PHYSICS 2, GENERAL RELATIVITY WINTER 2014
DEPARTMENT OF PHYSICS, UCSD
BENJAMIN GRINSTEIN

Assignment 3

February 12, 2014 (Due Monday February 24, 2014)

1. This problem is designed to flex your muscles with FRW cosmology:

(i) For a matter dominated universe, show that

$$\frac{k}{a^2} = (2q-1)H^2, \quad \text{and} \quad \frac{8\pi G\rho}{3} = 2qH^2, \tag{1}$$

while for a radiation dominated one,

$$\frac{k}{a^2} = (q-1)H^2, \quad \text{and} \quad \frac{8\pi G\rho}{3} = qH^2. \tag{2}$$

Here, as in class, a, H and q stand for the FRW-scale parameter, the Hubble parameter and the deceleration parameter, respectively.

(ii) The previous results can (trivially) be used in the case $k \neq 0$ to express the scale factor $a_0 = a(t_0)$ today in terms of the presently measured Hubble and deceleration constants, H_0 and q_0. Why is this not possible for $k = 0$ (give a physical explanation)?

(iii) Solve the Friedmann equation

$$H^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} \tag{3}$$

for a assuming the density is matter dominated. You can see that straightforward integration gives an equation for $t(a)$ which cannot be inverted analytically to give $a(t)$. However, if you compute a in terms of conformal time, $d\eta = dt/a(t)$, you will find $a(\eta)$ analytically. Give also $t = t(\eta)$. Express the result in terms of H_0 and q_0.
Not required, but not much more work, do the same for a radiation dominated universe.

(iv) Now assume $k = 1$ and the universe is always matter dominated. What is the total proper volume of the universe today?

(v) Under the same assumptions, what is the proper volume of the universe we can see? Let me explain what I mean by this. We "see" photons (or electromagnetic waves, if you want to think classically). A photon that we see today can originate from variously distant sources, with correspondingly varying travel times. For example, photons that were emitted form a spherical shell of comoving radius r and comoving thickness δr come from a shell of proper volume equal to the proper thickness of the shell times the proper area of the spherical shell (comoving volume $4\pi r^2 \delta r$). To determine this physical volume note that these photons arrive today, at t_0, if they were emitted at some specific time t, a condition that gives the scale factor $a(t)$ at the time of emission. Now sum over all shells from the initial time ($t = 0$) to now.

(vi) From the previous question you should know what is the farthest distance we can see today. What is the proper volume *today* of the universe we can see?

2. Consider the angular size of an object observed in the sky. If the universe were Euclidean (flat spacetime) objects of a given physical size would subtend an angle inversely proportional to the distance from us. But things are different in a curved universe. Let $\theta(z)$ be the angular size of an object at redshift z of proper size (diameter) D. Assume the universe is matter dominated and flat. Show that $\theta(z)$ has a minimum. For what redshift? Why?
Next do some numerics. Galaxies are typically (at least) 10 kpc in size. What is the minimum angular size of galaxies (in this matter dominated, flat universe)? You will need a value for the Hubble parameter. Use $H_0 = 100\,\mathrm{km/s/\,Mpc} \times h$, where $h \approx 0.7$ parametrizes our ignorance of a precise value of H_0.

3. Consider flat ($k = 0$), matter dominated FRW cosmology. Suppose it contains homogeneously distributed point sources of constant (in time) luminosity L and that the local number density of sources today is n_0. Calculate the brightness of the sky today, B_0 = energy per solid angle of sky per collection area per unit time. Look up Olber's paradox, and comment.

4. DeSitter and FRW:

(i) Show that the DeSitter metric

$$ds^2 = -d\hat{t}^2 + e^{2\hat{t}/\alpha}(d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2) \tag{4}$$

solves the Friedmann equations. What are the curvature parameter k and the Hubble parameter H? What are the energy and pressure?

(ii) When we discussed DeSitter spacetime in class, we found that the metric (4) by making a transformation to the hat coordinates $(\hat{t}, \hat{x}, \hat{y}, \hat{z})$ from no-hat coordinates and pointed out that the hat coordinates do not cover all of DeSitter space. We will check that here:

Find the geodesics for massive bodies in the spacetime (4). Invert them to give the proper time along the geodesic as a function of time, $\tau = \tau(t)$, and show that it tends to a constant as $t \to -\infty$, which shows what we already knew, that the coordinates fail to cover the whole spacetime. What happens as $t \to +\infty$?