FitzHugh – Nagumo Model (LANG CUI)

— Waves in Modes with mult-steady states.

I. Introduction.

$\Rightarrow$ spreading $\leftrightarrow$ Non-locality.

$\Rightarrow$ avalanching $\leftrightarrow$ burgers. etc

$\Rightarrow$ intensity field $\leftrightarrow$ "$k-\epsilon$", $\frac{\partial k}{\partial t} - \nabla \cdot D(k) \cdot \nabla k = S_0 - \frac{k^{3/2}}{\ell}$

$\Rightarrow$ reaction – diffusion

Fisher Eqn: $\frac{\partial}{\partial t}\varepsilon - \frac{\partial}{\partial x}\varepsilon\frac{\partial \varepsilon}{\partial x} = \gamma\varepsilon - \alpha\varepsilon^2$

i unistable;

ii 2^{nd} order transition

iii $C \sim \sqrt{\gamma D}$

iv leading edge.

$\Rightarrow$ Bi-stable $\rightarrow$ "threshold"

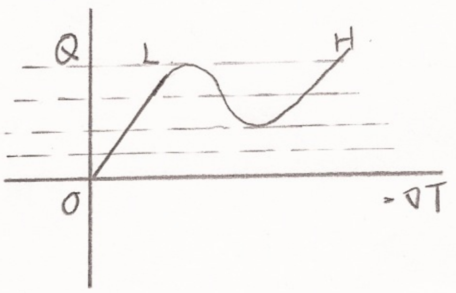
cubic: $\frac{\partial}{\partial t}\varepsilon - \frac{\partial}{\partial x} D_0 \varepsilon \frac{\partial \varepsilon}{\partial x} = \gamma(\nabla T)\varepsilon - \alpha\varepsilon^2 = -\alpha\varepsilon^2 + \left[\gamma_T L\left(\frac{Q}{\chi_0 + D_0\varepsilon} - \nabla T_{crit}\right)Q + \gamma_0\right]\varepsilon$

$Q = (\chi_0 + D_0\varepsilon)\nabla T$ $\rightarrow$ "switch"

Bi-stable Models:

$$Q = \frac{-\chi_T \nabla T}{1 + \alpha V_E'^2} - \chi_{neo} \nabla T$$

$$V_E' \rightarrow E + V \times B = -\frac{\nabla P}{\eta}.$$



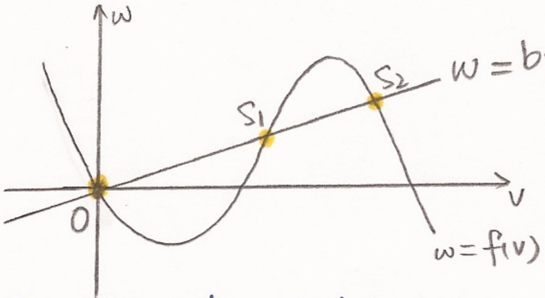
"What does bi-stable looks like?"

$\Rightarrow$ Fitzhugh – Nagumo Eqn. $\leftrightarrow$ reduction of "Hodgkin–Huxley"

$$\begin{cases} \dfrac{\partial V}{\partial t} = \overset{\text{"fast"}}{f(V) - W + I_a} \\[2ex] \dfrac{\partial W}{\partial t} = \overset{\text{"slow"}}{bV - \gamma W} \\[2ex] f(V) = V(a-V)(V-1) \end{cases}$$

($0 < a < 1$, b, γ positive constant).

$I_a \neq 0$:



three possible steady states;

1 unstable, S_1

2 stable but excitable, $(0,0)$, S_2

$\Rightarrow$ waves in Models with Multi-steady State : "Spread".

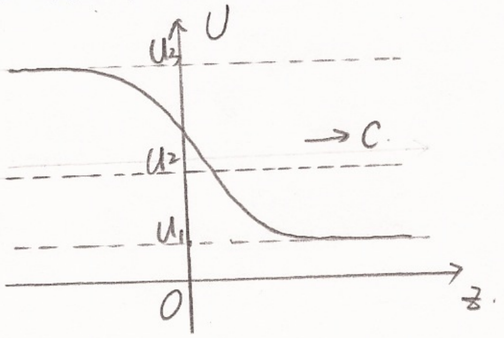
$$\frac{\partial u}{\partial t} = f(u) + D\frac{\partial^2 u}{\partial x^2}$$

$$f(u) = A(u-u_1)(u_2-u)(u-u_3)$$

in the case of: wave moves with a unique speed C:

the Solution $U(z)$:

$$U(-\infty) = U_3, \quad U(\infty) = U_1$$



Solution for c: (speed) : "travelling wave:"

$$u = u(x-ct).$$

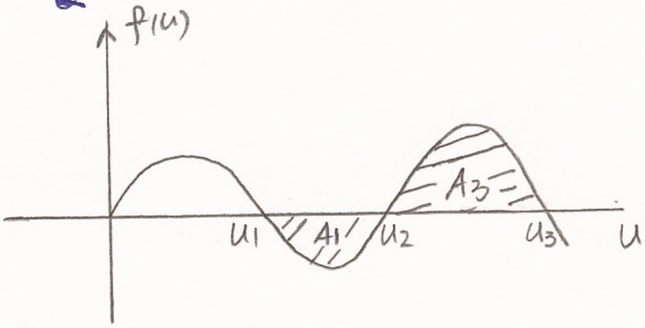
$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u) \Rightarrow -c\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2} + f(u) \cdots \quad (*)$$

$$\Rightarrow \; u' \cdot (*) \Rightarrow -c u'^2 = D u'' u' + u' f(u)$$

$$\Rightarrow -c\int_{u_3}^{u_1} u'^2 = D\int_{u_3}^{u_1} u'u'' + \int_{u_3}^{u_1} f(u)\,du$$

$$\boxed{c = \frac{-\int_{u_3}^{u_1} f(u)\,du}{\int_{u_3}^{u_1} [u']^2} = \text{"speed"} \quad !}$$

$$c \gtreqless 0 \text{ if } \int_{u_1}^{u_3} f(u)\,du \gtreqless 0.$$



Discussion: if $A_3 > A_1 \Rightarrow c > 0$: move to u_1

$A_3 < A_1 \Rightarrow c < 0$: move to u_3

$A_3 = A_1 \Rightarrow$ stationary.

* Akin to "Maxwell criterion", "phase coexsitance".

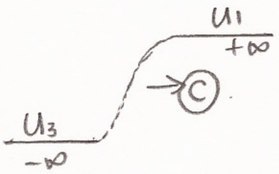
"How to Calculate speed $\underline{\underline{c}}$?"

$\Rightarrow$ reaction - diffusion eqn:

$$\begin{cases} \dfrac{\partial u}{\partial t} = A(u-u_1)(u_2-u)(u-u_3) + D\dfrac{\partial^2 u}{\partial x^2} \\ u(x,t) = U(z) = U(x-ct) \\ U(-\infty) = u_3 , \quad U(\infty) = u_1 \end{cases}$$

above gives:

$$L(u) = Du'' + cu' + A(u-u_1)(u_2-u)(u-u_3) = 0 \quad \cdots (\triangle)$$



try to make u fit:

$$\boxed{u' = a(u-u_1)(u-u_3)}$$

Substituting u' into $(\triangle)$:

$$L(u) = (u-u_1)(u-u_3)[Da^2(2u-u_1-u_3) + ca - A(u-u_2)]$$

$$= (u-u_1)(u-u_3)\{(2Da^2-A)u - [Da^2(u_1+u_3) - ca - Au_2]\}$$

For $L(u) = 0$: must have

$$\begin{cases} 2Da^2 - A = 0 \\ Da^2(u_1+u_3) - ca - Au_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a = \left(\dfrac{A}{2D}\right)^{1/2} \\ c = \left(\dfrac{AD}{2}\right)^{1/2}(u_1 - 2u_2 + u_3) \end{cases}$$

when $\dfrac{u_1+u_3}{2} = u_2 \Rightarrow$ steady coexistence.

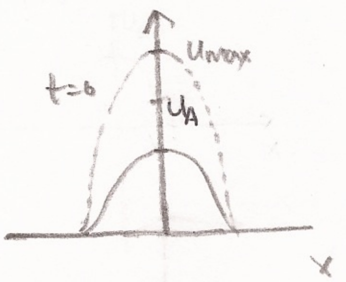
" Discussion of F–N models "

— waves in Excitable Media"

$$(\star)\quad \begin{cases} \dfrac{\partial u}{\partial t} = f(u) - v + D\dfrac{\partial^2 u}{\partial x^2} \\[2ex] \dfrac{\partial v}{\partial t} = bu - \gamma v \\[2ex] f(u) = (a-u)(u-1)u \end{cases} \qquad (0<a<1,\ b\ \&\ \gamma \text{ positive}).$$

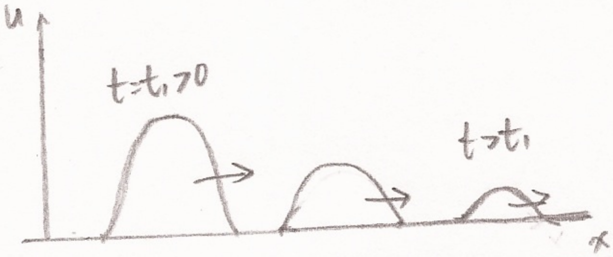
" to demonstrate how travelling wave solutions arise for reaction diffusion systems with excitable kinetics ".

B.C. satisfy: $u \to 0,\ u' \to 0,\ v \to 0$ as $|z| \to \infty$



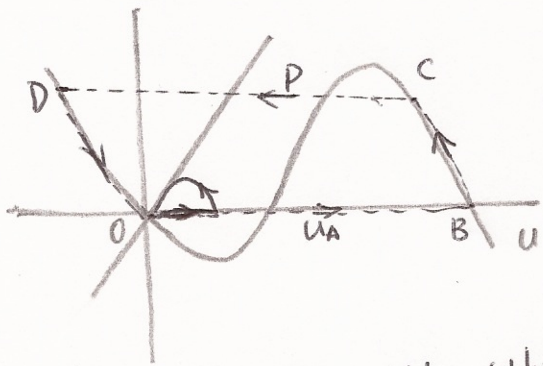
(a)

perturbation u: $u_{max} < u_A$



(b)

the solution is decaying transient



(c) solid line: $u_{max} < u_A$ (threshold).

dash line: $u_{max} > u_A$
(OBCDO)

(Cont).

— "wave in Excitable Media"

consider with b, γ small:

$b = \varepsilon L$, $\gamma = \varepsilon M$, $0 < \varepsilon \leq 1$

Eqn (✡) becomes:

$$\begin{cases} u_t = D u_{xx} + f(u) - v \\ v_t = \varepsilon (Lu - Mv) \end{cases}$$

1. in the limiting $\varepsilon \to 0$: $v \sim$ constant $\to 0$.

thus: $u_t = D u_{xx} + f(u)$

$f(u) = u(a-u)(u-1)$ $\Rightarrow$ $\begin{cases} u=0 \to \text{stable} \\ u=a \to \text{unstable} \\ u=1 \to \text{stable} \end{cases}$

$\Rightarrow$ wave speed:

$$\boxed{c = \left(\frac{D}{2}\right)^{1/2} (1-2a), \quad c \gtreqless 0 \text{ if } \int_0^1 f(u)\,du \gtreqless 0}$$

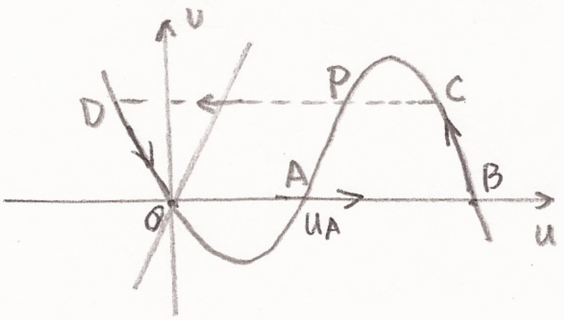
2. on the trajectory $v \approx v_c$:

$u_t = D u_{xx} + f(u) - v_c$:

the solution becomes:

$u = u(z)$, $z = x - ct$, $u(-\infty) = u_D$, $u(\infty) = u_C$.

$$c = \left(\frac{D}{2}\right)^{1/2} (u_C - 2u_P + u_D)$$



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