Hamilton-Jacobi I

Hamilton–Jacobi Theory

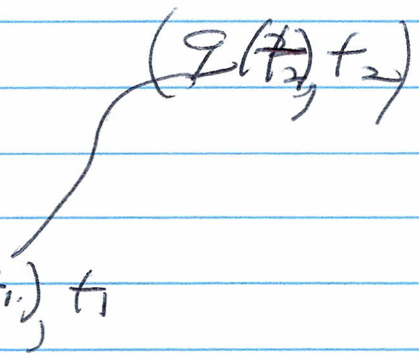
– here, three thrusts:

→ how does action evolve? → $S(q,t)$?
→ semi-classical limit QM ⟺ eikonal equation for Schroedinger Eqn?
→ when is motion integrable?

Now, can see (at least) two perspectives on Action and Principle of Least Action.

① "S as functional" ⟺ fixed end points

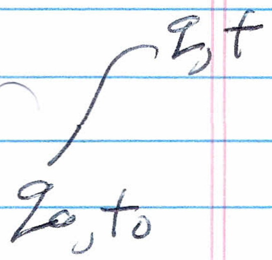
$$S = \int_{t_1}^{t_2} dt\, L(q,\dot{q},t)$$



$\delta S = 0 \Rightarrow$ Lagrange Eqns.

② $(S = S(q,t))$

"S as function" ⟺ variable upper end point



$S(q,t)$?

Approach by considering increments

i.e. $$dS = \left(\frac{\partial S}{\partial q}\right) dq + \left(\frac{\partial S}{\partial t}\right) dt$$

seek, for basic parametrization of $S(q,t)$

Outcome: ∂S –

$$\delta S = \frac{\partial L}{\partial \dot{q}} \, \delta q \Big|_{t_1}^{t}$$

$$= p(t)\, \delta q \qquad (\delta q(t_1) = 0)$$

so $$\boxed{\frac{\partial S}{\partial q} = p}$$

→ for time dependence,

$$\frac{dS}{dt} = \frac{\partial S}{\partial q} \dot{q} + \frac{\partial S}{\partial t}$$

but $dS/dt = L$

$\partial S / \partial q = p$

$$L = p\dot{q} + \partial S/\partial t$$

$$\Rightarrow \quad \frac{\partial S}{\partial t} = -(p\dot{q} - L) = -H$$

so $$dS = \sum_i p \, dq_i - H\,dt$$

Now, to H-J Eqn:

$H = H(q, p, t)$ $\left.\begin{array}{l}\dot{p} = -\partial H/\partial q \\ \dot{q} = \partial H/\partial p\end{array}\right\}$

but also showed

$H = H(q, \partial S/\partial q, t)$, as $p = \frac{\partial S}{\partial q}$

and

$$\frac{\partial S}{\partial t} = -H(p, q, t) = -H\left(q, \frac{\partial S}{\partial q}, t\right)$$

$\therefore$

$$\boxed{\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0}$$ $\left\{\begin{array}{l}\text{Hamilton-} \\ \text{Jacobi} \\ \text{Eqn.}\end{array}\right.$

$\rightarrow$ contains <u>all</u> info in Hamilton's Eqns.

$\rightarrow$ full info on dynamics

Now, if $\partial L/\partial t = 0$ so conservative
$H = E = \text{const.}$

$$H(p, q) = E = H\left(\frac{\partial S}{\partial q}, q\right)$$

$$\boxed{H\left(\frac{\partial S}{\partial q}, q\right) = E}$$

Time-Independent H-J Eqn. (for conservative system)

why care?:

i.) single, first order pde has full content of system

ii.) solvability (separability) of H-J eqn $\Rightarrow$ integrability of dynamical system (i.e. via geometrical system)

iii.) techniques to solve $S(q,t)$ $\Leftrightarrow$ equiv to solving Hamilton's Eqn.

iv.) H-J eqn. is eikonal equation for Schroedinger Eqn.

i.e.

S.E.: $\quad i\hbar \frac{\partial \psi}{\partial t} = H\psi$

$$= -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi$$

now, semi-classical limit appears as $\hbar \to 0$ limit, so

$$\psi = \psi_0 \, e^{i\phi(x,t)/\hbar}$$

$\hbar \to 0 \Rightarrow$ classical trajectory emerges as phase stationarity

$\hbar \sim$ action $\Rightarrow$ $\phi \sim$ action

$$+i\hbar \frac{i}{\hbar} \frac{\partial \phi}{\partial t} = \frac{-\hbar^2}{2m} \frac{-1}{\hbar^2} (\nabla \phi)^2 + V$$

$$-\frac{\partial \phi}{\partial t} = \frac{1}{2m} (\nabla \phi)^2 + V$$

$$= H(\nabla \phi, x, t)$$

so, if take $\phi \Rightarrow S$, by classical correspondence (i.e. $\delta S = 0 \Rightarrow$ classical trajectory), then eikonal equation is clearly H-J equation

$$\boxed{\frac{\partial S}{\partial t} = -H\left(\frac{\partial S}{\partial q}, q, t\right)}$$

and eikonal equation for T.I.S.E. is time independent H-J. equation

$$H = E \quad , \quad H = H\left(\frac{\partial S}{\partial q}, q\right)$$

→ Additional / Alternative Variational Principle (Abbreviated Action / Principle of Maupertuis).

Now, for eikonal theory: 2 results;

- ray <u>paths</u> : $\delta T = 0$ $\quad T = \int ds\, n(\underline{x})$

  i.e. paths trace ray, but dont give any time info.

- ray <u>trajectories</u> : $\delta \Phi = 0$

$$\frac{d\underline{x}}{dt} = \frac{\partial \omega}{\partial \underline{k}} \, ; \quad \frac{d\underline{k}}{dt} = -\frac{\partial \omega}{\partial \underline{x}}$$

  i.e. trajectories yield time info → at what t does wave packet pass?

Similarly, for particles:

position, trajectory : $\underline{r}(t)$ → usual

<u>path</u> : $r(\ell)$ → curve followed by particle. Does not tell when particle at a particular point.

e.g. free particle: paths are geodesics ⇒ contain geometry, only.

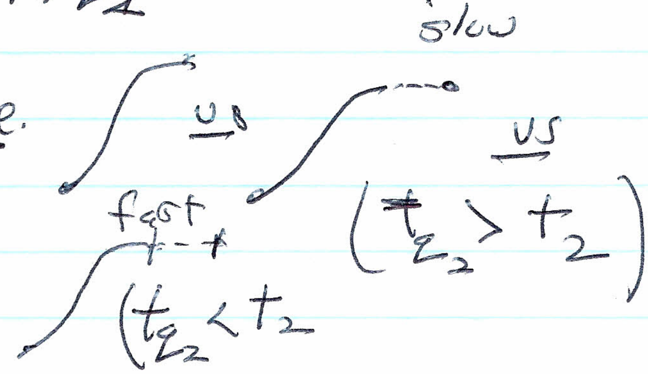
Now, for $\partial_t L = 0$; $H(p, q) = E$ conservative.

- $\delta \int_{q_1, t_1}^{q_2, t_2} L = 0 \Rightarrow \delta S = 0$ for fixed end points, (usual)

- now, allow t_2 vary; q_1, q_2 fixed

$$\delta \int_{q_1, t_1}^{q_2, t_2} L = -H \delta t$$

i.e.



$\Rightarrow$ defines virtual paths ....

i.e. particle passes thru q_2 but not necessarily at t_2.

- for energy conserving virtual paths

$$\delta S + H \delta t = 0 = \delta S + E \delta t$$

also know:

$$S = \int (p\dot{q} - H) dt$$

$$= \int (p\, dq - H\, dt) = \int (p\, dq - E\, dt)$$

so, in general:

$$\boxed{S = \int \sum_i p_i\, dq_i - E(t-t_0)}$$

define:

$$S_0 = \int \sum_i p_i\, dq_i \equiv \text{abbreviated action}$$

so, for **paths**:

$$\boxed{\delta S_0 = \delta \int \sum_i p_i\, dq_i = 0}$$

Principle of Maupertuis

$\Rightarrow$ abbreviated action has minimum with respect to all paths which conserve energy and pass thru final point at any t.

$\Rightarrow$ to use S_0, need express momenta in terms q, dq via:

$$p_i = \partial L / \partial \dot{q}_i \qquad L = L(q, \dot{q})$$

$$E(q, \dot{q}) = E$$

e.

$$L = \frac{1}{2} \sum_{i,k} a_{ik}(q)\, \dot{q}_i \dot{q}_k - U(q)$$

$\rightsquigarrow$ generic form (ala' HW)

$$dS_0 = \sum_i p_i \, dq_i$$

but

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \sum_k a_{ik}(q)\, \dot{q}_k$$

$$dS_0 = \sum_{k,i} a_{ik}(q)\, \dot{q}_k \, dq_i$$

$$= \sum_{k,i} a_{ik}(q) \frac{dq_k}{dt} dq_i$$

for dt;

$$E = \frac{1}{2} \sum_{i,k} a_{ik}(q)\, \dot{q}_i \dot{q}_k + U(q)$$

$$\frac{1}{2} \sum_{i,k} a_{ik}(q) \frac{dq_i \, dq_k}{(dt)^2} = E - U$$

$$\therefore \quad dt = \left[\sum a_{i,k} \, dq_i \, dq_k / 2(E-U) \right]^{1/2}$$

so using dt:

$$S = \int \Big[2(E-U) \underbrace{\sum_{i,k} a_{i,k} \, dq_i \, dq_k \Big]^{1/2}}_{\sim dl^2}$$

variational for path

For single particle: $T = \frac{1}{2} m (dl/dt)^2$

$$\delta S_0 = \delta \int_{q_1}^{q_2} \left[2m(E-U) \right]^{1/2} dl$$

- Jacobi's integral

- if $U = 0$ (free)

$$\delta S_0 = \delta \int dl = 0$$

→ minimal distance path of Least Action is geodesic

Ex. Derive equation for path
(n.b. ray !)

$$\delta \int (E-U)^{1/2} d\ell$$

$$= -\int \frac{\partial U}{\partial \underline{r}} \cdot \delta \underline{r} \frac{d\ell}{2(E-U)^{1/2}} + \int (E-U)^{1/2} d\,\delta\ell$$

so

$$d\ell^2 = d\underline{r}^2$$

$$d\ell\, d\delta\ell = d\underline{r} \cdot d\,\delta\underline{r}$$

$$d\,\delta\ell = \frac{d\underline{r}}{d\ell} \cdot d\,\delta\underline{r}$$

$$\Rightarrow \delta \int (E-U)^{1/2} d\ell =$$

$$-\int \left\{ \frac{\partial U}{\partial \underline{r}} \cdot \frac{\delta \underline{r}}{2(E-U)^{1/2}} d\ell - \sqrt{(E-U)} \frac{d\underline{r}}{d\ell} \cdot d\,\delta\underline{r} \right\}$$

$$d\,\delta\underline{r} = \left(\frac{d}{d\ell} \delta\underline{r} \right) d\ell$$

now, e.p.'s fixed, so IBP

$$0 = -\int \left\{ \frac{\partial U}{\partial \underline{r}} \cdot \frac{\delta \underline{r}}{2(E-U)^{1/2}}\, d\ell + \frac{d}{d\ell}\left[(E-U)^{1/2} \frac{d\underline{r}}{d\ell} \cdot \delta \underline{r}\, d\ell \right] \right\}$$

$\Rightarrow$

$$\boxed{2(E-U)^{1/2} \frac{d}{d\ell}\left[(E-U)^{1/2} \frac{d\underline{r}}{d\ell} \right] = -\frac{\partial U}{\partial \underline{r}}}$$

now, as for ray case:

$d\underline{r}/d\ell = \underline{t}$ unit tangent to path

so

$$\frac{d^2\underline{r}}{d\ell^2} = \frac{1}{2(E-U)}\left[-\frac{\partial U}{\partial \underline{r}} - \frac{d\underline{r}}{d\ell} \cdot \left(-\frac{\partial U}{\partial \underline{r}} \right) \underline{t} \right]$$

$$= \frac{1}{2(E-U)}\left[\underline{F} - (\underline{t} \cdot \underline{F})\underline{t} \right]$$

but

$$\underline{F} - \underline{t} \cdot \underline{F} = \underline{F}_n$$

normal force (to path)

$$\therefore \quad \frac{d\underline{t}}{d\ell} = \frac{1}{2(E-U)} \underline{F}_n$$

$$E - U = E_{kin} = \frac{1}{2} m v^2$$

$$d\underline{t}/d\ell = \frac{\hat{n}_0}{R_c}$$

$\hat{n}_0 \equiv$ normal to path

$R_c \equiv$ radius of curvature

$$\Rightarrow \boxed{\frac{m v^2}{R_c} \hat{n}_0 = \underline{F}_n}$$

normal acceleration on curved path.