Department of Physics, UCSD
Physics 225B, General Relativity
Winter 2014
Homework 1, solutions

1. (a) From the Killing equation, $\nabla_\rho K_\sigma + \nabla_\sigma K_\rho = 0$ taking one derivative,

$$\begin{aligned}
\nabla_\mu \nabla_\rho K_\sigma + \nabla_\mu \nabla_\sigma K_\rho &= 0 \\
-\nabla_\sigma \nabla_\mu K_\rho - \nabla_\sigma \nabla_\rho K_\mu &= 0 \\
\nabla_\rho \nabla_\mu K_\sigma + \nabla_\rho \nabla_\sigma K_\mu &= 0.
\end{aligned}$$

Adding these we have

$$(\nabla_\mu \nabla_\rho + \nabla_\rho \nabla_\mu) K_\sigma = [\nabla_\sigma, \nabla_\mu] K_\rho + [\nabla_\sigma, \nabla_\rho] K_\mu.$$

Add $[\nabla_\mu, \nabla_\rho] K_\sigma$ to both sides to obtain (twice) what we are looking for in terms of commutators of derivatives:

$$2\nabla_\mu \nabla_\rho K_\sigma = [\nabla_\sigma, \nabla_\mu] K_\rho + [\nabla_\sigma, \nabla_\rho] K_\mu + [\nabla_\mu, \nabla_\rho] K_\sigma.$$

Fro the very definition of curvature we have, for any vector field A^λ,

$$[\nabla_\mu, \nabla_\sigma] A_\rho = R_{\rho\lambda\mu\sigma} A^\lambda$$

so we have immediately that

$$\nabla_\mu \nabla_\rho K_\sigma = \tfrac{1}{2}(R_{\rho\lambda\sigma\mu} + R_{\mu\lambda\sigma\rho} + R_{\sigma\lambda\mu\rho}) K^\lambda.$$

Now we just need to shuffle indices around a bit. The second term is already of the form we want, $R_{\mu\lambda\sigma\rho} = R_{\sigma\rho\mu\lambda}$. Write the first term as $R_{\rho\lambda\sigma\mu} = R_{\sigma\mu\rho\lambda}$ and combine with the third term using anti-symmetry in the last three indices, $R_{\sigma\mu\rho\lambda} + R_{\sigma\lambda\mu\rho} = -R_{\sigma\rho\lambda\mu} = R_{\sigma\rho\mu\lambda}$, which doubles the second term and the result follows,

$$\nabla_\mu \nabla_\rho K_\sigma = R_{\sigma\rho\mu\lambda} K^\lambda. \tag{0.1}$$

(b) We are going to use the Bianchi identity $\frac{1}{2}\nabla_\mu R = \nabla_\lambda R^\lambda{}_\mu$. Contracting indices in the solution to part (a), Eq. (0.1) , we have $\nabla_\mu \nabla_\rho K_\mu = R_{\rho\lambda} K^\lambda$, and taking ∇^ρ of this,

$$\nabla^\rho \nabla_\mu \nabla_\rho K^\mu = \nabla^\rho (R_{\rho\lambda} K^\lambda) = (\nabla^\rho R_{\rho\lambda}) K^\lambda + R_{\rho\lambda} \nabla^\rho K^\lambda. \tag{0.2}$$

Since the Ricci tensor is symmetric in its indices we can replace $\frac{1}{2}\nabla^{(\rho}K^{\lambda)}$ for $\nabla^\rho K^\lambda$ in the last term on the right hand side, buy by Killing's equation $\nabla^{(\rho}K^{\lambda)} = 0$.

So to prove that $(\nabla^\rho R_{\rho\lambda})K^\lambda = 0$ we have to show that the left hand side in (0.2) vanishes:

$$\begin{aligned}
\nabla^\rho\nabla^\mu\nabla_\rho K_\mu &= [\nabla^\rho, \nabla^\mu]\nabla_\rho K_\mu && \text{(again, since } \nabla_{(\rho}K_{\mu)} = 0\text{)} \\
&= [\nabla_\rho, \nabla_\mu]\nabla^\rho K^\mu \\
&= R^\mu{}_{\rho\mu\lambda}\nabla^\rho K^\lambda + R^\lambda{}_{\rho\mu\lambda}\nabla^\mu K^\rho \\
&= R_{\rho\lambda}\nabla^\rho K^\lambda - R_{\rho\mu}\nabla^\mu K^\rho \\
&= 0. && \text{(once again, since } \nabla_{(\rho}K_{\mu)} = 0\text{)}
\end{aligned}$$

So $(\nabla^\rho R_{\rho\lambda})K^\lambda = 0$, or, by the Bianchi identity, $(\nabla_\lambda R)K^\lambda = 0$

2.(a) The pullback is

$$\hat{g}_{ij} = \frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j}g_{\mu\nu},$$

where x^μ are coordinates on $\mathbb{R}^3$ and y^i are coordinates on $\mathcal{P}$, and the map $x^\mu(y^i)$ is

$$\begin{aligned}
x &= \rho\cos\phi \\
y &= \rho\sin\phi \\
z &= \rho^2
\end{aligned}$$

Compute the derivatives:

$$\begin{aligned}
\frac{\partial x}{\partial\rho} &= \cos\phi & \frac{\partial y}{\partial\rho} &= \sin\phi & \frac{\partial z}{\partial\rho} &= 2\rho \\
\frac{\partial x}{\partial\phi} &= -\rho\sin\phi & \frac{\partial y}{\partial\phi} &= \rho\cos\phi & \frac{\partial z}{\partial\phi} &= 0
\end{aligned}$$

Using $g_{\mu\nu} = \delta_{\mu\nu}$ we have:

$$\begin{aligned}
\hat{g}_{\rho\rho} &= \frac{\partial x^\mu}{\partial\rho}\frac{\partial x^\nu}{\partial\rho}\delta_{\mu\nu} = \cos^2\phi + \sin^2\phi + 4\rho^2 = 1 + 4\rho^2 \\
\hat{g}_{\rho\phi} &= \hat{g}_{\phi\rho} = \frac{\partial x^\mu}{\partial\rho}\frac{\partial x^\nu}{\partial\phi}\delta_{\mu\nu} = \cos\phi(-\rho\sin\phi) + \sin\phi(\rho\cos\phi) + 2\rho(0) = 0 \\
\hat{g}_{\phi\phi} &= \frac{\partial x^\mu}{\partial\phi}\frac{\partial x^\nu}{\partial\phi}\delta_{\mu\nu} = (-\rho\sin\phi)^2 + (\rho\cos\phi)^2 + (0)^2 = \rho^2
\end{aligned}$$

which is summarized as

$$\boxed{d\hat{s}^2 = (1+4\rho^2)\mathrm{d}\rho^2 + \rho^2\mathrm{d}\phi^2}$$

(b)As a matrix, the inverse of $\hat{g}_{ij}$ is

$$\hat{g}^{ij} = \begin{pmatrix} \frac{1}{1+4\rho^2} & 0 \\ 0 & \frac{1}{\rho^2} \end{pmatrix}.$$

The push-forward, which i will denote by $\tilde{g}^{\mu\nu}$, generally, is given by

$$\tilde{g}^{\mu\nu} = \frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j}\hat{g}^{ij}.$$

Computing:

$$\tilde{g}^{xx} = \left(\frac{\partial x}{\partial \rho}\right)^2 \frac{1}{1+4\rho^2} + \left(\frac{\partial x}{\partial \phi}\right)^2 \frac{1}{\rho^2} = \frac{\cos^2\phi}{1+4\rho^2} + \sin^2\phi = \frac{1}{z}\left[\frac{x^2}{1+4z} + y^2\right]$$

$$\tilde{g}^{xy} = \frac{\partial x}{\partial \rho}\frac{\partial y}{\partial \rho}\frac{1}{1+4\rho^2} + \frac{\partial x}{\partial \rho}\frac{\partial y}{\partial \rho}\frac{1}{\rho^2} = \frac{\sin\phi\cos\phi}{1+4\rho^2} - \sin\phi\cos\phi = -\frac{4\rho^2\sin\phi\cos\phi}{1+4\rho^2} = -\frac{4xy}{1+4z}$$

$$\tilde{g}^{xz} = \frac{\partial x}{\partial \rho}\frac{\partial z}{\partial \rho}\frac{1}{1+4\rho^2} + \frac{\partial x}{\partial \rho}\frac{\partial z}{\partial \rho}\frac{1}{\rho^2} = \frac{2\rho\cos\phi}{1+4\rho^2} = \frac{2x}{1+4z}$$

$$\tilde{g}^{yz} = \frac{\partial y}{\partial \rho}\frac{\partial z}{\partial \rho}\frac{1}{1+4\rho^2} + \frac{\partial y}{\partial \rho}\frac{\partial z}{\partial \rho}\frac{1}{\rho^2} = \frac{2\rho\sin\phi}{1+4\rho^2} = \frac{2y}{1+4z}$$

$$\tilde{g}^{yy} = \left(\frac{\partial y}{\partial \rho}\right)^2 \frac{1}{1+4\rho^2} + \left(\frac{\partial y}{\partial \phi}\right)^2 \frac{1}{\rho^2} = \frac{\sin^2\phi}{1+4\rho^2} + \cos^2\phi = \frac{1}{z}\left[\frac{y^2}{1+4z} + x^2\right]$$

$$\tilde{g}^{zz} = \left(\frac{\partial z}{\partial \rho}\right)^2 \frac{1}{1+4\rho^2} + \left(\frac{\partial z}{\partial \phi}\right)^2 \frac{1}{\rho^2} = \frac{4\rho^2}{1+4\rho^2} = \frac{4z}{1+4z}$$

Keep in mind that these are defined only on the sub-manifold $\mathcal{P}$ (so, in a sense, it is better to keep the expressions for $\tilde{g}$ as given in terms of ρ and ϕ.

(c) Not much to do here. $g^{\mu\nu} = \delta^{\mu\nu}$ is very different from $\tilde{g}^{\mu\nu}$, but there was no reason to expect them to be the same.

3. (a) The integral curve of $V^\mu(x) = x^\mu$ is the solution to

$$\frac{dx^\mu(t)}{dt} = V^\mu(x(t)) = x^\mu(t).$$

We want a solution that satisfies $x^\mu(0) = x^\mu_o$. The integral is simple:

$$x^\mu(t) = x^\mu_o e^t.$$

For an integral curve through the origin, $x_o^\mu = 0$, the solution above is not a curve but just a map from the real line to a single point. The reason is that at that point the tangent field vanishes: the integral curve needs to know in what direction to move!

(b) The map $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ takes x_o^μ to $y^\mu = x^\mu(t) = x_o^\mu e^t$.

(c) The push forward is given by

$$(\phi_{-t*}W)^\mu|_{p_o} = \frac{\partial y^\mu}{\partial x^\nu}|_p W^\nu|_p$$

Explicitly, this is

$$(\phi_{-t*}W)^\mu(x_o) = \frac{\partial(x^\mu e^{-t})}{\partial x^\nu} W^\nu(x_o e^t) = e^{-t} W^\mu(x_o e^t).$$

Notice the minus sign in the exponential. This is because the map ϕ_{-t} takes $y^\mu = x^\mu(t) \mapsto x_o^\mu$ which we are writing as $x^\mu \mapsto e^{-t}x^\mu$. The Lie derivative is, by definition,

$$\mathcal{L}_V W = \lim_{t\to 0} \frac{1}{t}\left[(\phi_{-t*}W)^\mu(x_o) - W^\mu(x_o)\right] = \frac{\partial}{\partial t} e^{-t} W^\mu(x_o e^t) = -W^\mu(x_o) + x_o^\nu \partial_\nu W^\mu(x_o)$$

(d) $[V, W]^\mu = x^\nu \partial_\nu W^\mu - W^\nu \partial_\nu x^\mu = x^\nu \partial_\nu W^\mu - W^\mu$.
This, of course, agrees with the result of part (c).

4. First calculate integral curves of

$$A = \frac{y - x}{r}\frac{\partial}{\partial x} - \frac{y + x}{r}\frac{\partial}{\partial y}$$

That is, we look for solutions to

$$\frac{dx}{dt} = \frac{y - x}{r} \qquad\qquad \frac{dy}{dt} = -\frac{y + x}{r}$$

Note that the vector field A has magnitude $\sqrt{2}$ everywhere, is not defined at the origin and is tangential to a circle about the origin, pointing in the clockwise direction. So we expect the integral curves to grow towards the origin as they circulate clockwise.

Now, the fact that A^μ depends on x/r and y/r cries out for a description in a polar coordinate system, exactly what the whole formalism is suppose to do for us automatically. That is, if $\xi^m u$ is a new coordinate system, with $\xi^\mu = \xi^\mu(x^\nu)$ and if we denote the vector field A components in the new coordinate system by $\tilde{A}^\mu$, then

$$\tilde{A}^\mu = \frac{\partial \xi^\mu}{\partial x^\nu} A^\nu$$

So we take for new coordinates $\xi^\mu = (r, \phi)$ defined so that $x = r\cos\phi$ and $y = r\sin\phi$, that is

$$\phi = \arctan(y/x) \qquad r = \sqrt{x^2 + y^2}$$

If I can still compute derivatives,

$$\frac{\partial \xi^\mu}{\partial x^\nu} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\frac{1}{r}\sin\phi & \frac{1}{r}\cos\phi \end{pmatrix}.$$

I have written the result in terms of the coordinates ξ so we can write $\tilde{A}^\mu$ in terms of those coordinates:

$$\tilde{A}^r = \cos\phi(\sin\phi - \cos\phi) + \sin\phi(-\sin\phi - \cos\phi) = -1$$
$$\tilde{A}^\phi = -\frac{1}{r}\sin\phi(\sin\phi - \cos\phi) + \frac{1}{r}\cos\phi(-\sin\phi - \cos\phi) = -\frac{1}{r}$$

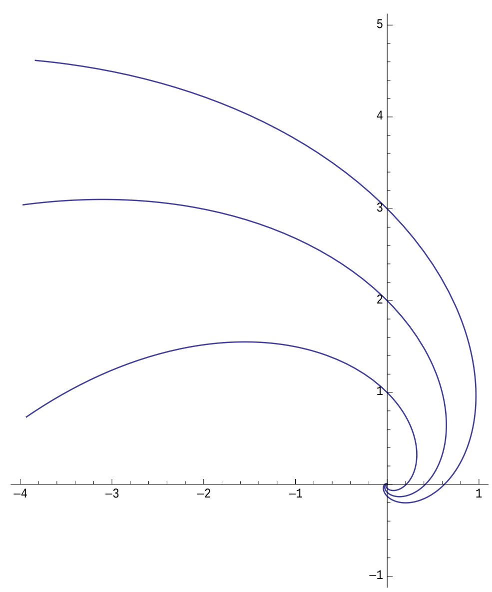
The equations for the integral curve are now simple,

$$\frac{dr}{dt} = -1, \qquad \frac{d\phi}{dt} = -\frac{1}{r}.$$

If the initial point is (r_0, ϕ_0) at $t = 0$, the solution is $r(t) = r_0 - t$ and $\phi(t) = \phi_0 + \ln(1 - t/r_0)$. In terms of the original coordinates we have then

$$x(t) = (r_0 - t)\cos(\phi_0 + \ln(1 - t/r_0)), \qquad y(t) = (r_0 - t)\sin(\phi_0 + \ln(1 - t/r_0)).$$

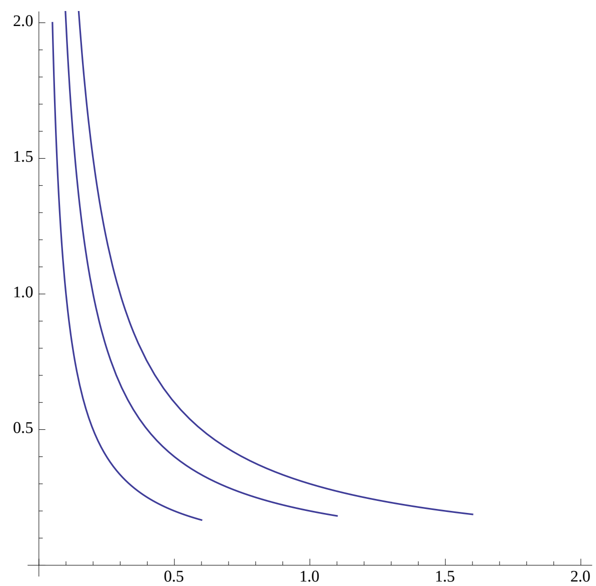
Here is a plot of the curves. I show three curves, going through $(0, 1)$, $(0, 2)$ and $(0, 3)$:



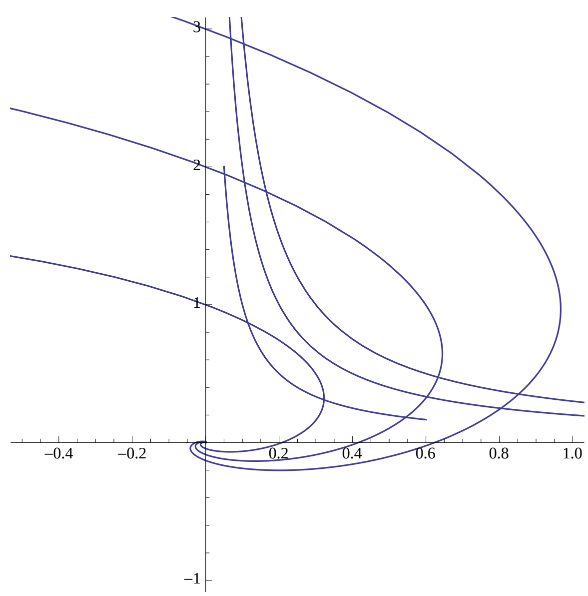
Things are much simpler for the B field, because it is easy to integrate directly. We have

$$\frac{dx}{dt} = xy, \qquad \frac{dy}{dt} = -y^2.$$

The equation for $y(t)$ can be integrated immediately, $y(t) = y_0/(1+y_0 t)$. Inseting this in the equation for $x(t)$ we have $x(t) = x_0(1+y_0 t)$. Note that this has $x(t)y(t) = x_0 y_0 =$ constant, so the parametric plot is easily drawn:



Here I have taken curves that go through $x = 0.1$ at $y = 1, 2, 3$. Of course, there are also analogous integral curves on the other three quadrants of the cartesian plane. We can also see both sets of integral curves together:



Finally, compute $C = \mathcal{L}_A B = [A, B]$, or $C^\mu = A^\nu \partial_\nu B^\mu - B^\nu \partial_\nu A^\mu$:

$$
\begin{aligned}
C^x &= \frac{y-x}{r}\partial_x(xy) + \left(-\frac{y+x}{r}\right)\partial_y(xy) - \left[xy\partial_x\left(\frac{y-x}{r}\right) + (-y^2)\partial_y\left(\frac{y-x}{r}\right)\right] \\
&= \frac{y(y-x)}{r} - \frac{x(y+x)}{r} + \frac{2xy^2(y+x)}{r^3} \\
&= \frac{y^4 + 2x^2y^2 - 2x^3y - x^4}{r^3} \\
C^y &= \frac{y-x}{r}\partial_x(-y^2) + \left(-\frac{y+x}{r}\right)\partial_y(-y^2) - \left[xy\partial_x\left(-\frac{y+x}{r}\right) + (-y^2)\partial_y\left(-\frac{y+x}{r}\right)\right] \\
&= \frac{2y(y+x)}{r} + \frac{2xy^2(y-x)}{r^3} \\
&= \frac{2y(x^3 + 2xy^2 + y^3)}{r^3}
\end{aligned}
$$

Below is a plot of the vector field C superimposed on the integral curves of A and B. You can sketch the integral curves of C in an obvious way (by stringing vectors together):