Hamilton – Jacobi II

⇒ a) Solving the Hamilton – Jacobi Equation... (See L&L: Chapt. VII)

Now goal of classical mechanics is to integrate equations of motion.

What does "integrability" mean?

- can reduce $p_i(t)$, $q_i(t)$ equations to solution by quadrature, each i.
- if N degree of freedom system, can find N COMs (IOMs) s/t $P_{i \dots N} \equiv$ const.

Now, a sufficient, but not necessary, condition for integrability is that the H-J equation be separable and solvable (N.B. "Solvable" ≡ can reduce pieces of separation to quadrature).

Best to proceed via examples:

i) trivial – 1D oscillator

$$\frac{p^2}{2m} + \frac{1}{2} k q^2 = E \Rightarrow \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2} k q^2 = E$$

$$\rightarrow \quad \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 = E - \frac{kq^2}{2}$$

$$\boxed{S = \sqrt{2m}\int dq\,\sqrt{E - kq^2/2}} \quad = S(q)$$

but also $\quad \frac{\partial S}{\partial q} = p = m\frac{dq}{dt}$

$$\therefore \quad \frac{dq}{dt} = \frac{\sqrt{2m}}{m}\left(E - kq^2/2\right)^{1/2} \qquad \left(t - t_0 = \frac{1}{2}\frac{\partial S}{\partial E} \right)$$

$$\text{so} \quad \boxed{\int dt = \sqrt{m}\int dq \Big/ \sqrt{2}\left(E - kq^2/2\right)^{1/2}} \quad \text{formal solution}$$

Rather clearly, obtaining S is equivalent to a solution for q.

ii.) Non-Trivial – 3D Potential

i.e. What form of $V(r, \theta, \phi)$ allows integrable motion in spherical coordinates?

⇒ If separable solution of H-J equation can be constructed, motion is integrable.

2. recall solution of PDE by separation of variables

$$\nabla^2 \psi + \frac{\omega^2}{c^2}\psi = 0$$

c const.
if $c^2(\underline{x})$, what is separable?
$1/c^2(\underline{x}) = \frac{1}{c^2(x)} + \frac{1}{c^2(y)} + \frac{1}{c^2(z)}$
and WKB.

$$\psi = X(x)\,Y(y)\,Z(z)$$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} + \frac{\omega^2}{c^2} = 0$$

Now, to each ratio i.e. X''/X, etc assign separation constant k_x^2, k_y^2, k_z^2

then $\frac{X''}{X} = -k_x^2$, etc.

Solutions from separation of variables are not most general.

$$-k_x^2 - k_y^2 - k_z^2 + \frac{\omega^2}{c^2} = 0$$

and determine separation constants by B.C.'s $\Rightarrow$ eigenvalues.
N.B. separation constants $\Rightarrow$ b.c.'s, symmetry.

Now;

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2m r^2} + \frac{p_\phi^2}{2m\sin^2\theta\, r^2} + V$$

$\therefore \quad H\left(\frac{\partial S}{\partial \xi}, \xi, E\right) = E$

is T. I. H-J eqn.

$\Rightarrow$

$$\boxed{\frac{1}{2m}\left\{\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2\right\} + V(r, \theta, \phi) = E}$$

Here, separation is additive:

$S = S_1(r) + S_2(\theta) + S_3(\phi)$

$\Rightarrow$

$$\frac{1}{2m}\left\{\left(\frac{\partial S_1(r)}{\partial r}\right)^2 + \frac{1}{r^2}\left[\left(\frac{\partial S_2(\theta)}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial S_3}{\partial \phi}\right)^2\right]\right\} + V(r, \theta, \phi) = E.$$

Now:

$\rightarrow$ structure of V must match the factors in kinetic energy

→ integrability set by metric ⇒ determines KE via $d\ell^2/dt^2$.

so, evident that

$$\boxed{V(r,\theta,\phi) = a(r) + \frac{b(\theta)}{r^2} + \frac{c(\phi)}{r^2\sin^2\theta}}$$

will allow solution by separation.

Now, to solve:

$$E = \left\{\frac{1}{2m}\left(\frac{\partial S_1(r)}{\partial r}\right)^2 + a(r)\right\} + \frac{1}{r^2}\left[\left\{\frac{1}{2m}\left(\frac{\partial S_2(\theta)}{\partial \theta}\right)^2 + b(\theta)\right\} + \frac{1}{\sin^2\theta}\left\{\frac{1}{2m}\left(\frac{\partial S_3(\phi)}{\partial \phi}\right)^2 + c(\phi)\right\}\right]$$

or

$$E = f_1(r) + \frac{1}{r^2}\left\{f_2(\theta) + \frac{1}{\sin^2\theta} f_3(\phi)\right\}$$

and can separate and solve f_i:

$$f_3(\phi) = C_\phi \quad \rightarrow \text{const}$$

$$f_2(\theta) + \frac{C_\phi}{\sin^2\theta} = C_\theta \quad \rightarrow \text{const}$$

$$f_1(r) + \frac{C_\theta}{r^2} = E \quad \rightarrow \text{const}$$

then:

- can solve azimuthal, polar, radial EOMs.

- separate and solve H-J.

Key points:

- in separation of H-J eqn., separation constants C_ϕ, C_θ, E
  - related to COMs P_ϕ, L^2, E
  - related to symmetry.

- separation solution related to ability to identify C.O.Ms.

Proceeding:

$$f_3(\phi) = c_\phi^2$$

$$\frac{1}{2m}\left(\frac{\partial S_3}{\partial \phi}\right)^2 + c(\phi) = c_\phi^2.$$

simplifying assumption $\Rightarrow$ take $c(\phi) = 0$, i.e. no azimuthal symmetry breaking in potential.

so

$$\frac{1}{2m}\left(\frac{\partial S_3}{\partial \phi}\right)^2 = c_\phi^2$$

Clearly $\left(\frac{\partial S_3}{\partial \phi}\right) = \text{const.} = p_\phi$

p_ϕ ↓ azimuthal momentum.

$$S_3 = p_\phi \phi + C_3.$$

$$c_\phi = \frac{p_\phi^2}{2m}$$

so, plugging in S_3 piece:

$$E = \left\{\frac{1}{2m}\left(\frac{\partial S_1}{\partial r}\right)^2 + a(r)\right\} + \frac{1}{r^2}\left\{\frac{1}{2m}\left(\frac{\partial S_2}{\partial \theta}\right)^2 + b(\theta) + p_\phi^2/2m\sin^2\theta\right\}$$

observe:

$$f_2(\theta) + \frac{f_3(\phi)}{\sin^2\theta} = f_2'(\theta)$$

$$f_3(\phi) \to p_\phi^2/2m$$

$$= \frac{1}{2m}\left(\frac{\partial S_2}{\partial \theta}\right)^2 + b(\theta) + \frac{p_\phi^2}{2m\sin^2\theta}$$

Now, need const. of sep. for f_2':

$$\frac{1}{2m}\left(\frac{\partial S_2}{\partial \theta}\right)^2 + b(\theta) + \frac{p_\phi^2}{2m\sin^2\theta} = C_\theta^2$$

C_θ^2: const. of separation, related to angular momentum.

so

$$\frac{\partial S_2}{\partial \theta} = \sqrt{2m}\left(C_\theta^2 - b(\theta) - \frac{p_\phi^2}{2m\sin^2\theta}\right)^{1/2}$$

$\therefore$

$$S_2(\theta) = \sqrt{2m}\int d\theta \left(C_\theta^2 - b(\theta) - \frac{p_\phi^2}{2m\sin^2\theta}\right)^{1/2} + C_2$$

observe:

→ $C_\theta^2 = L^2$ if $b(\theta) = 0$ (i.e. C_θ^2 = angular momentum if central potential)

$\Rightarrow \theta \leq \pi/2 \Rightarrow$ reality $S \Rightarrow C_\phi^2 \to p_\phi^2 \leq C_\theta^2$.

Then, for last step, absorb C_θ^2/r^2 into radial piece $f_1(r)$

$$\boxed{E = \frac{1}{2m}\left(\frac{\partial S_1}{\partial r}\right)^2 + a(r) + \frac{C_\theta^2}{2mr^2}}$$

final, universal COM.

From f_2'/r^2 (centrifugal potential bit of radial motion)

so

$$\boxed{S_1(r) = \sqrt{2m}\int dr\left(E - a(r) - \frac{C_\theta^2}{2mr^2}\right)^{1/2} + c_1.}$$

so finally:

$$S = S_1(r) + S_2(\theta) + S_3(\phi)$$

where:

$$S = \int dr \left[\sqrt{2m} \left(\overset{\text{COM/sep const}}{E} - a(r) - \frac{\overset{\text{sep const.}}{C_\Theta^2}}{2mr^2} \right)^{1/2} \right]$$

$$+ \int d\theta \left[\left(\underset{\text{sep const.}}{C_\Theta^2} - b(\theta) - \frac{p_\phi^2}{2m\sin^2\theta} \right)^{1/2} \sqrt{2m} \right] + \underset{\text{COM}}{p_\phi}\,\phi + \text{const.}$$

$$= S(r, \theta, \phi)$$

is separation solution of H-J equation for

$$H = a(r) + b(\theta)/r^2 + \underbrace{\cancel{c(\phi)/r^2\sin^2\theta}}_{0}$$

separation constants are:

$C_\phi^2 \rightarrow$ sep. const. for ϕ

$\quad \Rightarrow p_\phi^2/2m$ for $c(\phi) = 0$

$C_\Theta^2 \rightarrow$ sep const for θ

$\quad \Rightarrow L^2$ if $b(\theta) = 0$

$E \rightarrow$ sep constant for r

$\quad \rightarrow$ energy.

Finally, can obtain explicit $q(t)$ for r, θ, ϕ from:

$$p_i = \partial S / \partial q_i$$

and

$$p_r = m\dot{r}$$
$$p_\theta = mr^2\dot{\theta}$$
$$p_\phi = m \sin^2\theta \, \dot{\phi}$$