Black Holes

Start with Schwarzschild:
As seen briefly in 1^{st} quarter

$$\boxed{ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2}$$

is a solution of Einstein's equations in empty space

$$R_{\mu\nu} = 0$$

~~that~~ It has spherical symmetry and it is static.

Birkhoff's theorem asserts that the Schwarzschild metric is the unique static, spherically symmetric solution of Einstein's equations in empty space.

We won't go over the proof here. But the ingredients are
(1) Define spherical symmetry as having corresponding isometries: there are three killing vectors that generate the symmetries of the sphere. These are the generators of the Lie Algebra of the group of rotations, SO(3), that leave the sphere (S^2) invariant.
You are familiar with the algebra, it is just the same as angular momentum in Q.M.:

$$[L_i, L_j] = i\,\epsilon_{ijk} L_k$$

or, since we ~~don't~~ use anti-hermitean generators, let $X_i = iL_i \Rightarrow$

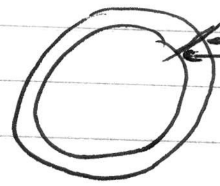
$$[X_i, X_j] = \epsilon_{ijk} X_k$$

Exercise: transforming to spherical coordinates $X_1 = y\partial_z - z\partial_y$, $X_2 = z\partial_x - x\partial_z$
$X_3 = x\partial_y - y\partial_x$ show that these killing vectors are

$R = \partial_\phi \qquad S = \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi \qquad T = -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi$

(ii) Frobenius theorem then allows one to show the space is folliated by 2-spheres. Basically the theorem says that if you have a set of vector fields that closes under commutation, $[X_i, X_j] =$ lin. combination of X_i's, then the integral curves form a submanifold of the manifold on which they are defined.

(iii) Put spherical coordinates θ, ϕ on one sphere. Extend to other neighboring spheres using orthogonal geodesics



and characterize the other spheres by two coordinates, say p & q, (the space of orthogonal geodesics through one point on a sphere is $4-2=2$ dimensional). Then ~~by suitable~~ one has

$$ds^2 = g_{pp}(p,q)\,dp^2 + 2g_{pq}(p,q)\,dp\,dq + g_{qq}(p,q)\,dq^2 + r^2(p,q)\,d\Omega_2^2$$

and by changing variables one can write

$$ds^2 = T(t,r)\,dt^2 + R(t,r)\,dr^2 + r^2 d\Omega_2^2$$

(iv) Plug this into Einstein's equations and solve. Impose the condition that the metric is static. This too has to be defined with some care. A metric is stationary if it has a timelike Killing vector near infinity, and a stationary metric is static if in addition the timelike Killing vector is orthogonal to a family of hypersurfaces.

Singularities in Schwarzschild.

It is difficult to define in general what is meant by a singularity. One common means of determining whether there is a singularity is to look for infinities in geometric quantities (coordinate independent), such as R, $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$, etc.

In the case at hand the metric

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2$$

is singular at $r = 2GM$ and at $r=0$. But are these real singularities or artifacts of the metric.

In this case $R=0$ and $R_{\mu\nu}=0$. But $R_{\mu\nu\lambda\sigma} \neq 0$ and computing explicitly one finds

$$R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} = \frac{48G^2M^2}{r^6}$$

$\Rightarrow$ there is no singularity at $r=2GM$ as far as this invariant can show, but there certainly is one at $r=0$.

In fact we will introduce coordinates that have a perfectly regular metric at $r=2GM$.

Another way of defining singularities is by finding inextendible geodesics that terminate at finite affine parameter. Let's study geodesics.

Geodesics

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$\Gamma^{\mu}_{\nu\lambda} = g^{\mu\rho}\Gamma_{\rho\nu\lambda} \qquad \Gamma_{\rho\nu\lambda} = \frac{1}{2}\left(g_{\rho\nu,\lambda} + g_{\lambda\rho,\nu} - g_{\nu\lambda,\rho}\right)$$

$$\Gamma_{rtt} = -\frac{1}{2}g_{tt,r} = \frac{GM}{r^2}$$

$$\Gamma_{ttr} = \Gamma_{trt} = -\frac{GM}{r^2}$$

$$\Gamma_{rrr} = \frac{1}{2}g_{rr,r} = -\left(1-\frac{2GM}{r}\right)^{-2}\frac{GM}{r^2}$$

$$\Gamma_{r\theta\theta} = -\frac{1}{2}g_{\theta\theta,r} = -r$$

$$\Gamma_{r\phi\phi} = -r\sin^2\theta$$

$$\Gamma_{\theta r\theta} = \Gamma_{\theta\theta r} = r$$

$$\Gamma_{\phi r\phi} = \Gamma_{\phi\phi r} = r\sin^2\theta$$

$$\Gamma_{\theta\phi\phi} = -\frac{1}{2}g_{\phi\phi,\theta} = -r^2\sin\theta\cos\theta$$

$$\Gamma_{\phi\phi\theta} = \Gamma_{\phi\theta\phi} = r^2\sin\theta\cos\theta$$

$$\Gamma^{r}_{tt} = \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)$$

$$\Gamma^{t}_{rt} = \Gamma^{t}_{tr} = \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-1}$$

$$\Gamma^{r}_{rr} = -\left(1-\frac{2GM}{r}\right)^{-1}\frac{GM}{r^2}$$

$$\Gamma^{r}_{\theta\theta} = -r\left(1-\frac{2GM}{r}\right)$$

$$\Gamma^{r}_{\phi\phi} = -r\sin^2\theta\left(1-\frac{2GM}{r}\right)$$

$$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r}$$

$$\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{r}$$

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$$

Geodesic Eqn:

$$\frac{d^2t}{d\lambda^2} + \frac{2GM}{r(r-2GM)}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0$$

$$\frac{d^2r}{d\lambda^2} + \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \frac{GM}{r(r-2GM)}\left(\frac{dr}{d\lambda}\right)^2 - r\left(1-\frac{2GM}{r}\right)\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right] = 0$$

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0$$

$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} + 2\cot\theta\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0$$

To solve these, use constants of the motion (1^{st} integrals). We have four Killing vectors, three from $SO(3)$ symmetry and one timelike Killing vector. For each

$$K_\mu \frac{dx^\mu}{d\lambda} = \text{constant}$$

along the geodesic. Moreover, for massive particles we can take $\lambda = \tau$ so that

$$\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} g_{\mu\nu} = -1 \qquad \text{timelike geodesic}$$

and for massless particles

$$\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} g_{\mu\nu} = 0 \qquad \text{null geodesic.}$$

The Killing vectors associated with $SO(3)$ are like angular momentum, $\vec{L}$. Just as in flat space, $\vec{L}$ = constant implies motion in a plane orthogonal to $\vec{L}$ and with fixed magnitude of $\vec{L}$. So we can fix the plane of motion choosing

$$\theta = \frac{\pi}{2}.$$

The magnitude of L corresponds to the Killing vector ∂_ϕ

$$(\partial_\phi)^\mu = (0,0,0,1)$$

In addition, the timelike Killing vector is

$$(\partial_t)^\mu = (1,0,0,0)$$

The conserved quantities are

$$E = -g_{\mu\nu}(\partial_t)^\mu \frac{dx^\nu}{d\lambda} = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\lambda}$$

and

$$L = (\partial_\phi)^\mu \frac{dx^\nu}{d\lambda} g_{\mu\nu} = r^2 \sin^2\theta \frac{d\phi}{d\lambda} = r^2 \frac{d\phi}{d\lambda} \qquad (\text{since } \theta = \tfrac{\pi}{2}).$$

The constants are named E & L, suggestively. But these are just labels. We can discuss energy and angular momentum later.

For timelike geodesics we have ($U^\mu U_\mu = -1$):

$$-\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2 = -1$$

or multiplying by $\left(1-\frac{2GM}{r}\right)$ and using E & L

$$-E^2 + \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2GM}{r}\right)\left(1+\frac{L^2}{r^2}\right) = 0$$

This is like a particle in a central potential

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V(r) = \mathcal{E}$$

with $\quad V(r) = \frac{1}{2}\left(1-\frac{2GM}{r}\right)\left(1+\frac{L^2}{r^2}\right) \qquad \mathcal{E} = \frac{1}{2}E^2 \qquad$ (I)

The null geodesic is similar, but the LHS -1 is replaced by 0:

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V_n(r) = \mathcal{E}$$

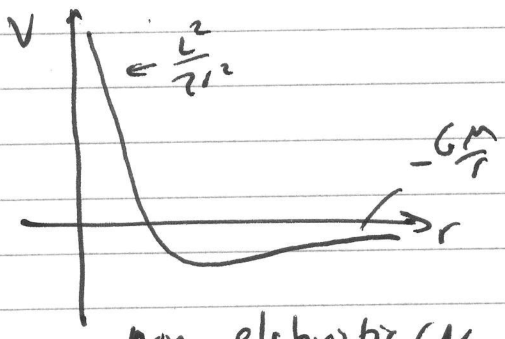
$$V_n(r) = \frac{1}{2}\left(1-\frac{2GM}{r}\right)\frac{L^2}{r^2} \qquad \mathcal{E} = \frac{1}{2}E^2 \qquad \text{(II)}$$

(or, together

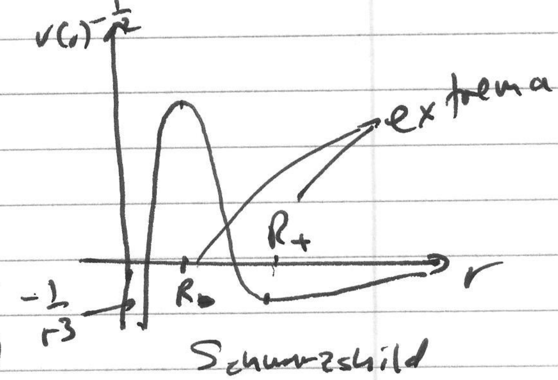
$$V(r) = \frac{1}{2}\left(1-\frac{2GM}{r}\right)\left(\kappa + \frac{L^2}{r^2}\right) \qquad \kappa = \begin{cases} 0 & \text{null-like} \\ 1 & \text{time-like} \end{cases}$$

Expanding (I):

$$V(r) = \underset{\text{irrelevant}}{\frac{1}{2}} - \underset{\text{Newtonian}}{\frac{GM}{r}} + \underset{\text{std. } L}{\frac{L^2}{2r^2}} - \underset{\underline{\underline{\text{new}}}}{\frac{GML^2}{r^3}}$$



non-relativistic (Newtonian)

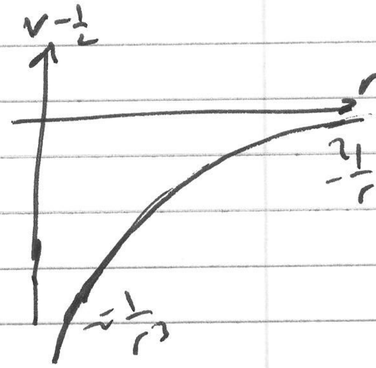


Schwarzschild

find extrema $\frac{\partial V}{\partial r} = 0 = GMr^2 - L^2 r + 3GML^2$

or $$R_\pm = \frac{1}{2GM}\left[L^2 \pm \sqrt{L^4 - 12(GML)^2}\right]$$

For $12G^2M^2 > L^2$ → no extrema of V



For $L^2 > 12G^2M^2$ R_+ a minimum, R_- a maximum as in figure.
⇒ stable circular orbit at $r = R_+$ (unstable at $r = R_-$).
Now ~~R_+~~ for $L^2 \gg 12(GM\epsilon)^2$ $R_+ \simeq \frac{L^2}{GM}$ the Newtonian formula.

Now, as we vary L^2, R_+ is smallest at $L^2 = 12(GM)^2$, where
$$R_+ = \frac{1}{2GM} 12(GM)^2 = 6GM$$ so

$$R_+ > 6GM$$

Here is a smallest stable circular orbit!

(Similarly unstable circular orbits radii are restricted to by $R_- < 6GM$
— same calculation — and on low end take $L \to \infty$
$$R_- \to \frac{1}{2GM}\left[L^2 - L^2\left[1 - \frac{1}{2}\frac{12(GM)^2}{L^2} + \cdots\right]\right] = 3GM$$

so $3GM < R_- < 6GM$).

(~~Note that this calculation also gives~~

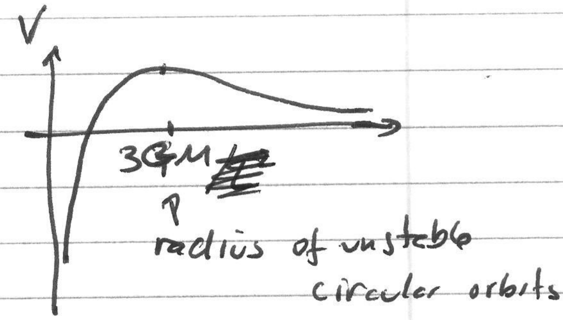
~~$L \to \infty \Rightarrow R_+ \to \frac{L^2}{GM}$~~

(Note: at this point comparison of ω_r (for small perturbations about circular orbit, with $\omega_r^2 = \frac{d^2V}{dr^2}$) and $\omega_\phi = \dot\phi$ gives precession of perihelion → mercury → classical test. This must have been covered?). (No time here).

Null geodesics:

$$V_n(r) = \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$$

$$\left(\frac{\partial V}{\partial r} = 0 = L^2 r - 3GML^2\right)$$



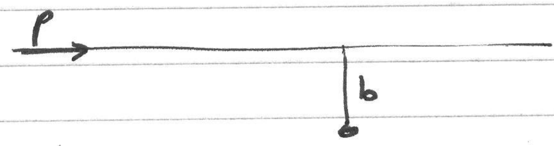
Now recall E & L are (in arbitrary units) the energy and angular momentum of the particle (photon?), the energy necessary for the particle to go over the potential barrier is the height of the barrier:

$$\frac{1}{2}E^2 = V_n(3GM) = \frac{L^2}{27(GM)^3}\left(\tfrac{1}{2}3GM - GM\right) = \frac{L^2}{54G^2M^2}$$

$$\Rightarrow \quad \frac{L}{E} = 3\sqrt{3}\,GM$$

But L/E has a simple interpretation. In the asymptotically flat region ($r \gg GM$) it corresponds to the impact parameter

$$b = \frac{L}{E}$$

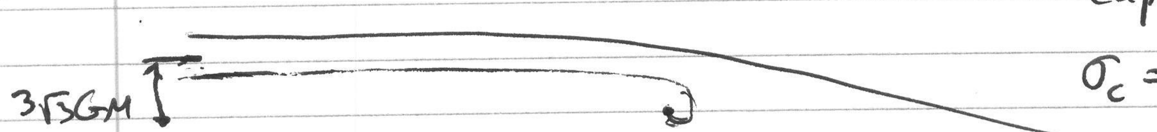


$L = bp$

and $E = p$ (massless)

For $b < 3\sqrt{3}\,GM$ the photon is captured

For $b > 3\sqrt{3}\,GM$ it is scattered

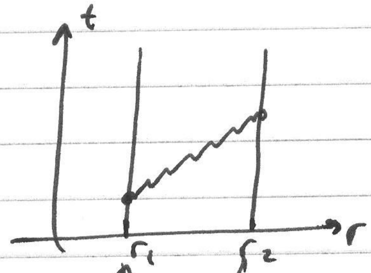


Capture cross section

$$\sigma_c = \pi b_{crit}^2 = 27\pi(GM)^2$$

Red-Shift

Similar to what we did before:



(Note, all we need is that U be of this form instantaneously)

two observers (not on geodesics)

both with $U^\mu = (U^0, 0, 0, 0)$

so $U \cdot U = -1 \Rightarrow U^0 = 1/\sqrt{-g_{tt}} = \dfrac{1}{\sqrt{1 - \frac{2GM}{r}}}$

Now, if $k^\mu = \dfrac{dx^\mu}{d\lambda}$ on the null geodesic for the photon then the observers measure $\omega = -U \cdot k = +\sqrt{1 - \frac{2GM}{r}}\,\dfrac{dt}{d\lambda}$

and $\dfrac{dt}{d\lambda} = E/(1 - \frac{2GM}{r}) \Rightarrow \omega = \dfrac{E}{\sqrt{1 - \frac{2GM}{r}}}$

Since E is constant we have

$$\omega_1 \sqrt{1 - \frac{2GM}{r_1}} = \omega_2 \sqrt{1 - \frac{2GM}{r_2}}$$

or

$$\boxed{\frac{\omega_2}{\omega_1} = \sqrt{\frac{1 - 2GM/r_1}{1 - 2GM/r_2}}}$$

Gravitational redshift.

For weak fields,

$$\frac{\omega_2}{\omega_1} \approx 1 - \frac{GM}{r_1} + \frac{GM}{r_2} = 1 + \Phi_1 - \Phi_2 = 1 - \Delta\Phi$$

which is the formula obtained last quarter (see Schwartz) on general grounds (principle of equivalence) for weak fields.

Kruskal Coordinates and extension

Coordinate singularities vs real singularities: toy models first (warm-up):

$$ds^2 = -\frac{1}{t^4}dt^2 + dx^2$$

defined for $x \in (-\infty, \infty)$ and $t \in (0, \infty)$. Seems singular but defining $t' = \frac{1}{t}$, we have

$$ds^2 = -dt'^2 + dx^2$$

$\Rightarrow$ original spacetime is a portion of Minkowski space with $t'>0$. Note that the original spacetime is not geodesically complete: geodesics approaching $t \to \infty$ take finite affine parameter to get there (even though approaching $t=0$ take infinite affine parameter)

(Check:

$$\Gamma_{ttt} = \frac{1}{2}g_{tt,t} = 2t^{-5} \qquad \Gamma^t_{tt} = -\frac{2}{t}$$

$$\frac{d^2t}{d\lambda^2} + \left(-\frac{2}{t}\right)\left(\frac{dt}{d\lambda}\right)^2 = 0 \qquad \frac{d^2x}{d\lambda^2} = 0 \Rightarrow \frac{dx}{d\tau} = v = \text{const}$$

Let $\cancel{v = \frac{dt}{d\lambda} \Rightarrow \frac{\dot{v}}{v^2} = \frac{2}{t} \quad \int\frac{dv}{v^2} = 2\int\frac{dt}{t} = -\frac{1}{v} = 2\ln t}$

$$\cancel{v = \frac{dt}{d\lambda} = \frac{1}{-2\ln t} \qquad \int dt \ln t = -\frac{1}{2}\int d\lambda}$$

$$\cancel{t(\ln t - 1) = -\tfrac{1}{2}\lambda}$$

But $U \cdot U = -1 \Rightarrow -\frac{1}{t^4}\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 = -1$

$$\frac{dt}{d\tau} = t^2\sqrt{1+v^2} \qquad \int\frac{dt}{t^2} = \sqrt{1+v^2}\int d\tau \Rightarrow \frac{1}{t_0} - \frac{1}{t} = \sqrt{1+v^2}\,\tau$$

So $t \to \infty$ as $\tau \to 1/t_0\sqrt{1+v^2}$. However $t \to 0$ as $\tau \to \infty$).

2nd example: Rindler spacetime

$$ds^2 = -x^2 dt^2 + dx^2$$

$t \in (-\infty, \infty)$ $\quad x \in (0, \infty)$

Singularity at $x = 0$?

Geodesics: $\left(\begin{array}{ll} \Gamma_{xtt} = -\frac{1}{2} g_{tt,x} = x & \Gamma^x{}_{tt} = x \\ \Gamma_{txt} = \Gamma_{ttx} = \frac{1}{2} g_{tt,x} = -x & \Gamma^t{}_{tx} = \Gamma^t{}_{xt} = \frac{1}{x} \end{array} \right)$

$p_t = \text{const} \Rightarrow g_{tt} \frac{dt}{d\tau} = E = \text{const.} \qquad \frac{dt}{d\tau} = -\frac{E}{x^2}$

$$-1 = -x^2 \left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 = -\frac{v^2}{x^2} + \left(\frac{dx}{d\tau}\right)^2$$

$$\frac{dx}{d\tau} = \sqrt{\frac{E^2}{x^2} - 1} \qquad \int \frac{x\,dx}{\sqrt{E^2 - x^2}} = \int d\tau$$

let $E^2 \xi = x^2$ $\quad E^2 d\xi = 2x\,dx$ $\quad \tau = \frac{E}{2} \int \frac{d\xi}{\sqrt{1-\xi}} = E\sqrt{1-\xi} = \frac{E}{\sqrt{\ }}\sqrt{1 - v^2 x^2}$
$= \sqrt{E^2 - x^2}$

so $\tau = \sqrt{E^2 - x^2} \qquad x^2 = E^2 - \tau^2 \qquad \frac{dt}{d\tau} = -\frac{E}{E^2 - \tau^2}$

$$t = E \int \frac{d\tau}{-\tau^2 + v^2} \qquad \cancel{\tau = v\sin\theta \quad v\int \frac{v\cos\theta\, d\theta}{v^2 \cos^2\theta} = \int \frac{d\theta}{\cos\theta}} =$$

$$\frac{1}{E - \tau} + \frac{1}{E + \tau} = \frac{2E}{E^2 - \tau^2} \quad \rightarrow t = \frac{1}{2} \ln \frac{E - \tau}{E + \tau} \quad t \to \infty \text{ in finite } \tau \; (\tau \to E)$$

Geodesically incomplete. How about curvature?

$$R^t{}_{xtx} = \partial_t \Gamma^t{}_{xx} - \partial_x \Gamma^t{}_{tx} + \Gamma^t{}_{t\lambda} \Gamma^\lambda{}_{xx} - \Gamma^t{}_{x\lambda} \Gamma^\lambda{}_{tx}$$
$$= 0 - \partial_x \frac{1}{x} + 0 - \frac{1}{x^2} = 0$$

so this is a portion of Minkowski space, again.

Q: How to find coordinates that are non-singular starting from this, not using the fact that this is Minkowski?

Use a family of geodesics that head towards the singularity, with affine parameter as one coordinate. Must avoid crossing of geodesics because this would give new coordinate singularities. In 2-DIM we can take null ingoing and outgoing geodesics (they never cross because it null geodesics have same tangent they agree everywhere) null geodesics:

$$0 = -x^2 \left(\frac{dt}{d\lambda}\right)^2 + \left(\frac{dx}{d\lambda}\right)^2$$

$$\Rightarrow \qquad x\frac{dt}{d\lambda} = \pm\frac{dx}{d\lambda}$$

$$\Rightarrow \qquad \pm\frac{dx}{x} = dt$$

$$\Rightarrow \qquad t = \pm \ln x + \text{const}$$

Define $\quad u = t - \ln x \qquad \Leftrightarrow \qquad t = \frac{1}{2}(u+v)$

$\qquad\quad v = t + \ln x \qquad\qquad\qquad x = e^{\frac{1}{2}(v-u)}$

So geodesics are $u = \text{constant}$ or $v = \text{constant}$. Then

$$ds^2 = -x^2dt^2 + dx^2 = -e^{(v-u)}\frac{1}{4}(du+dv)^2 + e^{(v-u)}\frac{1}{4}(dv-du)^2$$

or
$$ds^2 = -e^{v-u}\,du\,dv$$

We want to analyze the singularity at $x=0$. Can't do that yet since u,v in $(-\infty,\infty)$ still has $x>0$. But now we can extend the space beyond $x=0$, ie, beyond u,v infinite by ~~rep~~ introducing new coordinates $U(u)$ and $V(v)$. Calculate affine parameter along null geodesics. Since

$$g_{tt}\frac{dt}{d\lambda} = -E = \text{const} \quad \Rightarrow \quad \frac{dt}{d\lambda} = \frac{E}{x^2}$$

so, along $u=\text{constant}$ we have $\dfrac{dt}{d\lambda} = \dfrac{1}{2}\dfrac{dv}{d\lambda} = E e^{-(v-u)}$

or
$$\lambda = \frac{1}{2E}\int e^{v-u}dv = C_1 + \frac{e^{v-u}}{2E} \qquad (u = \text{constant})$$

Along outgoing null geodesics $\lambda_{out} = e^{v}$ is an affine parameter

while $\lambda_{in} = -e^{-u}$

So use $\quad U = -e^{-u} \qquad V = e^{v} \qquad ds^2 = -dU\,dV$

Now $ds^2 = -dUdV$ has $U<0$ $V>0$

but there is no obstruction to extending these to $(-\infty,\infty)$. and we get Minkowski space after

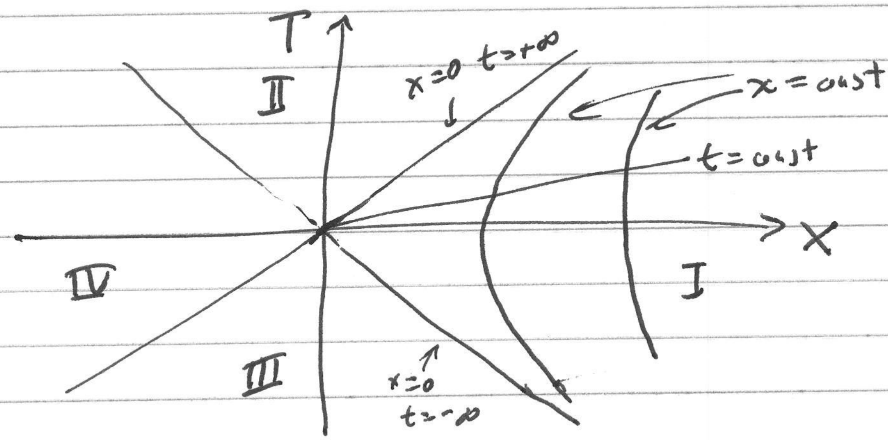
$$T = \tfrac{1}{2}(U+V) \qquad \Leftrightarrow \qquad U = T-X$$
$$X = \tfrac{1}{2}(V-U) \qquad\qquad V = T+X$$

$$ds^2 = -dT^2 + dX^2$$

The original coordinates are given, in terms of these, by

$$t = \tfrac{1}{2}(u+v) = \tfrac{1}{2}(-\ln(-U) + \ln V)$$
$$= \tfrac{1}{2}(-\ln(X-T) + \ln(X+T))$$
$$= \tfrac{1}{2}\ln\frac{X+T}{X-T} = \tanh^{-1}\left(\frac{T}{X}\right)$$

$$x = e^{\frac{1}{2}(v-u)} = \sqrt{-VU} = \sqrt{X^2-T^2}$$



original space is the wedge I in Minkowski space time. $(X>|T|)$.

Now do the same for Schwarzschild. We can ignore angular coordinates for most of the discussion. Consider

$$ds^2 = -(1-2GM/r)dt^2 + (1-\tfrac{2GM}{r})^{-1}dr^2$$

Null geodesics:

$$-(1-\tfrac{2GM}{r})\left(\frac{dt}{d\lambda}\right)^2 + (1-\tfrac{2GM}{r})^{-1}\left(\frac{dr}{d\lambda}\right)^2 = 0$$

$$\Rightarrow \quad \left(\frac{dt}{dr}\right)^2 = (1-\tfrac{2GM}{r})^{-2}$$

$$t = \pm r_* + \text{constant}$$

r_* is the "Regge-Wheeler tortoise coordinate" given by

$$r_* = \int \frac{dr}{1-\frac{2GM}{r}} = \int dr\left[\frac{r-2GM+2GM}{r-2GM}\right] = r + 2GM\ln\left(\frac{r}{2GM}-1\right)$$

Then define null coordinates

$$u = t - r_*$$

$$v = t + r_*$$

Calculate metric: $du = dt - dr_* = dt - (1-\frac{2GM}{r})^{-1}dr$

$$dv = dt + (1-\tfrac{2GM}{r})^{-1}dr$$

$$du\,dv = dt^2 - (1-\tfrac{2GM}{r})^{-2}dr^2$$

or

$$ds^2 = -(1-\tfrac{2GM}{r})\,du\,dv$$

with r understood as $r = r(u,v)$.

with $v-u = 2r_* = 2r + 4GM\ln(\frac{r}{2GM}-1)$

we have

$$e^{\frac{v-u}{4GM}} = e^{\frac{r}{2GM}}\left(\frac{r}{2GM}-1\right) = e^{\frac{r}{2GM}}\left(1-\frac{2GM}{r}\right)\frac{r}{2GM}$$

so

$$ds^2 = -\frac{2GM}{r}\,e^{-\frac{r}{2GM}}\,e^{\frac{v-u}{4GM}}\,du\,dv$$

This is useful because the factor $\frac{2GM}{r} e^{-\frac{r}{2GM}}$ is not singular as $r \to 2GM$.

Now, as in Rindler case, we introduce

$$U = -e^{-u/4GM}$$

$$V = e^{v/4GM}$$

$$\Rightarrow \cancel{ds^2} = dU = \frac{1}{4GM} e^{-u/4GM} du \quad dV = \frac{1}{4GM} e^{v/4GM} dv$$

and

$$ds^2 = -\frac{32(GM)^3 e^{-r/2GM}}{r} dU\, dV$$

While this is defined for $V > 0$ and $U < 0$, we can now extend to $(-\infty, \infty)$ and define as before

$$T = \tfrac{1}{2}(U + V)$$

$$X = \tfrac{1}{2}(V - U)$$

The full metric is now

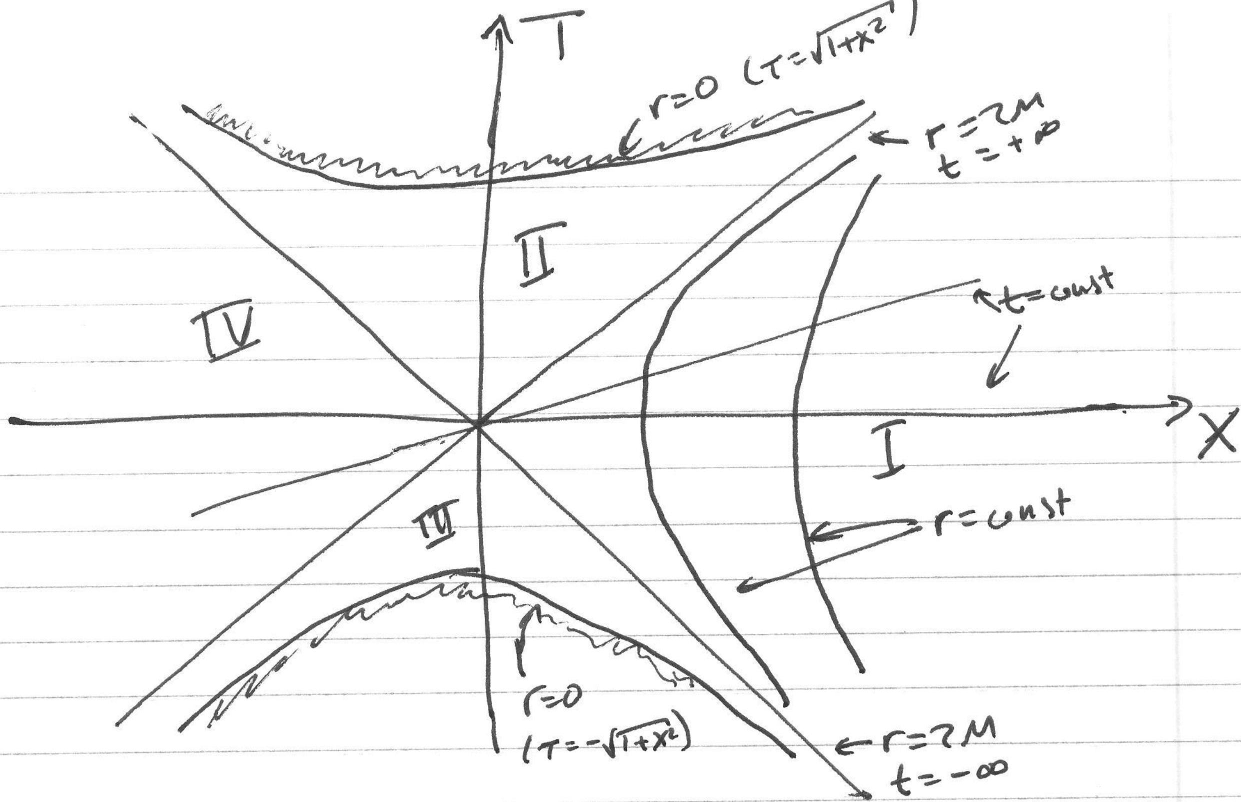
$$\boxed{ds^2 = \frac{32(GM)^3 e^{-r/2GM}}{r}\left(-dT^2 + dX^2\right) + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)}$$

Relating to original coordinates:

$$X^2 - T^2 = -UV = e^{\frac{v-u}{4GM}} = e^{\frac{2r_*}{4GM}} = e^{\frac{r}{2GM}}\left(\frac{r}{2GM} - 1\right) \quad (*)$$

$$\tanh^{-1}\frac{T}{X} = \frac{1}{2}\ln\left(\frac{T+X}{X-T}\right) = \frac{1}{2}\ln\frac{V}{-U} = \frac{1}{2}\ln e^{\frac{v+u}{4GM}} = \frac{t}{4GM}$$

which would have been hard to guess. Eq (*) also gives $r = r(T,X)$ for the metric. Note that $r > 0$ in (*) gives the allowed range for X, T: $X^2 - T^2 > -1$.



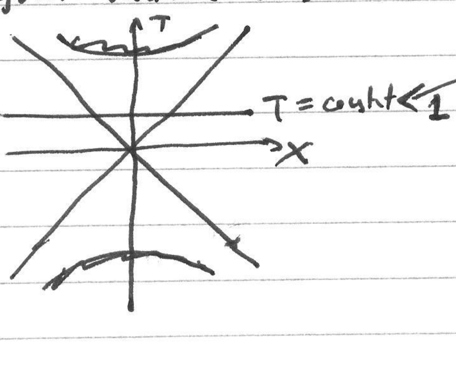
Keep in mind each point is a S^2 with radius r.

Causal structure: null geodesics are 45° lines.

- Singularities at $r=0$ are spacelike. Two of them
  - Future of region II
  - Past of region III

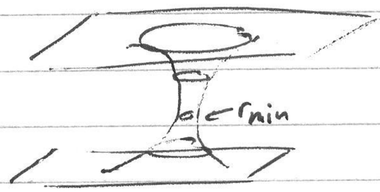
  NOT a timelike line at origin, as suggested by original coordinates.
- Region I corresponds to original $r > 2M$, exterior gravitational field of ~~body~~ a spherical body. Radially infalling observer that crosses $X=T$ can never escape back to region I AND will eventually hit singularity ergo "black hole"
- Region III has the time reversed properties of I ⇒ "white hole".
- Region IV has identical properties to I, asymptotically flat.

To see what's going on, consider hypersurfaces of $T=$ constant, restore one angular variable (θ):



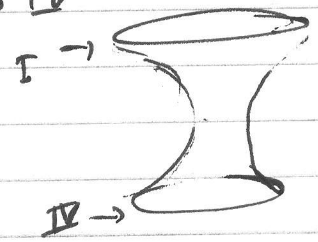
$$e^{\frac{r}{2GM}}\left(\frac{r}{2GM}-1\right) = X^2 - T^2$$

as X goes from $-\infty$ to $+\infty$ r goes from ∞ to a minimum and back to ∞.



for $T=1$
$r_{min}=0$

There is another space, on the other side of the blackhole. Can we communicate with our brothers here? No, as is clear from causality diagram. What happens in this picture is that as an observer wants to go from I to IV



the radius of the throat is shrinking and it necessarily pinches off before the observer makes it to the other side.

Penrose Diagram

Recall
$$ds^2 = -\frac{32(GM)^3}{r} e^{-\frac{r}{2GM}} \, dU \, dV + r^2 d\Omega_2^2$$

Now let
$$\hat{U} = \arctan\left(U\right) \qquad \hat{V} = \arctan\left(V\right)$$

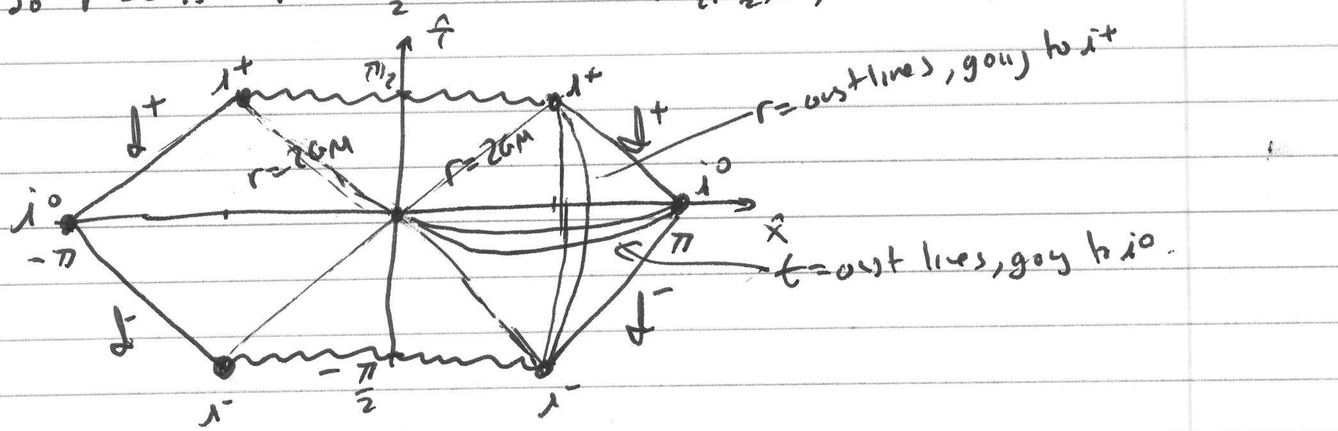
So $\hat{U} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\hat{V} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(It is quite irrelevant what the metric is in $\hat{U}, \hat{V}$, just matters that it has finite range). Also, since $-UV = e^{r/2GM}\left(\frac{r}{2GM} - 1\right) > -1 \Rightarrow UV < 1$

$\Rightarrow \tan\hat{U} \tan\hat{V} < 1 \Rightarrow \cos(\hat{U}+\hat{V}) > 0$ $\quad |\hat{U}+\hat{V}| < \frac{\pi}{2}$

Also $r=0$ is $UV = 1$ $\quad \hat{V}+\hat{U} = \pm\frac{\pi}{2}$. Now let $\hat{T} = \cancel{\tfrac{1}{2}}(\hat{V}+\hat{U})$ $\hat{X} = \cancel{\tfrac{1}{2}}(\hat{V}-\hat{U})$

So $r=0$ is $\hat{T} = \pm\frac{\pi}{2}$ and $\hat{T} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\hat{X} \in (-\pi, \pi)$



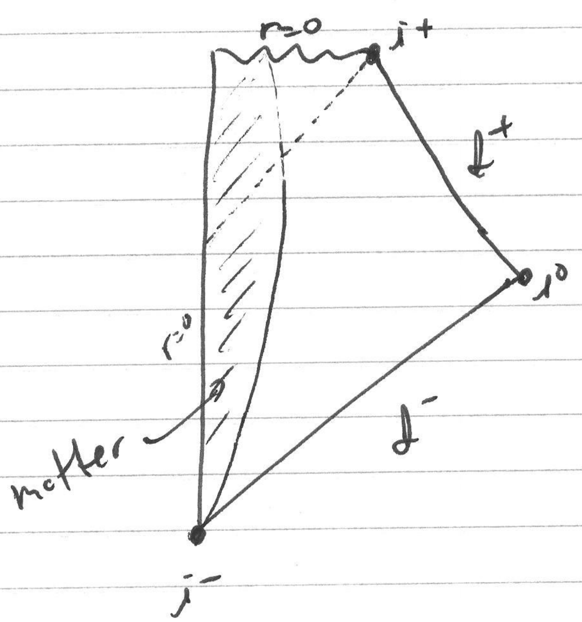
Black holes in nature arise from gravitational collapse of stars. ~~Even~~ This is ~~also~~ unlikely to be exactly spherically symmetric, but we can consider the idealized case. ~~Since at some time.~~

The metric is spherically symmetric with some mass/~~ene~~energy density distributed with spherical symmetry. Birkhoff's theorem gives that outside the region of mass the metric is Schwarzschild (well, not quite — that would be true if the situation were static, but let's imagine it is slow, so the metric is approximately Schwarzschild). Yet, inside the metric is not Schwarzschild and in fact is regular at $r=0$.

$\rho = 0$ $\quad \rho \neq 0$

≈ Schwarzschild

~~Moreover, at~~ As we let the star collapse (which <u>will</u> happen if the object is dense enough, in fact if the mass $GM > \frac{4R}{9}$) once the mass falls within $R = 2GM$ it will continue falling irreversibly towards the singularity. A picture (conformal diagram caricature) of the process is

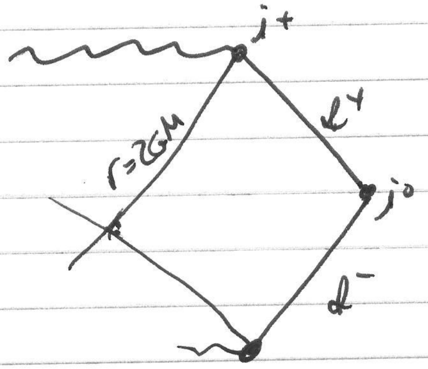


and we see such a spacetime has not white-hole nor a region IV.

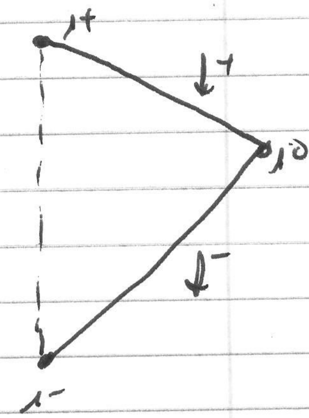
More General Black Holes

What characterizes black holes?

Recall Schwarzschild



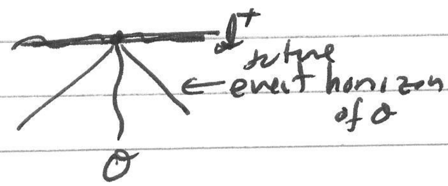
Flat Minkowski



Two important ingredients:

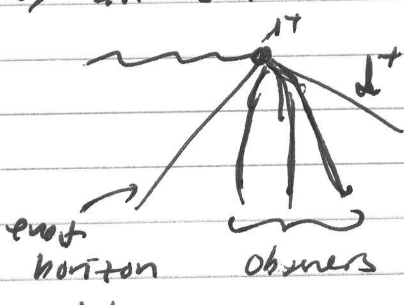
(i) Asymptotically flat, so it looks like Minkowski on the "outside".

(ii) Has an event horizon (future) (at $r=2GM$). Recall we had



in de-Sitter.

So in Schwarzschild all observers that remain in I go to i^+ at infinity, and they all share $r=2GM$ as a future event horizon



So, more general black hole