Physics 211B : Solution Set #2

[1] Rectangular Barrier – Consider a symmetric planar barrier consisting of a layer of Al$_x$Ga$_{1-x}$As of width 2$a$ imbedded in GaAs. The barrier height $V_0$ is simply the difference between conduction band minima $\Delta E_c$ at the $\Gamma$ point; energies are defined relative to $E_{\Gamma}^{GaAs}$. Derive the $S$-matrix for this problem. Show that

$$T(E) = \frac{1}{1 + \left[ \sinh \left( \frac{b\sqrt{2}}{\eta(1-\eta)} \right) \right]^2} \quad (\eta \leq 1)$$

and

$$T(E) = \frac{1}{1 + \left[ \sin \left( \frac{b\sqrt{2}}{\eta-1} \right) \right]^2} \quad (\eta \geq 1) ,$$

where $\eta = E/V_0$ and $b = a/\ell$ with $\ell = h/\sqrt{2m^*V_0}$. Sketch $T(E)$ versus $E/V_0$ for various values of the dimensionless thickness $b$.

Solution: Let the barrier extend from $x = 0$ to $x = d \equiv 2a$. The energy is

$$E = \frac{\hbar^2 k^2}{2m^*} = \frac{\hbar^2 q^2}{2m^*} + V_0 .$$

Thus, with $\eta = E/V_0$, and $\ell = h/\sqrt{2m^*V_0}$, the wavevectors $k$ and $q$ outside and inside the barrier region are given by $k = \ell^{-1}\sqrt{\eta}$ and $q = \ell^{-1}\sqrt{\eta-1}$, respectively.

The wavefunction in the three regions is written

$$\psi(x) = A e^{ikx} + B e^{-ikx} \quad (x \leq 0)$$

$$= C e^{iqx} + D e^{-iqx} \quad (0 \leq x \leq d)$$

$$= E e^{ikx} + F e^{-ikx} \quad (d \leq x) .$$

Matching the wavefunction and its derivative at the points $x = 0$ and $x = d$ gives four equations in the six unknowns $A$, $B$, $C$, $D$, $E$, and $F$:

$$A + B = C + D$$

$$k(A - B) = q(C - D)$$

$$C e^{iqd} + D e^{-iqd} = E e^{ikd} + F e^{-ikd}$$

$$q(C e^{iqd} + D e^{-iqd}) = k(E e^{ikd} - F e^{-ikd}) .$$

Solving the first two equations for $C$ and $D$ yields

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ q & -q \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ k & -k \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

The bottom pair says

$$\begin{pmatrix} E \\ F \end{pmatrix} = \begin{pmatrix} e^{ikd} & e^{-ikd} \\ k e^{ikd} & -k e^{ikd} \end{pmatrix}^{-1} \begin{pmatrix} e^{iqd} & e^{-iqd} \\ q e^{iqd} & -q e^{-iqd} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} .$$
Thus, the transfer matrix for this problem is
\[
M = \frac{1}{4kq} \begin{pmatrix}
  k e^{-ikd} & e^{-ikd} \\
  k e^{ikd} & e^{ikd}
\end{pmatrix}
\begin{pmatrix}
  e^{iqd} & e^{-iqd} \\
  q e^{iqd} & -q e^{-iqd}
\end{pmatrix}
\begin{pmatrix}
  q & 1 \\
  q & -1
\end{pmatrix}
\begin{pmatrix}
  1 & 1 \\
  k & -k
\end{pmatrix}
\]
\[
= \frac{1}{4kq} \begin{pmatrix}
  (k + q)^2 e^{-i(k-q)d} - (k - q)^2 e^{-i(k+q)d} & -2i(k^2 - q^2) e^{-ikd} \sin(qd) \\
  2i(k^2 - q^2) e^{ikd} \sin(qd) & (k + q)^2 e^{i(k-q)d} - (k - q)^2 e^{i(k+q)d}
\end{pmatrix}
\begin{pmatrix}
  1/t^* & -r^*/t^* \\
  -r/t' & 1/t'
\end{pmatrix}.
\]

Thus,
\[
t^* = \frac{4kq e^{ikd}}{(k + q)^2 e^{iqd} - (k - q)^2 e^{-iqd}}
\]
and (see sketch in figure 1):
\[
T(E) = |t|^2 = \frac{1}{1 + \left(\frac{k^2 - q^2}{2kq}\right)^2 \sin^2(qd)}
\]
\[
= \frac{1}{1 + \left[\sin\left(\frac{2b\sqrt{\eta - 1}}{2\sqrt{\eta(\eta - 1)}}\right)\right]^2} \quad (\eta \geq 1)
\]
\[
= \frac{1}{1 + \left[\sinh\left(\frac{2b\sqrt{1-\eta}}{2\sqrt{\eta(1-\eta)}}\right)\right]^2} \quad (\eta \leq 1).
\]

[2] Multichannel Scattering – Consider a multichannel scattering process defined by the Hamiltonian matrix
\[
\mathcal{H}_{ij} = \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \varepsilon_i\right) \delta_{ij} + \Omega_{ij} \delta(x),
\]
which describes the scattering among \(N\) channels by a \(\delta\)-function impurity at \(x = 0\). The matrix \(\Omega_{ij}\) allows a particle in channel \(j\) passing through \(x = 0\) to be scattered into channel \(i\). The \(\{\varepsilon_i\}\) are the internal (transverse) energies for the various channels. For \(x \neq 0\), we can write the channel \(j\) component of the wavefunction as
\[
\psi_j(x) = \begin{cases} 
  I_j e^{ik_jx} + O_j' e^{-ik_jx} & (x < 0) \\
  O_j e^{ik_jx} + I_j' e^{-ik_jx} & (x > 0)
\end{cases}
\]
where the \(k_j\) are positive and determined by
\[
\varepsilon_j' = \frac{\hbar^2 k_j^2}{2m} + \varepsilon_j.
\]

Show that the incoming and outgoing flux amplitudes are related by a \(2N \times 2N\) \(S\)-matrix:
\[
\begin{pmatrix}
  \sqrt{\nu} O' \\
  \sqrt{\nu} O
\end{pmatrix} = \begin{pmatrix}
  r & t' \\
  t & r'
\end{pmatrix} \begin{pmatrix}
  \sqrt{\nu} I \\
  \sqrt{\nu} I'
\end{pmatrix}
\]

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where \( v = \text{diag}(v_1, \ldots, v_N) \) with \( v_i = \hbar k_i/m > 0 \). Find explicit expressions for the component \( N \times N \) blocks \( r, t, t', r' \), and show that \( S \) is unitary, i.e. \( S^\dagger S = SS^\dagger = I \).

**Solution:** Continuity of the wavefunction at \( x = 0 \) requires
\[
I_j + O_j' = O_j + I_j'.
\]
Integrating the Schrödinger equation from \( x = 0^- \) to \( x = 0^+ \) yields
\[
-\frac{\hbar^2}{2m} \left[ \psi'_j(0^+) - \psi'_j(0^-) \right] + \Omega_{ij} \psi_j(0) = 0,
\]
which is equivalent to
\[
(ihV + \Omega)_{ij} (I_j + I'_j) = (ihV - \Omega)_{ij} (O_j + O'_j),
\]
with \( V_{ij} = v_i \delta_{ij} \). Thus,
\[
\begin{pmatrix}
1 & -1 \\
ihV - \Omega & i\hbar V - \Omega
\end{pmatrix}
\begin{pmatrix}
O' \\
O
\end{pmatrix} =
\begin{pmatrix}
-1 & 1 \\
(ihV + \Omega) & (i\hbar V + \Omega)
\end{pmatrix}
\begin{pmatrix}
I \\
I'
\end{pmatrix}.
\]
If \( A \) is any \( N \times N \) matrix, then
\[
\begin{pmatrix}
1 & -1 \\
A & A
\end{pmatrix} = \frac{1}{2}
\begin{pmatrix}
1 & A^{-1} \\
-1 & A
\end{pmatrix}.
\]
Consequently,

\[
\left( \begin{array}{c} O' \\ O \end{array} \right) = \frac{1}{2} \left( \begin{array}{cc} Q - 1 & Q + 1 \\ Q + 1 & Q - 1 \end{array} \right) \left( \begin{array}{c} I' \\ I \end{array} \right)
\]

with \( Q = (i\hbar V - \Omega)^{-1}(i\hbar V + \Omega) \). This immediately gives the \( S \)-matrix as

\[
S = \left( \begin{array}{c} O' \\ O \end{array} \right) = \frac{1}{2} \left( \begin{array}{cc} \tilde{Q} - 1 & \tilde{Q} + 1 \\ \tilde{Q} + 1 & \tilde{Q} - 1 \end{array} \right)
\]

where

\[
\tilde{Q} = V^{1/2} Q V^{-1/2} = (1 + i\hbar^{-1} \tilde{\Omega})^{-1}(1 - i\hbar^{-1} \tilde{\Omega}),
\]

with \( \tilde{\Omega} = V^{-1/2} \Omega V^{-1/2} \). Note that the product in the above equation may be taken in either order, as the two factors commute. Since \( \tilde{\Omega} = \tilde{\Omega}^\dagger \) is Hermitian, \( \tilde{Q} \) is unitary, which in turn guarantees the unitarity of \( S \):

\[
S^\dagger S = \frac{1}{2} \left( \begin{array}{cc} \tilde{Q}^\dagger \tilde{Q} + 1 & \tilde{Q}^\dagger \tilde{Q} - 1 \\ \tilde{Q}^\dagger \tilde{Q} - 1 & \tilde{Q}^\dagger \tilde{Q} + 1 \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).
\]

[3] Spin Valve – Consider a barrier between two halves of a ferromagnetic metallic wire. For \( x < 0 \) the magnetization lies in the \( \hat{z} \) direction, while for \( x > 0 \) the magnetization is directed along the unit vector \( \hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \). The Hamiltonian is given by

\[
\mathcal{H} = -\frac{\hbar^2}{2m^*} \frac{d^2}{dx^2} + \mu_B \mathbf{H}_{\text{int}} \cdot \mathbf{\sigma},
\]

where \( \mathbf{H}_{\text{int}} \) is the (spontaneously generated) internal magnetic field and \( \mu_B = e\hbar/2m_e c \) is the Bohr magneton\(^1\). The magnetization \( \mathbf{M} \) points along \( \mathbf{H}_{\text{int}} \). For \( x < 0 \) we therefore have

\[
E_F = \frac{\hbar^2 k_F^2}{2m^*} + \Delta = \frac{\hbar^2 k_F^2}{2m^*} - \Delta,
\]

where \( \Delta = \mu_B H_{\text{int}} \). A similar relation holds for the Fermi wavevectors corresponding to spin states \( |\hat{n}\rangle \) and \( |\hat{n}^-\rangle \) in the region \( x > 0 \).

Consider the \( S \)-matrix for this problem. The ‘in’ and ‘out’ states should be defined as local eigenstates, which means that they have different spin polarization axes for \( x < 0 \) and \( x > 0 \). Explicitly, for \( x < 0 \) we write

\[
\left( \begin{array}{c} \psi_1(x) \\ \psi_2(x) \end{array} \right) = \left\{ \begin{array}{c} A_1 e^{ik_1 x} + B_1 e^{-ik_1 x} \\ A_2 e^{i(k_1 - \phi) x} + B_2 e^{-i(k_1 - \phi) x} \end{array} \right\} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \left\{ \begin{array}{c} A_3 e^{ik_2 x} + B_3 e^{-ik_2 x} \\ A_4 e^{i(k_2 - \phi) x} + B_4 e^{-i(k_2 - \phi) x} \end{array} \right\} \left( \begin{array}{c} 0 \\ 1 \end{array} \right),
\]

\(^1\)Note that it is the bare electron mass \( m_e \) which appears in the formula for \( \mu_B \) and not the effective mass \( m^* \).

\(^2\)For weakly magnetized systems, the magnetization is \( \mathbf{M} = \mu_B^2 g(\epsilon_F) \mathbf{H}_{\text{int}} \), where \( g(\epsilon_F) \) is the total density of states per unit volume at the Fermi energy.
while for $x > 0$ we write
\[
\begin{pmatrix}
\psi^\uparrow(x) \\
\psi^\downarrow(x)
\end{pmatrix} = \left\{ C^\uparrow e^{ik^\uparrow x} + D^\uparrow e^{-ik^\downarrow x} \right\} \begin{pmatrix} u \\ v \end{pmatrix} + \left\{ C^\downarrow e^{ik^\downarrow x} + D^\downarrow e^{-ik^\uparrow x} \right\} \begin{pmatrix} -v^* \\ u \end{pmatrix},
\]
where $u = \cos(\theta/2)$ and $v = \sin(\theta/2) \exp(i\phi)$. The $S$-matrix relates the flux amplitudes of the in-states and out-states:
\[
\begin{pmatrix}
B^\uparrow \\
B^\downarrow \\
C^\uparrow \\
C^\downarrow
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & -u & v^* \\
0 & 1 & v & -u \\
k^\uparrow & 0 & k^\uparrow u & -k^\downarrow v \\
k^\downarrow & k^\downarrow v & k^\downarrow u & 0
\end{pmatrix}
\begin{pmatrix}
A^\uparrow \\
A^\downarrow \\
C^\uparrow \\
C^\downarrow
\end{pmatrix}.
\]
Derive the $2 \times 2$ transmission matrix $t$ (you don’t have to derive the entire $S$-matrix) and thereby obtain the dimensionless conductance $g = \text{Tr}(tt^\dagger)$. Define the polarization $P$ by
\[
P = \frac{n^\uparrow - n^\downarrow}{n^\uparrow + n^\downarrow},
\]
where $n_\sigma = k_\sigma/\pi$ is the electronic density. Find $g(P, \theta)$.

**Solution:** Continuity of the wavefunction and its derivatives at $x = 0$ yields four equations, conveniently written in matrix form:
\[
\begin{pmatrix}
1 & 0 & -u & v^* \\
0 & 1 & v & -u \\
k^\uparrow & 0 & k^\uparrow u & -k^\downarrow v \\
k^\downarrow & k^\downarrow v & k^\downarrow u & 0
\end{pmatrix}
\begin{pmatrix}
B^\uparrow \\
B^\downarrow \\
C^\uparrow \\
C^\downarrow
\end{pmatrix} =
\begin{pmatrix}
-1 & 0 & u & -v^* \\
0 & -1 & v & u \\
k^\uparrow & 0 & k^\uparrow u & -k^\downarrow v \\
k^\downarrow & k^\downarrow v & k^\downarrow u & 0
\end{pmatrix}
\begin{pmatrix}
A^\uparrow \\
A^\downarrow \\
C^\uparrow \\
C^\downarrow
\end{pmatrix}.
\]
Defining the $2 \times 2$ blocks,
\[
\Sigma \equiv \begin{pmatrix} u & -v^* \\ v & u \end{pmatrix}, \quad K \equiv \begin{pmatrix} k^\uparrow & 0 \\ 0 & k^\downarrow \end{pmatrix},
\]
we have
\[
\begin{pmatrix}
B \\
C
\end{pmatrix} = \left( \begin{pmatrix} 1 & -\Sigma \\ k & \Sigma K \end{pmatrix} \right)^{-1} \begin{pmatrix} -1 & \Sigma \\ K & \Sigma K \end{pmatrix}
\begin{pmatrix} A \\
D
\end{pmatrix}.
\]
Converting to flux amplitudes, we have
\[
S = \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \sqrt{K} \end{pmatrix} \left( \begin{pmatrix} 1 & -\Sigma \\ K & \Sigma K \end{pmatrix} \right)^{-1} \begin{pmatrix} -1 & \Sigma \\ K & \Sigma K \end{pmatrix} \begin{pmatrix} \sqrt{K^{-1}} & 0 \\ 0 & \sqrt{K^{-1}} \end{pmatrix}.
\]
We now invoke the general result
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} \\ (B - AC^{-1}D)^{-1} \end{pmatrix} \begin{pmatrix} (C - DB^{-1}A)^{-1} \\ (D - CA^{-1}B)^{-1} \end{pmatrix}.
\]
to obtain the blocks of $S$:

$$
\begin{align*}
    r &= K^{1/2}\left\{ (1 + K^{-1}\Sigma K^{-1})^{-1} - (1 + \Sigma K^{-1})^{-1} \right\} K^{-1/2} \\
    t' &= 2K^{1/2}(\Sigma^{-1} + K^{-1}\Sigma^{-1}K)^{-1}K^{-1/2} \\
    t &= 2K^{1/2}(\Sigma + K^{-1}\Sigma K)^{-1}K^{-1/2} \\
    r' &= K^{1/2}\left\{ (1 + K^{-1}\Sigma^{-1}K) - (1 + \Sigma^{-1}K^{-1})^{-1} \right\} K^{-1/2} .
\end{align*}
$$

We find

$$
    t = \frac{1}{u^2 + |v|^2 \cosh^2 y} \begin{pmatrix} u & v^* \cosh y \\ -v \cosh y & u \end{pmatrix}
$$

with $y = \frac{1}{2} \ln(k_1/k_\perp)$. The dimensionless conductance is

$$
    g(P, \theta) = \text{Tr} \left( t^\dagger t \right) = \frac{2}{u^2 + |v|^2 \cosh^2 y} = \frac{2 (1 - P^2)}{(1 - P^2) \cos^2 \frac{1}{2} \theta + \sin^2 \frac{1}{2} \theta} ,
$$

where $P$ is the polarization. Note that $g(P = \pm 1, \theta) = 0$, since it is impossible to match boundary conditions on the lower components. One can also compute the reflection matrix,

$$
    r = \frac{\sinh y \sin \frac{1}{2} \theta}{\cos^2 \frac{1}{2} \theta + \sin^2 \frac{1}{2} \theta \cosh^2 y} \begin{pmatrix} \cos \frac{1}{2} \theta & \cosh y \sin \frac{1}{2} \theta e^{-i\phi} \\ -\cosh y \sin \frac{1}{2} \theta e^{i\phi} & \cosh \frac{1}{2} \theta \end{pmatrix} .
$$

[4] Distribution of Resistances of a One-Dimensional Wire – In this problem you are asked to derive an equation governing the probability distribution $P(R, L)$ for the dimensionless resistance $R$ of a one-dimensional wire of length $L$. The equation is called the Fokker-Planck equation. Here’s a brief primer on how to derive Fokker-Planck equations.

Suppose $x(t)$ is a stochastic variable. We define the quantity

$$
    \delta x(t) \equiv x(t + \delta t) - x(t) ,
$$

and we assume

$$
    \langle \delta x(t) \rangle = F_1(x(t)) \delta t \\
    \langle [\delta x(t)]^n \rangle = 2 F_2(x(t)) \delta t
$$

but $\langle [\delta x(t)]^n \rangle = O((\delta t)^2)$ for $n > 2$. The $n = 1$ term is due to drift and the $n = 2$ term is due to diffusion. Now consider the conditional probability density, $P(x, t | x_0, t_0)$, defined to be the probability distribution for $x \equiv x(t)$ given that $x(t_0) = x_0$. The conditional probability density satisfies the composition rule,

$$
    P(x, t | x_0, t_0) = \int_{-\infty}^{\infty} dx' P(x, t | x', t') P(x', t | x_0, t_0) ,
$$

with
for any value of $t'$. Therefore, we must have

\[ P(x, t + \delta t | x_0, t_0) = \int_{-\infty}^{\infty} dx' P(x, t + \delta t | x', t) P(x', t | x_0, t_0). \]

Now we may write

\[
P(x, t + \delta t | x', t) = \langle \delta(x - x' - \delta x(t)) \rangle = \left\{ 1 + \langle \delta x(t) \rangle \frac{d}{dx'} + \frac{1}{2} \langle [\delta x(t)]^2 \rangle \frac{d^2}{dx'^2} + \ldots \right\} \delta(x - x'),
\]

where the average is over the random variables. Upon integrating by parts and expanding to $O(\delta t)$, we obtain the Fokker-Planck equation,

\[
\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} \left[ F_1(x) P(x, t) \right] + \frac{\partial^2}{\partial x^2} \left[ F_2(x) P(x, t) \right].
\]

That wasn’t so bad, now was it?

For our application, $x(t)$ is replaced by $R(L)$. We derived the composition rule for series quantum resistors in class:

\[
R(L + \delta L) = R(L) + R(\delta L) + 2R(L) R(\delta L) - 2 \cos \beta \sqrt{R(L) [1 + R(L)] \ R(\delta L) [1 + R(\delta L)]},
\]

where $\beta$ is a random phase. For small values of $\delta L$, we needn’t worry about quantum interference and we can use our Boltzmann equation result. Show that

\[
R(\delta L) = e^{2} \frac{m^*}{\hbar ne^2 \tau} \delta L = \frac{\delta L}{2 \ell},
\]

where $\ell = v_F \tau$ is the elastic mean free path. (Assume a single spin species throughout.)

Find the drift and diffusion functions $F_1(R)$ and $F_2(R)$. Show that the distribution function $P(R, L)$ obeys the equation

\[
\frac{\partial P}{\partial L} = \frac{1}{2 \ell} \frac{\partial}{\partial R} \left\{ R (1 + R) \frac{\partial P}{\partial R} \right\}.
\]

Show that this equation may be solved in the limits $R \ll 1$ and $R \gg 1$, with

\[
P(R, z) = \frac{1}{z} e^{-R/z}
\]

for $R \ll 1$, and

\[
P(R, z) = \left(4 \pi z\right)^{-1/2} \frac{1}{R} e^{-\left(\ln R - z\right)^2 / 4z}
\]

for $R \gg 1$, where $z = L/2 \ell$ is the dimensionless length of the wire. Compute $\langle R \rangle$ in the former case, and $\langle \ln R \rangle$ in the latter case.
Solution: We have
\[
\mathcal{R}(\delta L) = \frac{e^2}{\hbar} \rho \delta L = \frac{e^2}{\hbar} \frac{m^*}{ne^2 \tau} \delta L = \frac{e^2}{\hbar} \frac{m^* v_F}{ne^2 \ell} \delta L
\]
\[
= \frac{k_F}{2\pi n} \frac{\delta L}{\ell} = \frac{\delta L}{2\ell}.
\]
From the composition rule for series quantum resistances, we derive the phase averages
\[
\langle \delta \mathcal{R} \rangle = \left( 1 + 2 \mathcal{R}(L) \right) \frac{\delta L}{2\ell}
\]
\[
\langle (\delta \mathcal{R})^2 \rangle = \left( 1 + 2 \mathcal{R}(L) \right)^2 \left( \frac{\delta L}{2\ell} \right)^2 + 2 \mathcal{R}(L) \left( 1 + \mathcal{R}(L) \right) \frac{\delta L}{2\ell} \left( 1 + \frac{\delta L}{2\ell} \right)
\]
\[
= 2 \mathcal{R}(L) \left( 1 + \mathcal{R}(L) \right) \frac{\delta L}{2\ell} + \mathcal{O}((\delta L)^2),
\]
whence we obtain the drift and diffusion terms
\[
F_1(\mathcal{R}) = \frac{2 \mathcal{R} + 1}{2\ell}, \quad F_2(\mathcal{R}) = \frac{\mathcal{R}(1 + \mathcal{R})}{2\ell}.
\]
Note that \( F_1(\mathcal{R}) = dF_2/d\mathcal{R} \), which allows us to write the Fokker-Planck equation as
\[
\frac{\partial P}{\partial L} = \frac{\partial}{\partial \mathcal{R}} \left\{ \mathcal{R} \left( 1 + \mathcal{R} \right) \frac{\partial P}{\partial \mathcal{R}} \right\}.
\]
Defining the dimensionless length \( z = L/2\ell \), we have
\[
\frac{\partial P}{\partial z} = \frac{\partial}{\partial \mathcal{R}} \left\{ \mathcal{R} \left( 1 + \mathcal{R} \right) \frac{\partial P}{\partial \mathcal{R}} \right\}.
\]
In the limit \( \mathcal{R} \ll 1 \), this reduces to
\[
\frac{\partial P}{\partial z} = \mathcal{R} \frac{\partial^2 P}{\partial \mathcal{R}^2} + \frac{\partial P}{\partial \mathcal{R}}
\]
which is satisfied by \( P(\mathcal{R}, z) = z^{-1} \exp(-\mathcal{R}/z) \). In the opposite limit, \( \mathcal{R} \gg 1 \), we have
\[
\frac{\partial P}{\partial z} = \mathcal{R}^2 \frac{\partial^2 P}{\partial \mathcal{R}^2} + 2 \mathcal{R} \frac{\partial P}{\partial \mathcal{R}}
\]
\[
= \frac{\partial^2 P}{\partial \nu^2} + \frac{\partial P}{\partial \nu},
\]
where \( \nu \equiv \ln \mathcal{R} \). This is solved by the log-normal distribution,
\[
P(\mathcal{R}, z) = (4\pi z)^{-1/2} e^{-(\nu+z)^2/4z}.
\]
Note that
\[
P(\mathcal{R}, z) d\mathcal{R} = (4\pi z)^{-1/2} \exp \left\{ -\frac{(\ln \mathcal{R} - z)^2}{4z} \right\} d\ln \mathcal{R}.
\]