Physics 211B : Assignment #2

[1] Rectangular Barrier – Consider a symmetric planar barrier consisting of a layer of Al$_x$Ga$_{1-x}$As of width $2a$ imbedded in GaAs. The barrier height $V_0$ is simply the difference between conduction band minima $\Delta E_c$ at the $\Gamma$ point; energies are defined relative to $E^\text{GaAs}_\Gamma$. Derive the $S$-matrix for this problem. Show that

$$T(E) = \frac{1}{1 + \left[ \sinh \left( \frac{b\sqrt{1-\eta}}{2\sqrt{\eta(1-\eta)}} \right) \right]^2} \quad (\eta \leq 1)$$

and

$$T(E) = \frac{1}{1 + \left[ \sin \left( \frac{b\sqrt{\eta-1}}{2\sqrt{\eta}(\eta-1)} \right) \right]^2} \quad (\eta \geq 1) ,$$

where $\eta = E/V_0$ and $b = a/\ell$ with $\ell = \hbar/\sqrt{2m^*V_0}$. Sketch $T(E)$ versus $E/V_0$ for various values of the dimensionless thickness $b$.

[2] Multichannel Scattering – Consider a multichannel scattering process defined by the Hamiltonian matrix

$$\mathcal{H}_{ij} = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \epsilon_i \right) \delta_{ij} + \Omega_{ij} \delta(x) ,$$

which describes the scattering among $N$ channels by a $\delta$-function impurity at $x = 0$. The matrix $\Omega_{ij}$ allows a particle in channel $j$ passing through $x = 0$ to be scattered into channel $i$. The $\{\epsilon_i\}$ are the internal (transverse) energies for the various channels. For $x \neq 0$, we can write the channel $j$ component of the wavefunction as

$$\psi_j(x) = I_j e^{ik_jx} + O'_j e^{-ik_jx} \quad (x < 0)$$
$$= O_j e^{ik_jx} + I'_j e^{-ik_jx} \quad (x > 0) ,$$

where the $k_j$ are positive and determined by

$$\epsilon = \frac{\hbar^2 k_j^2}{2m} + \epsilon_j .$$

Show that the incoming and outgoing flux amplitudes are related by a $2N \times 2N$ $S$-matrix:

$$\mathcal{S} = \begin{pmatrix} \sqrt{v} O' \cr \sqrt{v} O \end{pmatrix} = \begin{pmatrix} r & t' \end{pmatrix} \begin{pmatrix} \sqrt{v} I \cr t' \end{pmatrix} \begin{pmatrix} \sqrt{v} I' \cr t \end{pmatrix}$$

where $v = \text{diag}(v_1, \ldots, v_N)$ with $v_i = \hbar k_i/m > 0$. Find explicit expressions for the component $N \times N$ blocks $r$, $t$, $t'$, $r'$, and show that $S$ is unitary, i.e. $S^\dagger S = SS^\dagger = \mathbb{I}$.

[3] Spin Valve – Consider a barrier between two halves of a ferromagnetic metallic wire. For $x < 0$ the magnetization lies in the $\hat{z}$ direction, while for $x > 0$ the magnetization is
directed along the unit vector $\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. The Hamiltonian is given by

$$H = -\frac{\hbar^2}{2m^*} \frac{d^2}{dx^2} + \mu_B H_{\text{int}} \cdot \sigma,$$

where $H_{\text{int}}$ is the (spontaneously generated) internal magnetic field and $\mu_B = e\hbar/2m_e c$ is the Bohr magneton\(^1\). The magnetization $M$ points along $H_{\text{int}}$. For $x < 0$ we therefore have

$$E_F = \frac{\hbar^2 k_\uparrow^2}{2m^*} + \Delta = \frac{\hbar^2 k_\downarrow^2}{2m^*} - \Delta,$$

where $\Delta = \mu_B H_{\text{int}}$. A similar relation holds for the Fermi wavevectors corresponding to spin states $|\hat{n}\rangle$ and $|-\hat{n}\rangle$ in the region $x > 0$.

Consider the $S$-matrix for this problem. The 'in' and 'out' states should be defined as local eigenstates, which means that they have different spin polarization axes for $x < 0$ and $x > 0$. Explicitly, for $x < 0$ we write

$$\begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix} = \begin{pmatrix} A_\uparrow e^{ik_\uparrow x} + B_\uparrow e^{-ik_\uparrow x} \\ 0 \end{pmatrix} + \begin{pmatrix} A_\downarrow e^{ik_\downarrow x} + B_\downarrow e^{-ik_\downarrow x} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

while for $x > 0$ we write

$$\begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix} = \begin{pmatrix} C_\uparrow e^{ik_\uparrow x} + D_\uparrow e^{-ik_\uparrow x} \\ -v^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} C_\downarrow e^{ik_\downarrow x} + D_\downarrow e^{-ik_\downarrow x} \end{pmatrix} \begin{pmatrix} -v^* \\ u \end{pmatrix},$$

where $u = \cos(\theta/2)$ and $v = \sin(\theta/2) \exp(i\phi)$. The $S$-matrix relates the flux amplitudes of the in-states and out-states:

$$\begin{pmatrix} b_\uparrow \\ b_\downarrow \\ c_\uparrow \\ c_\downarrow \end{pmatrix} = S \begin{pmatrix} r_{11} & r_{12} & t_{11}' & t_{12}' \\ r_{21} & r_{22} & t_{21}' & t_{22}' \\ t_{11} & t_{12} & r_{11}' & r_{12}' \\ t_{21} & t_{22} & r_{21}' & r_{22}' \end{pmatrix} \begin{pmatrix} a_\uparrow \\ a_\downarrow \\ d_\uparrow \\ d_\downarrow \end{pmatrix}.$$

Derive the $2 \times 2$ transmission matrix $t$ (you don’t have to derive the entire $S$-matrix) and thereby obtain the dimensionless conductance $g = \text{Tr} (t^\dagger t)$. Define the polarization $P$ by

$$P = \frac{n_\uparrow - n_\downarrow}{n_\uparrow + n_\downarrow},$$

where $n_\sigma = k_\sigma / \pi$ is the electronic density. Find $g(P, \theta)$.

\(^1\)Note that it is the bare electron mass $m_e$ which appears in the formula for $\mu_B$ and not the effective mass $m^\star$.

\(^2\)For weakly magnetized systems, the magnetization is $M = \mu_B^2 g(\varepsilon_F) H_{\text{int}}$, where $g(\varepsilon_F)$ is the total density of states per unit volume at the Fermi energy.
Distribution of Resistances of a One-Dimensional Wire – In this problem you are asked to derive an equation governing the probability distribution $P(R, L)$ for the dimensionless resistance $R$ of a one-dimensional wire of length $L$. The equation is called the Fokker-Planck equation. Here’s a brief primer on how to derive Fokker-Planck equations.

Suppose $x(t)$ is a stochastic variable. We define the quantity

$$\delta x(t) \equiv x(t + \delta t) - x(t),$$

and we assume

$$\langle \delta x(t) \rangle = F_1(x(t)) \delta t$$

$$\langle [\delta x(t)]^2 \rangle = 2 F_2(x(t)) \delta t$$

but $\langle [\delta x(t)]^n \rangle = \mathcal{O}((\delta t)^2)$ for $n > 2$. The $n = 1$ term is due to drift and the $n = 2$ term is due to diffusion. Now consider the conditional probability density, $P(x, t \mid x_0, t_0)$, defined to be the probability distribution for $x \equiv x(t)$ given that $x(t_0) = x_0$. The conditional probability density satisfies the composition rule,

$$P(x, t \mid x_0, t_0) = \int_{-\infty}^{\infty} dx' P(x, t \mid x', t') P(x', t' \mid x_0, t_0),$$

for any value of $t'$. Therefore, we must have

$$P(x, t + \delta t \mid x_0, t_0) = \int_{-\infty}^{\infty} dx' P(x, t + \delta t \mid x', t) P(x', t \mid x_0, t_0).$$

Now we may write

$$P(x, t + \delta t \mid x', t) = \langle \delta(x - x' - \delta x(t)) \rangle$$

$$= \left\{ 1 + \langle \delta x(t) \rangle \frac{d}{dx'} + \frac{1}{2} \langle [\delta x(t)]^2 \rangle \frac{d^2}{dx'^2} + \ldots \right\} \delta(x - x'),$$

where the average is over the random variables. Upon integrating by parts and expanding to $\mathcal{O}(\delta t)$, we obtain the Fokker-Planck equation,

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} [F_1(x(t)) P(x, t)] + \frac{\partial^2}{\partial x^2} [F_2(x(t)) P(x, t)].$$

That wasn’t so bad, now was it?

For our application, $x(t)$ is replaced by $R(L)$. We derived the composition rule for series quantum resistors in class:

$$R(L + \delta L) = R(L) + R(\delta L) + 2 R(L) R(\delta L) \left[ 1 + R(\delta L) \right] \frac{\partial}{\partial x} \left[ R(L) \right] \frac{\partial}{\partial x} \left[ R(\delta L) \right],$$

$$- 2 \cos \beta \sqrt{R(L) [1 + R(L)] R(\delta L) [1 + R(\delta L)]},$$
where $\beta$ is a random phase. For small values of $\delta L$, we needn’t worry about quantum interference and we can use our Boltzmann equation result. Show that

$$R(\delta L) = e^{2\hbar m^*} \delta L = \frac{\delta L}{2\ell},$$

where $\ell = v_F \tau$ is the elastic mean free path. (Assume a single spin species throughout.)

Find the drift and diffusion functions $F_1(R)$ and $F_2(R)$. Show that the distribution function $P(R, L)$ obeys the equation

$$\frac{\partial P}{\partial L} = \frac{1}{2\ell} \frac{\partial}{\partial R} \left\{ R (1 + R) \frac{\partial P}{\partial R} \right\}.$$  

Show that this equation may be solved in the limits $R \ll 1$ and $R \gg 1$, with

$$P(R, z) = \frac{1}{z} e^{-R/z}$$

for $R \ll 1$, and

$$P(R, z) = (4\pi z)^{-1/2} \frac{1}{R} e^{-(\ln R - z)^2/4z}$$

for $R \gg 1$, where $z = L/2\ell$ is the dimensionless length of the wire. Compute $\langle R \rangle$ in the former case, and $\langle \ln R \rangle$ in the latter case.