum

(rock must be thrown against rotation of bh). To see this lets figure out the condition that the rock \( r_i \) crosses the exact horizon \( R_h \). We must be slightly careful since \( r = R \) is a null surface.
Killing Horizons: if a Killing vector $\chi^m$ is null on a null hypersurface $\Sigma$, then we say $\Sigma$ is a Killing Horizon.

For Kerr, $\partial^\mu$ is not null on the event horizon; it is null on the SLS (stationary limit surface) by def'n.

The event horizon is null, and

$$\chi^m = \partial^0 + \rho \partial_r$$

is null for some constant $R_+$. Exercise: show $R_+ = \frac{g_{tt}}{a^2 + \rho^2}$

Calculate: $\chi^2 = 0 = g^m_n \chi_n $ $= \frac{g_{tt}^2 + 2 \rho g_{t\theta} g^\theta + \rho^2 g_{\theta \theta}^2}{g_{tt}}$

so $R_+ = - \frac{g_{tt}}{g_{tt}} + \frac{\sqrt{(g_{tt})^2 - g_{tt}}}{g_{tt}}$

Now on $r = R_+$, $\Delta = r^2 - 2GMr + a^2 = 0$ and

$$g_{tt} = -(1 - \frac{2GM}{\rho^2}) = - \frac{1}{\rho^2} (r^2 - 2GM) = \frac{1}{\rho^2} (r^2 + a^2 \cos^2 \theta - r^2 - a^2) = + \frac{1}{\rho^2} a^2 \sin^2 \theta$$

$$g_{\phi \phi} = \frac{\sin^2 \theta}{\rho^2} [ (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta] = \frac{\sin^2 \theta}{\rho^2} (r^2 + a^2)^2$$

$$g_{rt} = - 2GM a \frac{\sin^2 \theta}{\rho^2} = - \frac{a}{\rho^2} a (r^2 + a^2) \sin^2 \theta$$

so $R_+ = \frac{a}{a^2 + \rho^2} + \sqrt{\frac{a^2}{(a^2 + \rho^2)^2} - \frac{a^2}{(a^2 + \rho^2)^2}} = \frac{a}{a^2 + \rho^2}$
Exercise: show $r = R^+$ is null.

Note: if $\Sigma$ is defined through $f(x^i) = \text{constant}$, then $\Sigma$ is null $\iff \nabla f$ is null.

[Calculate: $r = R^+$ is the same as $f(\theta, \phi, r) = r$. Then

$$\frac{\partial r}{\partial \theta} g^{rr} \frac{\partial r}{\partial r} = 0 \text{ since } g^{rr} = 0 \text{ at } r = R^+.\]
To see what condition we want to impose on $p^{(i)}$ (that signals $p^{(i)}$ crosses $R_+$), look at a null curve in Minkowski space first

The surface is defined by $x - t = \text{constant}$, and $\nabla_v (x - t)$ is a null vector both normal and tangent to it.

$$\nabla_v (x - t) = \eta^{\mu} (1, 1)$$

Now, if we have a particle moving along $x^{\mu}(\lambda)$, then

$$n \cdot \frac{dx}{d\lambda} = -\frac{dt}{d\lambda} + \frac{dx}{d\lambda}$$

So, if $n \cdot \frac{dx}{d\lambda} < 0 \Rightarrow \frac{dx}{d\tau} < 0 \text{ or } \frac{dx}{dt} < 0$

while if $n \cdot \frac{dx}{d\lambda} > 0 \text{ then } \frac{dx}{d\lambda} > 0$

Going back to our problem, if $X^{\mu}$ is a null tangent to $r = R_+$ then $X^{\mu} p^{(i)} < 0 \text{ at } r = R_+$ signals motion inwards but moreover since $X$ is a Killing vector $X^{\mu} p^{(i)} = 0$ is not.

Now $\bar{X} = \bar{\partial}_t + \bar{R}_+ \bar{\partial}_r$ with $\bar{R}_+ = \frac{a}{r^2 + \bar{a}^2}$ is a null Killing vector on $r = R_+$. $\bar{R}_+$ can be interpreted as the angular velocity of the black hole at the event horizon $R_+$, so can be $a$, since it corresponds to the minimum $a$ for a massive particle at $r = R_+$.

Then the condition is $X^{\mu} p^{(i)} < 0$

$$\Rightarrow \bar{X}^{\mu} p^{(i)} = \bar{\partial}_t^{(i)} + \bar{R}_+ \bar{\partial}_r^{(i)} p^{(i)} = -E^{(i)} + \bar{R}_+ L^{(i)} < 0$$

$$\Rightarrow \frac{L^{(i)}}{E^{(i)}} < 0$$

So the angular momentum of the black hole decreases by $L^{(i)}$. 
\[ \delta M = E^{(2)} \]
\[ \delta J = L^{(2)} \]

\[ a \text{ and } \delta J < \frac{\delta M}{Q_H} \]

Conclusion: Energy is extracted from the black hole. As a result, the black hole loses mass and spin.

However, the process cannot violate the area theorem, that the area of the event horizon is non-decreasing.

The area of the event horizon is

\[ A = 4\pi (R_+^2 + a^2) \]

Calculate for \[ ds^2 = g_{ij} dx^i dx^j = ds^2_{\text{at } dt = dr = 0, r = R_+} \].

Then

\[ A = \int |\text{det} g| d\theta d\phi \]

\[ = \int \sqrt{ \rho^i \frac{\sin \Theta}{r^2} \left[ (R^2 + a^2) \Delta \sin \Theta \right] } d\theta d\phi \]

Get \( r = R_+ \), we have \( \Delta = 0 \) so

\[ A = (R_+^2 + a^2) \int \sin \Theta d\theta d\phi \].

To show that \( A \) is non-decreasing, define the "impeccable mass" through

\[ M_{\text{impe}}^2 = \frac{A}{16\pi G^2} = \frac{1}{4G^2} (R_+^2 + a^2) \]

\( \Delta = 0 \) \( \Rightarrow \) \( R_+^2 + a^2 = 2GM_+ \)

\( R_+ = GM + \sqrt{(GM)^2 - a^2} \)

\[ = \frac{1}{4G^2} 2GM \left[ GM + \sqrt{(GM)^2 - a^2} \right] \]

or

\[ M_{\text{impe}} = \frac{1}{2} \left[ \sqrt{M_+^2 + (\sqrt{M_+^2 - (J/M_+)^2})^2} \right] \quad (J = Ma) \]
Now
\[ 2 \cdot 2 M_{\text{irr}} \delta M_{\text{irr}} = 2 M \delta M + \frac{\sqrt{M^4 - (J/\ell)^2}}{\sqrt{M^4 - (J/\ell)^2}} \cdot \frac{2 M^2 \delta M - 2 J \delta J / \ell^2}{\sqrt{M^4 - (J/\ell)^2}} = \frac{2 (M \sqrt{M^4 - (J/\ell)^2} + M^2) \delta M - J \delta J / \ell^2}{\sqrt{M^4 - (J/\ell)^2}} \]

we recognize
\[ M^3 + M \sqrt{M^4 - (J/\ell)^2} = M \left( M^2 + \sqrt{M^4 - (J/\ell)^2} \right) = 2 M M_{\text{irr}}^2 \]
\[ = 2 M \frac{1}{4 \ell^2} \left( R_+^2 + a^2 \right) \]
\[ = 2 M \frac{a}{4 \ell^2} \frac{1}{R_+} \]
\[ = \frac{J}{4 \ell^2} \frac{1}{\Omega_+} \]

so
\[ \delta M_{\text{irr}} = \frac{J/\ell^2}{4 M_{\text{irr}} \sqrt{M^4 - (J/\ell)^2}} \left[ \delta M / \Omega_+ - \delta J \right] \]

so our bound that \( \delta J < \delta M / \Omega_+ \) implies \( \delta M_{\text{irr}} > 0 \)

Now \( \delta A = 16 \pi G^2 \delta M_{\text{irr}} = 8 \pi J \left( \delta M / \Omega_+ - \delta J \right) / \sqrt{M^4 - (J/\ell)^2} \)

or
\[ \delta M = \frac{\kappa}{8 \pi G} \delta A + \Omega_+ \delta J \]

where \( \kappa = \left( \frac{G^2 M^2 - J^2}{J/M} \right) \frac{R_+}{J/M} = \sqrt{G^2 M^2 - a^2} / \sqrt{R_+^2 + a^2} \)

or
\[ \kappa = \frac{\sqrt{G^2 M^2 - a^2}}{2 G M (G M + \sqrt{(G M)^2 - a^2})} \]
[Note: \( \kappa \) is the surface gravity of the Kerr metric. For a Killing horizon with Killing (null) vector \( \chi \), the surface gravity is \( \kappa^2 = -\frac{1}{2} (\nabla \chi_\nu)(\nabla^\nu \chi^\mu) \).]

Now \( \delta M = \frac{k}{8\pi G} \delta A + m \delta J \)

is just like \( dE = TdS - p dV \)

for a thermodynamic system, with the association

\[ E \leftrightarrow M \]
\[ A \leftrightarrow S \]
\[ T \leftrightarrow \frac{k}{G} \]

The ambiguity in the association of \( A \leftrightarrow T \) (where do we put the \( 8\pi G \))

is settled by Hawking's black hole evaporation.

Thermodynamics

\( T \) is constant in thermal equilibrium.

Steady state black holes have constant \( k \).

\( 1^{st} \) Law:

\[ dE = dQ + dW \]

\( 2^{nd} \) Law:

\[ \delta S > 0 \]
\[ \delta A > 0 \]

Generalized 2nd law \( \delta (S + \frac{A}{8\pi G}) > 0 \).

Note: To make sense of units, \( S \) is dimensionless (\( k_B = 1 \)) but

\[ \frac{A}{8\pi G} \]

has units of mass \( \times \) length, same as \( S \). So it really should be

\[ S \rightarrow \frac{A}{8\pi G} \]
\[ T \rightarrow \frac{k}{8\pi G} \]

(or \( k \rightarrow 4\pi \)).
Stationary axisymmetric space: general observations,

(i) General case: require \( g_{\mu \nu} = g_{\mu \nu}(r, \phi) \) (not of \( t, \phi \)),

and symmetry \( t \to -t, \phi \to -\phi \) (so \( g_{tt} = g_{\phi \phi} = 0 = g_{t \phi} = g_{\phi t} \))

\[ ds^2 = -\tilde{A} dt^2 + B d\phi^2 - 2Bw dt d\phi + C dr^2 + D d\theta^2 \]

\[ = -\tilde{A} dt^2 + B (d\phi - \omega dt)^2 + C dr^2 + D d\theta^2 \]

\[ \tilde{A} = A - Bw^2 \]

Note that

\[ g^{tt} = \frac{1}{\tilde{A}}, \quad g^{\phi \phi} = \frac{1}{B}, \quad G = \begin{pmatrix} g_{tt} & g_{t \phi} \\ g_{t \phi} & g_{\phi \phi} \end{pmatrix} \Rightarrow G^{-1} = \begin{pmatrix} \frac{1}{\tilde{A}} & -\frac{w}{A} \\ -\frac{w}{A} & -\frac{1}{B} \end{pmatrix}, \quad det(G) = \frac{g_{tt}g_{\phi \phi} - g_{t \phi}^2}{-AB} \]

For Kerr, plug into \( P_\mu = 0 \), hence results.

(ii) Killing vectors \( \partial_\phi, \partial_\theta = \) conserved quantities

\[ L = P_\phi = mg_{\mu \phi} \frac{dx^\mu}{dt} \quad \text{and} \quad E = P_\phi = mg_{\mu \phi} \frac{dx^\mu}{dt} \]

\( r \), replace \( \phi/r \to \lambda \), for massless.

More explicitly

\[ L = g_{\phi \phi} \frac{dy}{d\lambda} + g_{\phi t} \frac{dt}{d\lambda} \]

\[ E = g_{t \phi} \frac{dy}{d\lambda} + g_{t t} \frac{dt}{d\lambda} \]

\[ L = 0 : \quad \frac{d\phi}{dt} = -\frac{g_{t \phi}}{g_{\phi \phi}} = c(r, \phi) \quad \text{ANGULAR VELOCITY WITHOUT ANGULAR MOMENTUM} \]

Suppose metric is asymptotically flat (as in Kerr). Then "drop" body from \( \infty \) towards center (from \( r \to \infty \) towards \( r = 0 \)) with \( \frac{dr}{dt} = 0 \) originally (since \( c(r, \phi) = 0 \) at \( r = 0 \)).

Then \( \frac{dr}{dt} \) will change as body drops.

"Dragging of inertial frames": our test body is free falling, so locally it is moving in straight line 
interpret \( \frac{dr}{dt} \to 0 \) as moving/rotating inertial frames.

2014-03-12 13:10:52 1/6  kerr (#4)
iii) Stationary limit surface

Consider photon emitted in φ-direction (from (r, θ, ϕ))

At emission \( dθ = 0 = dr \) ⇒ \( ds^2 = 0 = g_{tt} \, dt^2 + 2 \, g_{tθ} \, t \, dθ + g_{θθ} \, dθ^2 \)

\[ \frac{dθ}{dt} = \frac{g_{tt}}{g_{θθ}} \sqrt{\left( \frac{g_{tt}}{g_{θθ}} \right)^2 \frac{g_{tt} - g_{θθ}}{g_{θθ}}} = \frac{ω}{B} \]

- While \( g_{tt}/g_{θθ} < 0 \) get \( \frac{dθ}{dt} > 0 \) \( \Rightarrow \) emitted out + in-
- \( \frac{dθ}{dt} < 0 \) \( \Rightarrow \) emitted in - out

On a \( g_{tt} = 0 \) surface \( \frac{ω}{B}(\mathbf{v}) \) \( \frac{dθ}{dt} = \frac{1}{ℓ} \) \( \mathbf{v} \) going nowhere?

Massive particles all dragged in same direction on \( g_{tt} = 0 \) surface

"Stationary limit surface" = any surface with \( g_{tt} = 0 \)

Q: Schwarzschild? (leave for student to ponder)

Inside stationary limit surface all bodies and radiation are forced to move in same direction, cannot remain fixed.
Suppose \( u^a \) is null of body, \( u^a u_\alpha = 0 \). If we take \( u^a = (u, 0, 0, 0) \)
\[ g_{\mu
u} = \frac{1}{(u^a u_\alpha)^2} < 0, \] in compatible with interior of limit surface.

But requiring wrong \( g_{\mu
u} + g_{\nu\xi} u^\xi u^\mu + g_{\mu\nu} u^\beta u^\beta = -1 \) since \( g_{\mu\nu} = \omega_{\mu\nu} \)
and the relative sign of \( u^\mu, u^\nu \) not fixed. But if \( g_{\mu\nu} = 0 \) we neglect \( g_{\mu\nu} \)
and get \( u^\mu (u^2 - 2w^2 u^\mu) = \frac{1}{\omega^2} < 0 \) which is easily satisfied.

(u) Redshift: recall for comoving observers (fixed coordinates)
\[ \frac{\lambda_{rec}}{\lambda_{em}} = \sqrt{\frac{g_{\mu\nu}(rec)}{g_{\mu\nu}(em)}} \]

For an observer at stationary limit surface, \( g_{\mu\nu}(em) \rightarrow 0 \) \( \rightarrow \lambda_{rec} \rightarrow \infty \).
This is just \( g \) with Schwarzschild

(v) Event Horizons. Again we look for null 3-surfaces.
\[ f(x^a) = 0 \] defines surface
\[ \partial f = \) gradient = normal to surface = \( n_s \)

Tangent, \( f(x^a(\lambda)) \rightarrow 0 \) \( \Rightarrow \) \( df(x^a(\lambda)) = \frac{df}{d\lambda} \partial f \rightarrow \frac{df}{d\lambda} n_s = 0 \) \( \Rightarrow \) vector \( \frac{1}{n_s} \)

In particular, if \( n_s \) is null then \( n_s^a = g^a_\mu n_\mu \) \( \equiv \) \( n_\mu (g_{\mu\nu} n_\nu = 0) \)
\( \Rightarrow \) Look for \( G_{\mu\nu} \) \( \partial f = 0 \)
Recall in spherically symmetric case we take \( f = f(r) \):
\[ g_{\mu\nu} \partial f \partial f = 0 \Rightarrow g^r (\partial f)^2 = 0 \Rightarrow g^r = 0 \]
Now with axial symmetry, \( f = f(r, \theta) \),
\[ g_{rr} (\partial f)^2 + 2 g_{r\theta} \partial f \partial \theta + g_{\theta\theta} (\partial f)^2 = 0 \]
We can still look for solutions with \( f = f(r) \) and \( g_{rr} = 0 \). Let's
look at this (and others) in Kerr metric.
Back to Kerr

(i) Singularities. From $\mathbb{R}^+ \times \mathbb{R}$ one finds $p=0$ is a singularity.

Now $p^2 = r^2 + a^2 \sin^2 \theta = 0 \Rightarrow r=0, \theta = \frac{\pi}{2}$

Recall Boyer-Lindquist coordinates in Cartesian!

$$x = \sqrt{r^2 + a^2 \sin^2 \theta} \cos \phi$$
$$y = \sqrt{r^2 + a^2 \sin^2 \theta} \sin \phi$$
$$z = r \cos \theta$$
$$\theta = \frac{\pi}{2} \Rightarrow x^2 + y^2 = a^2 \text{ circle ("equator")}$$

The singularity is not a point but $S^2$.

(ii) Event Horizon: look for $g^{rr} = 0$. Now $g^{rr} = \frac{\Delta}{\rho^2}$, so need $\Delta = 0$.

$$\Rightarrow r^2 - 2GMr + a^2 = 0 \Rightarrow r_+ = GM \pm \sqrt{(GM)^2 - a^2}$$

Note, if $|a| < GM \Rightarrow r < a < r_+$ and the singularity is behind the horizon (singularity at $r=0$, horizon $r_+$).

For $|a| > GM$, no horizon, naked singularity.

$|a| = GM - r_+$ is "extreme Kerr B.H." (It is believed, through calculation, that realistic BH's are near extremal Kerr BH's, since accretion increases $a \approx 3/M$. Limited only by accreting matter radiating away some angular momentum. Calculations give $a \approx 0.99G M$ - see text by Hobson, Ellis and Wiltshire, p. 324).

Geometry: take $r = \text{const. (w.r.t)}$ $t = \text{const.}$ 2-dim surface. Line element is

$$ds^2 = \frac{\rho^2}{\Delta} d\theta^2 + \left(\frac{2GM}{\rho^2}\right) \sin^2 \theta d\rho^2$$

Not the geometry of $S^2$ embedded in $\mathbb{R}^3$. Rather a pancake, or more technically, an axisymmetric ellipsoid (embedded in $\mathbb{R}^3$).
(iii) Stationary Limit Surface: $g_{tt} = 0$

$$1 - \frac{2GM}{\rho^2} = 0 \implies r^4 + a^2 \cos^4 \theta - 2GMr = 0$$

$$\rho^2(\theta) = GM \pm \sqrt{(GM)^2 - a^4 \cos^2 \theta}$$

(Not the same $r$ as before, exactly Minkowskian)

Geometry (a) previous:

$$ds^2 = \rho^2 \, d\theta^2 + \frac{2GMr(2GMr + 2a^2 \sin^2 \theta)}{\rho^2} \sin^4 \theta \, d\phi^2 + d\rho^2$$

with $\rho_1 = r_+(\theta)$

and $\rho_2 = r_-(\theta)$$$

Also $r(\pi) = 0 = \text{singularity}$.

'$\text{ergosphere}'$

region with $g_{tt} > 0$

not behind event horizon.

In ergosphere every mass is forced to move. As before $u^2 = 1$ with $u^\nu = (u^t, u^r, u^\theta, u^\phi)$

$$\Rightarrow u^\nu (\mathcal{L} u^\nu + 2 g_{\nu\lambda} \nabla^\lambda u^\rho + g_{\nu\rho} \nabla^\rho u^\lambda) = 0$$

where

$$\mathcal{L} = \frac{\partial}{\partial t} + u^\rho \frac{\partial}{\partial \rho}$$

$u^2$ real $\Rightarrow g_{\nu\lambda} \nabla^\lambda u^\rho + g_{\nu\rho} \nabla^\rho u^\lambda < 0 \Rightarrow \mathcal{L} \in (\Omega, \Omega)$

with

$$\Omega = - \frac{g_{\nu\lambda}}{g_{\rho\beta}} + \sqrt{ \frac{g_{\mu\lambda} g_{\nu\beta}}{g_{\rho\beta} g_{\mu\lambda}}} = \mathcal{L} + \sqrt{\frac{A}{B}}$$

Special case:

(1) $A \neq 0$ (yet $=0$) $\Rightarrow \mathcal{L} = 0$ , $\Omega = \text{const}$. This occurs at $r = r_+(\theta)$ (stat. lim. surface)

(already $A \neq 0 \Rightarrow \Omega$)

(2) $\mathcal{L}^2 = \frac{g_{tt}}{g_{\lambda\lambda}} \Rightarrow \mathcal{L} = \pm \Omega$. This occurs at $r = r_-$, so call $\Omega_\pm$.
We have \( \Omega_h = \omega(r_h, \theta) = \frac{a}{2GM_r} \) (from Kerr metric).

At the horizon, the angular velocity is limited to the one value \( \Omega_h \).
Hawking radiation: baby version

In QFT, vacuum fluctuations: vac → γ + γ → vac are virtual processes \( E^{(0)} + E^{(1)} = 0 \) so \( E^{(0)} > 0 \) and \( E^{(1)} < 0 \) and photons cannot propagate freely. But in our backgroun vicinity at horizon, imagine 2nd γ crosses horizon. Then even if \( E^{(1)} < 0 \) it must fall into singularity. The 1st photon escapes, taking energy away to asymptotically flat region.

Take a freely falling observer with 4-velocity \( U \). In his locally flat system \( \Sigma \) he is inertial, so he sees normal laws of quantum electrodynamics. He sees vacuum fluctuations. A fluctuation of energy \( E \) lasts no longer than:

\[
\Delta t \sim \frac{\hbar}{\hbar E}
\]

(This is a bit backwards. To measure energy \( E \) need at least time \( \Delta t \), \( \Delta t \cdot E \geq \hbar \)). If he is close to horizon, then he has limited time to measure and this sets the scale for every 2 photons he sees are pair created (virtual photons).

Say he starts free falling from \( r = R + \epsilon \) \( (R = 2GM) \). Then,

\[
-1 = -(1 - \frac{2GM}{r}) \left( \frac{d\epsilon}{dt} \right)^2 + (1 - \frac{2GM}{r})^{-1} \left( \frac{dr}{dt} \right)^2
\]

and

\[
\dot{E} = -3E \cdot U = (1 - \frac{2GM}{r}) \frac{dE}{dt} \quad \text{so}
\]

\[
\left( \frac{dr}{d\tau} \right)^2 = \sqrt{E^2 - (1 - \frac{2GM}{r})}
\]

Since \( \frac{dr}{d\tau} \to 0 \) at \( \delta r = R + \epsilon \), we have

\[
E^2 = 1 - \frac{2GM}{r} = 1 - \frac{2GM}{R + \epsilon} = \frac{2GM}{2GM + \epsilon} \frac{\epsilon}{2GM}
\]
How much time does one have to observe photon creation before crossing $r = R$?

$$\Delta \tau = \int_{r_{\infty}}^{R} \frac{dr}{ac} = \int_{r_{\infty}}^{R} dr \frac{1}{\sqrt{\frac{r}{c} - 1}} = \int_{r_{\infty}}^{R} \frac{dr}{\sqrt{2GM \sqrt{\frac{r}{c} - 1} - R_{\infty}}}$$

Making some good use of trial, we have the approximation:

With $\epsilon \ll 2GM$, change variables to $\xi = r - 2GM$

$$\frac{1}{2GM + \epsilon} - \frac{1}{2GM + \epsilon} = \frac{\epsilon - \xi}{(2GM + \epsilon)(2GM + \epsilon)} = \frac{\epsilon - \xi}{(2GM)^2}$$

$$\Delta \tau = \int_{\epsilon}^{0} \frac{d\xi}{\sqrt{\xi - \epsilon}} = 2\sqrt{2GM\epsilon}$$

So the energy of the photon created which escapes to $\infty$ is

$$E = \frac{\hbar}{\Delta \tau} = \frac{\hbar}{2\sqrt{2GM\epsilon}} \quad \text{as observed in his frame.}$$

Now

$$E = -\vec{p} \cdot \vec{v}$$

where $\vec{v}$ is for our falling observer. The energy $E$ of the photon as observed at $\infty$ is

$$E = -\vec{p} \cdot \hat{a}_e$$

Or

$$\begin{align*}
E &= \vec{p} \cdot \vec{U} = \sqrt{\frac{2GM}{c^2}} \vec{v} \\
&= \frac{\sqrt{2GM}}{c} \hat{e} \\
&= \frac{\sqrt{2GM}}{c} \\
&= \frac{\sqrt{2GM}}{c}
\end{align*}$$

At $r = R_{\infty}$

$$E = \frac{\sqrt{2GM}}{c}.$$
Completing the rho at $r = R + \epsilon$

\[
\frac{E}{E} = \frac{\gamma \cdot \beta_k}{\gamma \cdot \beta_U} = \frac{q \epsilon + p^t}{q \epsilon + p^U} = \frac{1}{\gamma} = \frac{1}{\gamma} - \frac{\epsilon}{\epsilon} = E = \sqrt{\frac{E}{2GM}}
\]

\[
\Rightarrow E = \sqrt{\frac{E}{2GM}} \cdot \frac{\hbar}{2\sqrt{2GM}} = \frac{\hbar}{4GM}
\]

An observer at $\infty$ sees an even photon with energy

\[
E = \frac{\hbar}{4GM}
\]

This is independent of $\gamma$ in the argument above. So we don't know exactly where the photon was emitted, but it does not matter.

A complete calculation shows the spectrum of photons is thermal with temperature $T = E/k_B$ with $E$ as above.

Recall we had before that for black hole thermodynamics

\[
T = \frac{k}{2\hbar} \quad \text{where} \quad k \text{ for Schwarzschild is} \quad k = \frac{1}{4GM}
\]

In units of $\hbar = 1$, we see that this agrees with the previous.
Causal Structure

The following definitions are for any spacetime \((M, g)\).

\((M, g)\) is \underline{time orientable} if as you vary continuously \(p \in M\) the future lightcone at \(p\) can be continuously defined.

**Example:** non-time-orientable

![Diagram of non-time-orientable](image)

We assume \((M, g)\) is time orientable from here on.

\(\text{time orientable} \iff \text{there is a continuous timelike vector field} \)

\(I^+(p)\): Chronological future of \(p\)

set of points that can be reached from \(p\) by a future directed timelike curve

![Diagram of I^+(p) and null geodesic](image)

**Notes**

- \(I^+(p)\) is open (the null geodesic cone is not in \(I^+(p)\)).
- Generally \(p \notin I^+(p)\), but \(p\) may be in \(I^+(p)\) if \(p\) is a closed timelike curve from \(p\) to \(p\).
For any set $S$ define $I^+(S) = \bigcup_{p \in S} I^+(p)$.

Particularly, $I^+(S)$ is also open and $\partial I^+(S)$ will be referred to as $\partial S$.

Define also $I^-(p) = I^+(S)$ (replace "future" by "past")
and $J^+(p) = J^+(S)$ (replace "timelike curve" by "causal curve")

$\partial S = \text{boundary of } I^+(S)$

In Minkowski spacetime $I^+(p)$ is the set of points that can be reached by future directed timelike geodesics starting at $p$. and $\partial I^+(p)$ is the set of future light cone (generated by future null geodesics).

This is locally true in any spacetime, but not necessarily globally. An artificial example

$\partial I^+(p)$

$\partial I^+(p)$ is not contained in $I^+(p)$ but cannot be reached by geodesic from $p$.

Remove point from spacetime

Still for UCM small enough, $\partial U$ still holds here.
Some fun "theorems"

\* \( I^+(s) \) cannot be timelike:

\[ \exists q \in I^+(s) \]

but \( q \not\in I^+(s) \), a contradiction

\* \( I^+(s) \) cannot be spacelike, except for the set \( S \) itself

\[ I^+(s) = \text{part of } I^+(s) \text{ not in } S \]

no point \( p \) in \( q \)'s past light cone is in \( S \)

\* Therefore \( I^+(S) \) is null, apart from \( S \) itself.

Moreover, to emphasize:

\( \forall q \in I^+(S) \) but \( q \not\in S \Rightarrow \exists \) past directed null geodesic through \( q \)

\( I^+(S) \)

\[ \text{null geodesic segment} \]

Extra information: \( I^+(S) \) is generated by null geodesic segments. We did not get this above, we just got that \( I^+(S) \) is null (up to \( S \)). The proof uses the fact that it is locally true (see previous page) and that one can show one can find a finite one with \( p \neq q \)
This does not conflict with their points out of space if one phrases it carefully; for example:

- If \( q \in J^+(p) - I^+(p) \) ⇒ any causal curve joining \( q \) is null geodesic or

- If \( q \in I^+(p) \) but \( q \notin S \) ⇒ there is a past-directed null geodesic segment through \( q \) lying on \( I^+(p) \).

So in our example

If there is none such one past directed null geodesic segment through \( q \) (lying on \( I^+(p) \) ⇒ \( q \) is the endpoint of the segments

So the structure of \( I^+(p) \) is as follows:

- It is saturated by null geodesic segments that
  - have past endpoints only on \( S \)
  - have future endpoints in the boundary (and would thus pass
    through the interior of \( I^+(p) \)) if they intersect another generator,
  - may have no endpoints
Example: Minkowski space with a horizontal line segment (in 1+1 dim) removed.

\[ \text{null, null end at } s \]

\[ \text{removal from space} \]