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Derivation of the harmonic oscillator propagator using the Feynman path integral and recursive relations

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Abstract
We present the simplest and most straightforward derivation of the one-dimensional harmonic oscillator propagator, using the Feynman path integral and recursive relations. Our calculations have pedagogical benefits for those undergraduate students beginning to learn the path integral in quantum mechanics, in that they can follow its calculations very simply with only elementary mathematical manipulation. Further, our calculations do not require cumbersome matrix algebra.

(Some figures may appear in colour only in the online journal)

1. Introduction

Feynman constructed the alternative description of quantum mechanics in terms of the path integral [1] on the basis of a suggestion originating from Dirac [2]. Since then, as is well known, the Feynman path integral has been behind brilliant achievements in quantum mechanics and quantum field theory. Physics undergraduates are now obliged to learn it.

In quantum mechanics the exact solutions for Schrödinger equations are quite numerous, in contrast to the small number of exact solutions for path integrals we are familiar with. The one-dimensional harmonic oscillator has an exactly solvable path integral. The simple harmonic oscillator (SHO) is important, not only because it can be solved exactly, but also because a free electromagnetic field is equivalent to a system consisting of an infinite number of SHOs, and the simple harmonic oscillator plays a fundamental role in quantizing electromagnetic field. It also has practical applications in a variety of domains of modern physics, such as molecular spectroscopy, solid state physics, nuclear structure, quantum field theory, quantum statistical mechanics and so on.

A variety of techniques to derive the one-dimensional SHO propagator using the Feynman path integral have been presented in journals [7–13] and textbooks [3–5] and online [14, 15]. Some of the authors emphasize that their derivations are easily accessible and pedagogical for advanced undergraduate students beginning to learn the path integral in quantum mechanics.

Their assertions might be thought to be not quite so easily comprehended by students as the authors state. Since the continuous fraction English and Winter [8] employed to solve the SHO is unfamiliar and very technical for undergraduate students, it might be thought to be not very easy for them to manipulate. The details of the calculations by Itzykson *et al* [4, 5] and Cohen [9], which make use of diagonalizing a matrix, requiring cumbersome matrix algebra, are also involved. Their calculations are therefore not very simple for novice students to follow.

As the techniques for solving path integrals in quantum mechanics have made remarkable progress [6] in recent years, we have been able to solve a great many of them.

Although the problems concerning harmonic oscillator seem to be mature and established, a wide variety of them [16–24] have continued to be addressed in this journal until quite recently. They range broadly through many areas, such as propagators, Laplace transforms, operator methods, a variational Monte Carlo method, and so on.

Nevertheless, we decided to take up the one-dimensional harmonic oscillator again to calculate the SHO with the path integral with the following motivation. Our calculations must be the simplest, be very straightforward to follow and be very easily accessible to undergraduate students. Readers are required to have few prerequisites to trace our approach. To achieve this our approach makes use of recursive relations to perform the multiple integration of the path integral.

By virtue of Feynman [1], the quantum propagator, $K(x', x)$, for the SHO in one dimension from the position x at time t_i to the position x' at time t_f is given as

$$K(x', x) = \int D[x(t)] \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t_i}^{t_f} L(\dot{x}, x)\,\mathrm{d}t\right\}, \tag{1}$$

where

$$L(\dot{x}, x) = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}\omega^2 x^2 \tag{2}$$

is the classical Lagrangian. The symbol $\int D[x(t)]$ represents the integration over all the paths in configuration space joining between x at t_i and x' at t_f .

To perform this integral practically, according to the discretization recipe, we divide time interval $\tau = t_f - t_i$ into N intervals of width ε each such that $\varepsilon = \tau/N$ and we denote $t_j = t_i + (j-1)\varepsilon$ $(j = 1, 2, \ldots, N+1)$. For each point $(x_2, \ldots, x_N)$ in $(N-1)$ dimensional real space R^{N-1}, a so-called path function $x(t)$ is defined by corresponding every t_j $(j = 1, \ldots, N+1)$ to $x_j = x(t_j)$ with $x(t_1) = x$ and $x(t_{N+1}) = x'$ fixed, thereby we get a possible path by joining the successive points (x_j, t_j) on the $x - t$ plane with the segments.

Then equation (1) may be written to

$$K(x', x) = \lim_{N\to\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{d}x_2\cdots\mathrm{d}x_N\left(\frac{1}{2\pi\mathrm{i}\varepsilon}\right)^{N/2}\exp\left\{\mathrm{i}\sum_{n=1}^{N}\varepsilon\mathrm{L}_n\right\}, \tag{3}$$

where

$$\int_{t_i}^{t_f} L(\dot{x}, x)\,\mathrm{d}t = \varepsilon\sum_{n=1}^{N}\mathrm{L}_n$$

$$\mathrm{L}_n = \frac{1}{2}\left(\frac{x_{n+1} - x_n}{\varepsilon}\right)^2 - \frac{1}{2}\omega^2\left(\frac{x_{n+1} + x_n}{2}\right)^2. \tag{4}$$

We have set $\hbar = 1$ and $m = 1$ here for convenience of calculation.

Our aim is to calculate equation (3) as easily as possible. For that purpose let us perform our calculations step by step. Our calculations are made up of a series of elementary mathematical techniques.

The outline is summarized in the two sections after the introduction and the detailed calculations for the outline are performed in the following sections.

2. The quadratic form

According to equation (3),the argument of the exponential is the quadratic form

$$\mathrm{i}\sum_{n=1}^{N}\varepsilon \mathrm{L}_n = \mathrm{i}\varepsilon\left\{\frac{1}{2}\left(\frac{x'-x_N}{\varepsilon}\right)^2-\frac{1}{2}\omega^2\left(\frac{x'+x_N}{2}\right)^2+\cdots+\frac{1}{2}\left(\frac{x_2-x}{\varepsilon}\right)^2-\frac{1}{2}\omega^2\left(\frac{x_2+x}{2}\right)^2\right\}$$
$$= \frac{\mathrm{i}}{2\varepsilon}\left\{(x'-x_N)^2+\cdots+(x_2-x)^2-\frac{\omega^2\varepsilon^2}{4}[(x'+x_N)^2+\cdots+(x_2+x)^2]\right\}. \quad (5)$$

If we put $\alpha = -\omega^2\varepsilon^2/4$, we have the identity

$$(x'-x_N)^2+\cdots+(x_2-x)^2+\alpha[(x'+x_N)^2+\cdots+(x_2+x)^2]$$
$$= (1+\alpha)(x'^2+x^2)+2(1+\alpha)\left[x_N^2+\cdots+x_2^2+2a(x'x_N+\cdots+x_2x)\right], \quad (6)$$

where we have set $2a = -(1-\alpha)/(1+\alpha)$.

Now we consider the identity with the following form:

$$x_N^2+\cdots+x_2^2+2a(x'x_N+\cdots+x_2x) = (a_1x_N+b_1x_{N-1}+c_1x')^2$$
$$+(a_2x_{N-1}+b_2x_{N-2}+c_2x')^2+\cdots$$
$$+(a_{N-1}x_2+b_{N-1}x+c_{N-1}x')^2+g(x,x'). \quad (7)$$

We have to show in later sections that the constant sequential coefficients a_n, b_n, c_n $(n = 1, 2, \ldots, N-1)$ and the function $g(x, x')$ in this identity explicitly exist.

Let us rewrite this expression to the convenient form with the new variables $t_N, t_{N-1}, \ldots, t_2$. To do so we set t_ns as follows:

$$\begin{aligned} t_N &= a_1x_N+b_1x_{N-1}+c_1x' \\ t_{N-1} &= a_2x_{N-1}+b_2x_{N-2}+c_2x' \\ &\vdots \\ t_2 &= a_{N-1}x_2+b_{N-1}x+c_{N-1}x'. \end{aligned} \quad (8)$$

Using ts variables with these equations, equation (7) is rewritten as

$$x_N^2+\cdots+x_2^2+2a(x'x_N+\cdots+x_2x) = t_N^2+t_{N-1}^2+\cdots+t_2^2+g(x,x'). \quad (9)$$

Furthermore, using this relation, equation (6) is rewritten as

$$(x'-x_N)^2+\cdots+(x_2-x)^2+\alpha[(x'+x_N)^2+\cdots+(x_2+x)^2]$$
$$= (1+\alpha)(x'^2+x^2)+2(1+\alpha)\left[x_N^2+\cdots+x_2^2+2a(x'x_N+\cdots+x_2x)\right]$$
$$= (1+\alpha)(x'^2+x^2)+2(1+\alpha)\left[t_N^2+t_{N-1}^2+\cdots+t_2^2+g(x,x')\right]. \quad (10)$$

3. The formulation of the harmonic oscillator propagator

It follows from the transformation (8) and equation (10) that the integral (3) becomes

$$\lim_{\varepsilon\to 0}\int \mathrm{d}x_2\cdots \mathrm{d}x_N\left(\frac{1}{2\pi \mathrm{i}\varepsilon}\right)^{\frac{1}{2}N}\exp\left\{\sum_{n=1}^{N}\mathrm{i}\varepsilon\mathrm{L}_n\right\}$$
$$= \lim_{\varepsilon\to 0}\int\left|\frac{\partial(x_2,\ldots,x_N)}{\partial(t_2,\ldots,t_N)}\right|\mathrm{d}t_2\cdots \mathrm{d}t_N\left(\frac{1}{2\pi\mathrm{i}\varepsilon}\right)^{\frac{1}{2}N}$$
$$\times\exp\left\{\frac{\mathrm{i}}{2\epsilon}\left[(1+\alpha)(x'^2+x^2+2g(x,x'))+2(1+\alpha)\left(t_N^2+t_{N-1}^2+\cdots+t_2^2\right)\right]\right\}, \quad (11)$$

where $\partial(x_2,\ldots,x_N)/\partial(t_2,\ldots,t_N)$ is Jacobian matrix for the transformation (8). From the transformation (8), we have

$$\frac{\partial(t_2,\ldots,t_N)}{\partial(x_2,\ldots,x_N)}=\begin{vmatrix} a_1 & b_1 & 0 & \cdots & 0\\ 0 & a_2 & b_2 & \cdots & 0\\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\\ \cdots\cdots & & 0 & a_{N-2} & b_{N-2}\\ \cdots\cdots & & & 0 & a_{N-1}\end{vmatrix}=a_1a_2\cdots a_{N-1}.$$

With the help of this relation, equation (11) becomes

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int\frac{1}{|a_1a_2\cdots a_{N-1}|}\mathrm{d}t_2\cdots\mathrm{d}t_N\left(\frac{1}{2\pi\mathrm{i}\varepsilon}\right)^{\frac{1}{2}N}\\&\quad\times\exp\left\{\frac{\mathrm{i}}{2\epsilon}\left[(1+\alpha)(x'^2+x^2+2g(x,x'))+2(1+\alpha)\left(t_N^2+t_{N-1}^2+\cdots+t_2^2\right)\right]\right\}\\&\quad=\lim_{\varepsilon\to 0}\frac{1}{|a_1a_2\cdots a_{N-1}|}\left(\frac{1}{2\pi\mathrm{i}\varepsilon}\right)^{\frac{1}{2}N}\\&\quad\times\exp\left\{\frac{\mathrm{i}}{2\epsilon}(1+\alpha)(x'^2+x^2+2g(x,x'))\right\}\left(\int_{-\infty}^{\infty}\mathrm{d}t\,\exp\left\{\frac{\mathrm{i}}{\varepsilon}(1+\alpha)t^2\right\}\right)^{N-1}\\&\quad=\lim_{\varepsilon\to 0}\frac{1}{|a_1a_2\cdots a_{N-1}|}\left(\frac{1}{2\pi\mathrm{i}\varepsilon}\right)^{\frac{1}{2}N}\exp\left\{\frac{\mathrm{i}}{2\varepsilon}(1+\alpha)(x'^2+x^2+2g(x,x'))\right\}\\&\quad\times\left(\sqrt{\frac{\mathrm{i}\pi\varepsilon}{1+\alpha}}\right)^{N-1},\end{aligned}\tag{12}$$

where, to get the last line, we have used the well-known Gaussian integral

$$\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\gamma t^2}\mathrm{d}t=\sqrt{\frac{\mathrm{i}\pi}{\gamma}}.$$

Here you need to keep in mind that the next two sections are devoted to finalizing equation (12).

4. To solve the recursive relations

Now by comparing the both sides of equation (7) we obtain the simultaneous recursive relations:

$$a_1^2=1,\quad a_1b_1=a,\quad a_1c_1=a,\tag{13}$$

$$a_{n+1}^2+b_n^2=1\quad (n=1,2,\ldots,N-2),\tag{14}$$

$$a_nb_n=a\quad (n=1,2,\ldots,N-1),\tag{15}$$

$$b_nc_n+a_{n+1}c_{n+1}=0\quad (n=1,2,\ldots,N-2).\tag{16}$$

Also, if we set $x_N=x_{N-1}=\cdots=x_2=0$ in equation (7), then we obtain

$$g(x,x')=-\left\{b_{N-1}^2x^2+2b_{N-1}c_{N-1}xx'+\left(c_1^2+c_2^2+\cdots+c_{N-1}^2\right)x'^2\right\}.\tag{17}$$

Now let us solve the simultaneous recursive relations equations (13)–(16).

If we eliminate b_n from equation (14) using equation (15), equation (14) becomes

$$a_{n+1}^2 + \frac{a^2}{a_n^2} = 1 \qquad (n = 1, 2, \ldots, N-2). \tag{18}$$

We define a new sequence A_n as $A_n = a_n^2$. By equation (13) we have

$$A_1 = a_1^2 = 1.$$

Then, equation (18) becomes

$$A_{n+1} + \frac{a^2}{A_n} = 1. \tag{19}$$

We take B_n as

$$B_n = \frac{1}{A_n - \beta}, \tag{20}$$

where β satisfies

$$\beta^2 - \beta + a^2 = 0. \tag{21}$$

If we eliminate A_n from equation (19) with equation (20) after substitution of $\beta - \beta^2$ for a^2 in equation (19), we obtain

$$B_{n+1} = \frac{\beta}{1-\beta} B_n + \frac{1}{1-\beta}. \tag{22}$$

With $A_1 = 1$, we have

$$B_1 = \frac{1}{A_1 - \beta} = \frac{1}{1-\beta}. \tag{23}$$

Now we can solve equation (22) for B_n easily

$$B_n = \frac{1 - \left(\frac{\beta}{1-\beta}\right)^n}{1 - 2\beta}. \tag{24}$$

And we also get from equation (20)

$$A_n = \frac{1}{B_n} + \beta = (1-\beta) \frac{1 - \left(\frac{\beta}{1-\beta}\right)^{n+1}}{1 - \left(\frac{\beta}{1-\beta}\right)^n}. \tag{25}$$

We introduce a new variable y defined by

$$y = \beta/(1-\beta). \tag{26}$$

With $A_n = a_n^2$, we obtain

$$a_n = \sqrt{(1-\beta)\frac{1-y^{n+1}}{1-y^n}}, \tag{27}$$

where we have taken the phase as $a_n > 0$.

And also with equation (15), we obtain

$$b_n = \frac{a}{a_n} = a\sqrt{\frac{1-y^n}{(1-\beta)(1-y^{n+1})}}. \tag{28}$$

We shall obtain c_n. With equation (16),

$$\frac{c_{n+1}}{c_n} = -\frac{b_n}{a_{n+1}},$$

then

$$\begin{aligned}c_n &= \frac{c_n}{c_{n-1}} \cdot \frac{c_{n-1}}{c_{n-2}} \cdots \frac{c_2}{c_1} \cdot \frac{c_1}{1} \\ &= \left(-\frac{b_{n-1}}{a_n}\right) \cdots \left(-\frac{b_1}{a_2}\right) \cdot \frac{a}{a_1} \\ &\quad \text{(we have used } a_1 c_1 = a) \\ &= (-1)^{n-1} \frac{b_{n-1} b_{n-2} \cdots b_1}{a_n a_{n-1} \cdots a_1} a \\ &= (-1)^{n-1} \frac{a^n}{a_n (a_{n-1} \cdots a_1)^2},\end{aligned} \tag{29}$$

where we have made use of equation (15) to obtain the last line. By using equation (27), we obtain

$$a_1 a_2 \cdots a_{N-1} = (1-\beta)^{\frac{N-1}{2}} \sqrt{\frac{1-y^N}{1-y}} \tag{30}$$

with the help of equations (27), (29) and (30), we obtain

$$\begin{aligned}c_n &= (-1)^{n-1} \frac{a^n}{a_n (a_{n-1} \cdots a_1)^2} \\ &= (-1)^{n-1} \left(\frac{a}{1-\beta}\right)^n (1-2\beta) \cdot \sqrt{\frac{1}{(1-\beta)(1-y^n)(1-y^{n+1})}},\end{aligned} \tag{31}$$

where we have used $1 - y = (1-2\beta)/(1-\beta)$. Furthermore, we have

$$\begin{aligned}c_n^2 &= \frac{(1-2\beta)^2}{1-\beta} \cdot \frac{\left(\frac{a}{1-\beta}\right)^{2n}}{(1-y^n)(1-y^{n+1})} \\ &= \frac{(1-2\beta)^2}{1-\beta} \cdot \frac{y^n}{(1-y^n)(1-y^{n+1})} \\ &= (1-2\beta) \left(\frac{1}{1-y^n} - \frac{1}{1-y^{n+1}}\right),\end{aligned} \tag{32}$$

where we have used $\{a/(1-\beta)\}^2 = y$ and $(1-\beta)(1-y) = 1-2\beta$. We shall obtain other useful relations:

$$\begin{aligned}\sum_{n=1}^{N-1} c_n^2 &= (1-2\beta) \sum_{n=1}^{N-1} \left(\frac{1}{1-y^n} - \frac{1}{1-y^{n+1}}\right) \\ &= (1-\beta) \frac{y-y^N}{1-y^N},\end{aligned} \tag{33}$$

where we have used $(1-2\beta)/(1-y) = 1-\beta$. From equation (28)

$$\begin{aligned}b_{N-1}^2 &= \frac{a^2}{1-\beta} \frac{1-y^{N-1}}{1-y^N} \\ &= (1-\beta) \cdot \frac{y-y^N}{1-y^N},\end{aligned} \tag{34}$$

where we have used $\beta^2 - \beta + a^2 = 0$ and $\beta = (1-\beta)y$.

With equations (28) and (31),

$$2b_{N-1} c_{N-1} = 2(-1)^N (1-2\beta) \left(\frac{a}{1-\beta}\right)^N \frac{1}{1-y^N}. \tag{35}$$

5. Preliminary computations

We shall calculate the function $g(x, x')$ given by equation (17) explicitly, using the results of equations (33)–(35),

$$g(x,x') = -\left\{b_{N-1}^2 x^2 + 2b_{N-1}c_{N-1}xx' + \left(c_1^2 + c_2^2 + \cdots + c_{N-1}^2\right)x'^2\right\}$$
$$= -\left\{(x^2+x'^2)(1-\beta)\cdot\frac{y-y^N}{1-y^N} + 2(-1)^N(1-2\beta)\left(\frac{a}{1-\beta}\right)^N\frac{1}{1-y^N}\cdot xx'\right\}. \tag{36}$$

Note that we before defined as $\alpha = -\omega^2\varepsilon^2/4$ and $2a = -(1-\alpha)/(1+\alpha)$. Since $\alpha < 0$ and $4a^2 = \{(1-\alpha)/(1+\alpha)\}^2 > 1$, we have $1 - 4a^2 < 0$.

$\beta^2 - \beta + a^2 = 0$ yields

$$\beta = \frac{1 \pm \mathrm{i}\sqrt{4a^2-1}}{2} = |a|\,\mathrm{e}^{\pm\mathrm{i}\theta}, \tag{37}$$

where $|\beta| = |a|$, $2|a| = (4+\omega^2\varepsilon^2)/(4-\omega^2\varepsilon^2)$ and we have set

$$|a|\cos\theta = \frac{1}{2}, \qquad |a|\sin\theta = \frac{\sqrt{4a^2-1}}{2} \tag{38}$$

and also $1 - \beta = a^2/\beta$ yields

$$1-\beta = |a|\mathrm{e}^{\mp\mathrm{i}\theta} \tag{39}$$

$$y = \frac{\beta}{1-\beta} = \mathrm{e}^{\pm 2\mathrm{i}\theta}. \tag{40}$$

Now we shall represent each term of the $g(x, x')$, equation (36), with θ using equations (37), (39) and (40):

$$(1-\beta)\cdot\frac{y-y^N}{1-y^N} = |a|\frac{\sin(N-1)\theta}{\sin N\theta} \tag{41}$$

and also,

$$2(-1)^N(1-2\beta)\left(\frac{a}{1-\beta}\right)^N\frac{1}{1-y^N} = 2|a|\frac{\sin\theta}{\sin N\theta}. \tag{42}$$

Next we go to equation (30),

$$a_1a_2\cdots a_{N-1} = (1-\beta)^{\frac{N-1}{2}}\sqrt{\frac{1-y^N}{1-y}} = |a|^{\frac{N-1}{2}}\sqrt{\frac{\sin N\theta}{\sin\theta}}. \tag{43}$$

We go on to equation (36). Using equations (41) and (42), we obtain

$$g(x,x') = -\left\{(x^2+x'^2)(1-\beta)\cdot\frac{y-y^N}{1-y^N} + 2(-1)^N(1-2\beta)\left(\frac{a}{1-\beta}\right)^N\frac{1}{1-y^N}\cdot xx'\right\}$$
$$= -\left\{(x^2+x'^2)|a|\frac{\sin(N-1)\theta}{\sin N\theta} + 2xx'|a|\frac{\sin\theta}{\sin N\theta}\right\}. \tag{44}$$

Then

$$(x^2+x'^2) + 2g(x,x') = (x^2+x'^2)\left(1 - 2|a|\frac{\sin(N-1)\theta}{\sin N\theta}\right) - 4xx'|a|\frac{\sin\theta}{\sin N\theta}. \tag{45}$$

If we relate this expression to equation (11), then we have

$$\frac{\mathrm{i}}{2\varepsilon}(1+\alpha)\{(x^2+x'^2)+2g(x,x')\}$$
$$= \frac{\mathrm{i}}{2\varepsilon}\left(1-\frac{\omega^2\varepsilon^2}{4}\right)\left\{(x^2+x'^2)\left(1-2|a|\frac{\sin(N-1)\theta}{\sin N\theta}\right) - 4xx'|a|\frac{\sin\theta}{\sin N\theta}\right\}. \tag{46}$$

And we also have with equation (43)

$$\frac{1}{|a_1a_2\cdots a_{N-1}|}\left(\frac{1}{2\pi\mathrm{i}\varepsilon}\right)^{\frac{1}{2}N}\left(\sqrt{\frac{\mathrm{i}\pi\varepsilon}{1+\alpha}}\right)^{N-1}=\left(\frac{1}{1+\frac{\omega^2\varepsilon^2}{4}}\right)^{\frac{N-1}{2}}\sqrt{\frac{\sin\theta}{2\pi\mathrm{i}\varepsilon\sin N\theta}}. \tag{47}$$

6. The harmonic oscillator propagator

Now let us get back to equation (12) to obtain the final solution. By combining equations (46) and (47), equation (12) becomes

$$\lim_{\varepsilon\to 0}\frac{1}{|a_1a_2\cdots a_{N-1}|}\left(\frac{1}{2\pi\mathrm{i}\varepsilon}\right)^{\frac{1}{2}N}\left(\sqrt{\frac{\mathrm{i}\pi\varepsilon}{1+\alpha}}\right)^{N-1}\exp\left\{\frac{\mathrm{i}}{2\epsilon}(1+\alpha)(x'^2+x^2+2g(x,x'))\right\}$$
$$=\left(\frac{\omega}{2\mathrm{i}\pi\sin\omega\tau}\right)^{\frac{1}{2}}\exp\left(\frac{\mathrm{i}\omega}{2\sin\omega\tau}\{(x^2+x'^2)\cos\omega\tau-2xx'\}\right), \tag{48}$$

where to get the last line, we have used the fact that for large N,

$$\theta\cong\omega\varepsilon=\omega\frac{\tau}{N}. \tag{49}$$

This is the final result we want to obtain.

7. Concluding remarks

Although our analytical approach requires many steps of calculation, these are simple and straightforward and we believe that most undergraduates could follow our approach easily with elementary mathematical manipulation and it may be of interest to those instructors who would like to introduce the path integral into their courses. That is why we stress the accessibility of our calculation. The essential points in this paper are equation (7) and the determination of its sequential coefficients, a_n, b_n, and $c_n (n = 1, 2, \ldots, N-1)$.

Itzykson *et al* [4, 14], Cohen [9] and we are conceptually equivalent in terms of diagonalizing the quadratic form, the left side of equation (7). But we think our procedure is more self-contained and less involved than the others and does not require cumbersome matrix algebra.

References

[1] Feynman R P 1948 Space-time approach to non relativistic quantum mechanics *Rev. Mod. Phys.* **20** 367–87
[2] Dirac P A M 1933 The Lagrangian in quantum mechanics *Phys. Z. Sowjetunion* **3** 64–72
[3] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)
[4] Itzykson C and Zuber J-B 1985 *Quantum Field Theory* (New York: McGraw-Hill) pp 430–2 (International Student Edition)
[5] Das A 1993 *Field Theory: A Path Integral Approach* (Singapore: World Scientific) pp 43–70
[6] Grosche C and Steiner F 1995 How to solve path integral in quantum mechanics *J. Math. Phys.* **36** 2354–85
[7] Marshall J T and Pell J L 1979 Path-integral evaluation of the space-time propagator for quadratic Hamiltonian systems *J. Math. Phys.* **20** 1297–302
[8] English L Q and Winters R R 1997 Continued fractions and the harmonic oscillator using Feynman path integral *Am. J. Phys.* **65** 390–3
[9] Cohen S M 1997 Path integral for the quantum harmonic oscillator using elementary methods *Am. J. Phys.* **66** 537–40
[10] Holstein B R 1998 The harmonic oscillator propagator *Am. J. Phys.* **66** 583–9
[11] Barone F A, Boschi-Filho H and Farina C 2003 Three methods for calculating the Feynman propagator *Am. J. Phys.* **71** 483–91
[12] Moriconi L 2004 An elementary derivation of the harmonic oscillator propagator *Am. J. Phys.* **72** 1258

[13] Moshinsky M, Sadurn E and del Campo A 2007 Alternative method for determining the Feynman propagator of a non-relativistic quantum mechanical problem *SIGMA* **3** 110–21

[14] Murayama H *221A Lecture Notes Path Integral* (Berkeley, CA: Department of Physics, University of California) http://hitoshi.berkeley.edu/221a/pathintegral.pdf

[15] Wikberg E 2006 *Path Integral in Quantum Mechanics Project Work 4p* (Stockholm: Department of Physics, Stockholm University) www.physto.se/∼emma/PathIntegrals.pdf

[16] de Souza Dutra A and de Castro A S 1989 An alternative method to calculate propagators *Eur. J. Phys.* **10** 194–6

[17] Witschel W 1996 Harmonic oscillator integrals by operator methods, a simple unified approach *Eur. J. Phys.* **17** 357–62

[18] Pottorf S, Pudzer A, Chou M Y and Hasbun J E 1999 The simple harmonic oscillator ground state using a variational Monte Carlo method *Eur. J. Phys.* **20** 205–12

[19] Denny M 2002 Harmonic oscillator quantization: kinetic theory approach *Eur. J. Phys.* **23** 183–90

[20] Andrews D L and Romero L C D 2009 A back-to-front derivation: the equal spacing of quantum levels is a proof of simple harmonic oscillator physics *Eur. J. Phys.* **30** 1371–80

[21] Fernndez F M 2010 Simple one-dimensional quantum-mechanical model for a particle attached to a surface *Eur. J. Phys.* **31** 961–8

[22] Nita G M 2010 A simple mechanical model for the isotropic harmonic oscillator *Eur. J. Phys.* **31** 1031–6

[23] Pimentel D R M and de Castro A S 2013 A Laplace transform approach to the quantum harmonic oscillator *Eur. J. Phys.* **34** 199–206

[24] Banerjee D and Bhattacharjee J K 2013 Study of some model quantum systems in two dimensions *Eur. J. Phys.* **34** 435–48