225B - Winter 2014

REVIEW

Maps of Manifolds

$\phi : \mathcal{M} \to \mathcal{M}'$ is a map between manifolds

(is a C^r map if the corresponding map of coordinates is C^r

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\phi"] \arrow[d, "x"'] & \mathcal{M}' \arrow[d, "y"] \\ \mathbb{R}^n & \mathbb{R}^m \end{tikzcd}$$

so $y \circ \phi \circ x^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ is C^r)

Notes:

- In gnl not one-to-one
- Even if one-to-one may not have an inverse

So it goes one way.

Let $f : \mathcal{M}' \to \mathbb{R}$ a function on $\mathcal{M}'$

(a "scalar field")

then ϕ defines a function on $\mathcal{M}$, $\phi^* f$

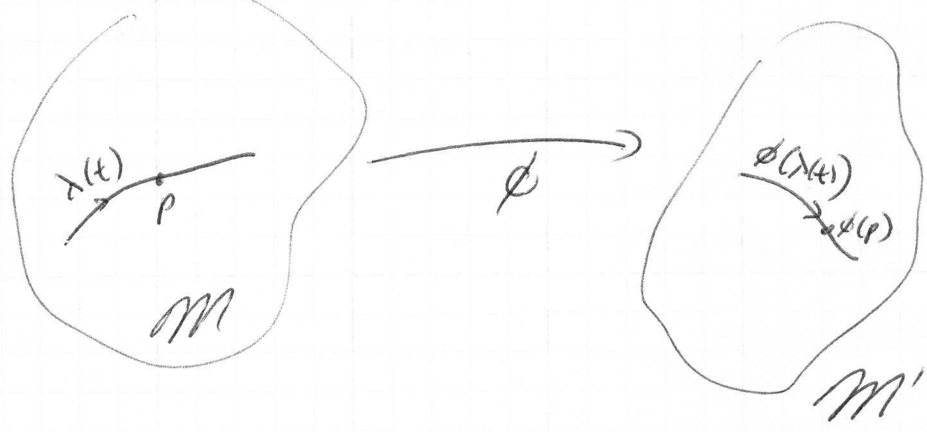
$$\phi^* f : \mathcal{M} \to \mathbb{R}$$

defined by $\mathcal{M} \xrightarrow{\phi} \mathcal{M}' \to \mathbb{R}$

ie if $p \in \mathcal{M}$ then $\phi(p) \in \mathcal{M}'$ and $f(\phi(p))$ is defined.

(This is a "pull-back" of a zero form)

Go the other direction



By mapping curves $\lambda(t)$ in $\mathcal{M}$ into $\mathcal{M}'$ we can get maps of tangent vectors.

If $T_p(\mathcal{M})$ is the tangent space to $\mathcal{M}$ at p then

push-forward

$$\phi_* : T_p(\mathcal{M}) \to T_{\phi(p)}(\mathcal{M}')$$

defined by mapping $\left(\frac{\partial}{\partial t}\right)_\lambda \longrightarrow \left(\frac{\partial}{\partial t}\right)_{\phi(\lambda)}$ $\leftarrow$ denote this by $\phi_*\left(\frac{\partial}{\partial t}\right)_\lambda\Big|_{\phi(p)}$

This is a linear transformation between the vector spaces: if x^μ and y^a are local coordinates on patches of $\mathcal{M}$ & $\mathcal{M}'$, then the curve is $x^\mu(t)$, mapped into $y^a(x^\mu(t))$ and

$$\frac{dy^a}{dt}\Big|_0 = \frac{\partial y^a}{\partial x^\mu}\Big|_p \frac{dx^\mu}{dt}\Big|_0$$

or

$$N^a = \frac{\partial y^a}{\partial x^\mu}\Big|_p M^\mu \qquad \text{where } \vec{N} \in T_{\phi(p)}(\mathcal{M}'),\quad \vec{M} \in T_p(\mathcal{M})$$

so ϕ_* is just the matrix $\frac{\partial y^a}{\partial x^\mu}\Big|_p$ and we write $\vec{N} = \phi_* \vec{M}$

Since a vector $\vec{M}$ is a directional derivative, we have $\vec{M}(f)$ defined.

(A vector gives a map of any function f at p into a number.

If $\vec{M} = \frac{\partial}{\partial t}$ then $\vec{M}(f) = \frac{\partial f}{\partial t}\Big|_{p=\lambda(t_0)}$ ie the derivative along $\lambda(t)$

Explicitly $\frac{df(x^\gamma(t))}{dt} = \frac{\partial f}{\partial x^\mu}\dot{x}^\mu$ so the action of the vector $\vec{A}$ with coordinates a^μ on f is $\vec{A}(f) = a^\mu \partial_\mu f|_p$).

$$\begin{tikzcd}
T_p(M) \arrow[r, "\phi_*"] & T_{\phi(p)}(M')
\end{tikzcd}$$

defines $\vec{N}(f) = \phi_*\vec{M}(f)$

$$\begin{tikzcd}
M \arrow[r, "\phi"] \arrow[rr, bend right, "\phi^* f"'] & M' \arrow[r, "f"] & R
\end{tikzcd}$$

defines $\vec{M}(\phi^* f)$

$$\vec{M}(\phi^* f)\Big|_p = \phi_*\vec{M}(f)\Big|_{\phi(p)}$$

check:

$$m^\mu \partial_\mu(f(y(x)))\Big|_p = m^\mu \frac{\partial f}{\partial y^a}\Big|_{\phi(p)} \frac{\partial y^a}{\partial x^\mu}\Big|_p = \Big(m^\mu \frac{\partial y^a}{\partial x^\mu}\Big)\frac{\partial f}{\partial y^a}\Big|_{\phi(p)}$$

n^a components of $\phi_*\vec{M}$.

Students: should always flesh out relations in terms of coordinate patches, to make sure they understand.

Go on to 1-forms: define pull-back

$$\phi^*: T^*_{\phi(p)}(M') \to T^*_p(M)$$

by requiring the contraction is mapped properly: $\phi^*: \tilde{\omega} \to \phi^*\tilde{\omega}$

$$\left(\phi^*: \quad \tilde{\omega} \in T^*_{\phi(p)}(M') \longrightarrow \phi^*\tilde{\omega} \in T^*_p(M) \right)$$

with $\boxed{\tilde{\omega}(\phi_* \vec{M}) = \phi^*\tilde{\omega}(\vec{M})}$ ~~defines $\phi^*\tilde{\omega}$~~

~~Recall~~

$$T_p \xrightarrow{\phi_*} T_{\phi(p)}$$

$$M \xrightarrow{\phi} M'$$

$$T^*_p \xleftarrow{\phi^*} T^*_{\phi(p)}$$

Recall $\tilde{\omega}(\vec{N})$ is a number, ie $\tilde{\omega}$ is a map from $T_p \to R$.
(In components $\tilde{\omega}(\vec{N})\big|_p = \omega_\mu\big|_p N^\mu\big|_p$, the index contraction.
Some texts write $\langle \tilde{\omega}, \vec{N} \rangle$).

So the ~~def~~ above gives the action of $\phi^*\tilde{\omega}$ on vectors $\vec{M} \in T(M)$ in terms of the action of $\tilde{\omega}$ on vectors $\vec{N} \in T(M')$. ~~which is~~

(In components $(\phi^*\tilde{\omega})_\mu M^\mu = \omega_a N^a = \omega_a \dfrac{\partial y^a}{\partial x^\mu} M^\mu$

that is $(\phi^*\tilde{\omega})_\mu = \omega_a \dfrac{\partial y^a}{\partial x^\mu}$).

In particular $\phi^*(df) = d(\phi^* f)$

(In components $df = f_{,a}\, dy^a$ $\quad \phi^*(df) = f_{,a} \dfrac{\partial y^a}{\partial x^\mu} dx^\mu$

while $d(\phi^* f) = df(y(x)) = \left(\dfrac{\partial f}{\partial y^a} \dfrac{\partial y^a}{\partial x^\mu} \right) dx^\mu$).

Clearly this can be extended to tensors of type T^r_0 and T^0_r

$$\phi_* : T^r_0(p) \to T^r_0(\phi(p))$$

recall $T \in T^r_0(p)$ ~~in~~ acts on r 1-forms $T(\tilde{\omega}^1, \ldots, \tilde{\omega}^r) \in \mathbb{R}$

so $T \to \phi_* T$ by $T(\phi^*\tilde{\omega}^1, \ldots, \phi^*\tilde{\omega}^r) = \phi_* T(\tilde{\omega}^1, \ldots, \tilde{\omega}^r)$

And $T \in T^0_r$ acts on r vectors so $\phi^* : T^0_r(\phi(p)) \to T^0_r(p)$

$$\phi^* T(\vec{M}_1, \ldots, \vec{M}_r) = T(\phi_* \vec{M}_1, \ldots, \phi_* \vec{M}_r)$$

In components ϕ_*: $$T^{a_1 \ldots a_r} = \frac{\partial y^{a_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{a_r}}{\partial x^{\mu_r}} T^{\mu_1 \ldots \mu_r}$$

$$\phi^* : \quad T_{\mu_1 \ldots \mu_r} = \frac{\partial y^{a_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{a_r}}{\partial x^{\mu_r}} T_{a_1 \ldots a_r}$$

Defs

Rank : ϕ is $\mathcal{M} \to \mathcal{M}'$ is rank k (at p) if the dimension of the tangent space at $\phi(p)$ $(\phi_*(T_p(\mathcal{M}))$ is k.

Injective: ϕ above is injective (at p) if rank = dimension of $\mathcal{M}$

$$k = n.$$

(In this case $n \leq n'$).

Exercise: If ϕ is injective then no non-zero vectors in $T_p(\mathcal{M})$ are mapped to zero by ϕ_*

<u>Surjective</u>: ϕ is surjective at p if rank of ϕ = dimension of M'

$k = n'$

(So that $n \geq n'$).

(~~<u>Immersion</u>: ϕ is an immersion if it has an inverse ϕ^{-1},~~ ~~(with same differentiability as ϕ) such that~~ ~~for each $p \in M$ there is $U \subset M$~~ with $p \in U$ $\ni$ ~~$\phi^{-1}: \phi(U) \to U$~~

(Skip immersion: it ~~dep~~ is subtle only when C^r properties matter)

If ϕ is injective $\forall p \in M$ we say ϕ is an immersion (actually, def'n of immersion is ~~of~~ given in terms of existance of differentiable inverse of ϕ, and then equivalence of statements is proved) $\Longrightarrow$ $\phi_* : T_p \to \phi_*(T_p) \subset T_{\phi(p)}$ is an isomorphism.

Then $\phi(M) \subset M'$ is an n-dimensional <u>immersed</u> submanifold in M'.

This is one-one locally, but may not be so globally.

An imbedding is, basically, an immersion that is one-one (actually a homeomorphism onto its image).

<u>Diffeomorphism:</u> one-to-one map $\phi : \mathcal{M} \to \mathcal{M}'$ with inverse $\phi^{-1} : \mathcal{M}' \to \mathcal{M}$.

Then $n = n' = k$, ϕ is injective and surjective.

Thm: If ϕ_* is injective and surjective at p then there is an open $\mathcal{U} \in \mathcal{M}$, $p \in \mathcal{U}$ + $\phi : \mathcal{U} \to \phi(\mathcal{U})$ is a diffeomorphism.

That is if $\phi_* : T_p \to T_{\phi(p)}$ is an isomorphism

then ϕ is a local diffeomorphism.

With a diffeomorphism we can go with $\phi_* : T_p(\mathcal{M}) \to T_{\phi(p)}(\mathcal{M}')$

and with $(\phi^{-1})^* : T_p^*(\mathcal{M}) \to T^*_{\phi(p)}(\mathcal{M}')$

So for any tensor in T^r_s

$$T(\tilde{\omega}^1, \dots, \tilde{\omega}^s, \vec{M}_1, \dots, \vec{M}_r)\Big|_p = \phi_* T\left((\phi^{-1})^*\tilde{\omega}^1, \dots, (\phi^{-1})^*\tilde{\omega}^s, \phi_* M_1, \dots \phi_* M_r\right)\Big|_{\phi(p)}$$

Differentiation without a connection

Two types arise naturally:

- Exterior derivative
- Lie Derivative

Exterior derivative $d : \Omega_s \to \Omega_{s+1}$

Ω_s : linear space of s-forms $\tilde{a} = a_{\mu_1 \cdots \mu_s} dx^{\mu_1} \wedge \cdots dx^{\mu_s}$

($\Omega_s \subset T^0_s$, is the totally antisymmetric T^0_s tensors).

Recall if $\tilde{a}$ & $\tilde{b}$ are p & q forms, $\tilde{a} \wedge \tilde{b} = (-1)^{pq} \tilde{b} \wedge \tilde{a}$.

d acts by

$$d\tilde{a} = da_{\mu_1 \cdots \mu_s} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_s}$$

$$= \frac{\partial a_{\mu_1 \cdots \mu_s}}{\partial x^{\sigma}} dx^{\sigma} \wedge dx^{\mu_1} \cdots \wedge dx^{\mu_s}$$

Exercise: show

- this is indeed a T^0_{s+1} (tensor) (obvious from first line).
- $d(a \wedge b) = d\tilde{a} \wedge \tilde{b} + (-1)^s a \wedge db$ if a is an s-form
- $d(d\tilde{a}) = 0$
- $d(\phi^* \tilde{a}) = \phi^*(d\tilde{a})$

Useful integration results (reminder)

if ϕ is a diffeomorphism and $\tilde{a}$ is an n-form ($n = \dim M$)

$$\int_M \tilde{a} = \int_{M' = \phi(M)} \phi_* \tilde{a}$$

If $\tilde{b}$ is an $n-1$ form

$$\int_{\partial M} \tilde{b} = \int_M d\tilde{b} \qquad \text{Stoke's theorem.}$$

Lie derivative

~~Let~~ $\vec{M}$ vector field on $\mathcal{M}$

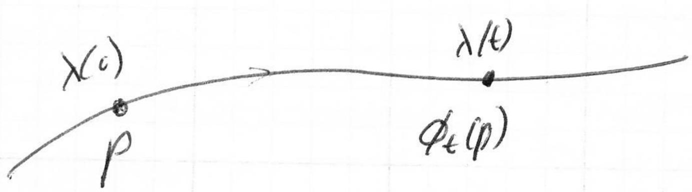
Thm. (Fundamental th of diff eqs) $\Longleftrightarrow$ unique ~~point~~ curve $\lambda(t)$ through p with $\lambda(0) = p$ and $\vec{M} = \frac{\partial}{\partial t}$

~~With~~ Locally, with coordinates x^μ, $\lambda(t)$ is $x^\mu(t)$ ~~and~~ with tangent $\frac{dx^\mu}{dt}$; so the theorem above is the statement of uniqueness of solution of

$$\frac{dx^\mu(t)}{dt} = M^\mu(x(t))$$

$\lambda(t)$ is the "integral curve of $\vec{M}$"

Given $\vec{M}$ we can construct a diffeomorphism ϕ_t of $\mathcal{M}$ into itself (actually from small open neighborhoods $U \ni p$ into $\mathcal{M}$), that maps p into the point along the curve a ~~distance t~~ parameter t away

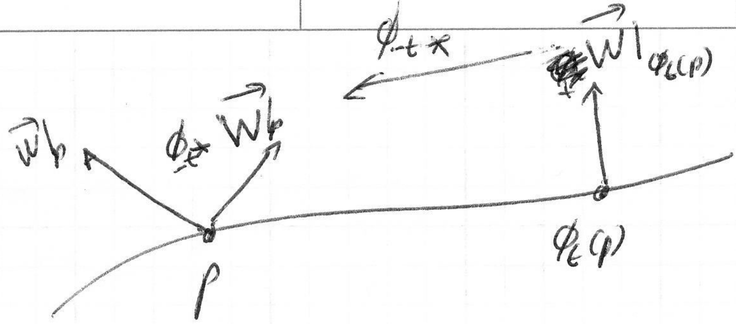


(Note ϕ_t forms a one parameter local group of diffeomorphisms

$$\phi_{t+s} = \phi_t \circ \phi_s = \phi_s \circ \phi_t \qquad \phi_{-t} = (\phi_t)^{-1} \qquad \phi_0 = e \text{ identity})$$

From ϕ_t construct $\phi_{t*} : T^r_s(p) \to T^r_s(\phi(p))$

$$T|_p \to \phi_{t*} T|_{\phi_t(p)}$$



Since ϕ_{t*} is a diffeomorphism, ϕ_{t*} is an isomorphism, we can directly compare $\phi_{t*}\vec{W}$ with T. Define Lie derivative at p by

$$\mathcal{L}_{\vec{M}} T = \lim_{t \to 0} \frac{1}{t}\left[\phi_{-t*} T_p - T_p\right]$$

Note: both at p.

Properties:

(i) If $T \in T^r_s(p) \Rightarrow \mathcal{L}_M T \in T^r_s(p)$

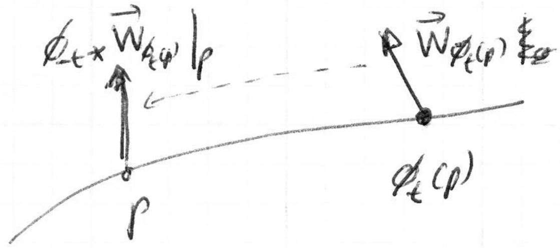
(ii) $\mathcal{L}_{\vec{M}}$ is linear

(iii) $\mathcal{L}_{\vec{M}}$ preserves contractions

(iv) $\mathcal{L}_{\vec{M}}(T \otimes S) = \mathcal{L}_{\vec{M}} T \otimes S + T \otimes \mathcal{L}_{\vec{M}} S$

(v) $\mathcal{L}_{\vec{M}} f = \vec{M}(f)$ (f a function $f: M \to \mathbb{R}$).

Get $\mathcal{L}_{\vec M} \vec W$ explicitly:



Recall $\qquad \mathcal{M} \xrightarrow{\phi} \mathcal{M}'$

then $\qquad \vec W_p \to \phi_* \vec W|_{\phi(p)}$

means $\qquad W^\mu \to (\phi_* W)^a\Big|_{\phi(p)} = \dfrac{\partial y^a}{\partial x^\mu}\Big|_p W^\mu\Big|_p$

Moreover, for our case ϕ_{-t*} at p is what? Take

$$\mathcal{M} \xrightarrow{\phi_{-t}} \mathcal{M}'$$

$$\vec W_{\phi_t(p)} \to \phi_{-t*}\vec W|_p$$

$$(\phi_{-t*}W)^a|_p = \frac{\partial x^a}{\partial x^\mu}\Big|_{\phi_t(p)} W^\mu\Big|_{\phi_t(p)}$$

But $y^a(x^\mu)$ is just the shift in coordinates along the curve: x^a are the coordinates of p, in terms of x^μ, the coordinates of $\phi_t(p)$. If the curve is the integral of $\dfrac{dx^\mu}{dt} = M^\mu$ $\quad$ ($\vec M$ a vector field).

then, $x^\mu(t) = x^\mu(0) + tM^\mu$ to order t, and $y^a(x^\mu)$ is just $x^\mu(0) = x^\mu(t) - tM^\mu$ so $\dfrac{\partial y^\nu}{\partial x^\mu}\Big|_{\phi_t(p)} = \delta^\nu_\mu - M^\nu{}_{,\mu}\, t$

$W^\mu|_{\phi_t(p)}$ is just $W^\mu(x^\mu(t)) = W^\mu(x^\mu(0) + tM^\mu) = W^\mu|_p + tM^\nu W^\mu{}_{,\nu}|$

so $\phi_{-t*}W|_p - W|_p = (\delta^\nu_\mu - M^\nu{}_{,\mu}t)(W^\mu + tM^\nu W^\mu{}_{,\nu}) - W^\nu$

and $(\mathcal{L}_{\vec M}\vec W)^\mu = M^\nu W^\mu{}_{,\nu} - W^\nu M^\mu{}_{,\nu} = [M, W]^\mu$ $\quad$ "Lie bracket" "commutator"

In particular, this shows $\mathcal{L}_{\vec M}\vec W = -\mathcal{L}_{\vec W}\vec M$

From this, ~~two~~ one can obtain ~~the res~~ the action of $\mathcal{L}_{\vec M}$ on other tensors:

$$\mathcal{L}_{\vec M}(\tilde\omega\otimes\vec W) = \mathcal{L}_{\vec M}\tilde\omega\otimes\vec W + \tilde\omega\otimes\mathcal{L}_{\vec M}\vec W$$

now, contracting $\Rightarrow$

$$\mathcal{L}_{\vec M}(\tilde\omega(\vec W)) = \mathcal{L}_{\vec M}\tilde\omega(\vec W) + \tilde\omega(\mathcal{L}_{\vec M}(\vec W))$$

Now if we use $\vec W = \vec E_\mu$, a basis vector we can get $\mathcal{L}_{\vec M}\tilde\omega$.

In particular, if $\vec E_\mu = \frac{\partial}{\partial x^\mu}$, the coordinate basis, then

$$\mathcal{L}_{\vec M}(\tilde\omega)(\vec E_\mu) = (\mathcal{L}_{\vec M}(\tilde\omega))_\mu$$ the components we are looking for.

$$\mathcal{L}_{\vec M}(\tilde\omega(\vec E_\mu)) = \mathcal{L}_{\vec M}(\omega_\mu) = \vec M(\omega_\mu) \quad \text{(property (i))}$$

$$= \frac{\partial\omega_\mu}{\partial x^\nu}M^\nu = \omega_{\mu,\nu}M^\nu$$

and $\tilde\omega(\mathcal{L}_{\vec M}(\vec E_\mu))^\nu = \dfrac{\partial(\vec E_\mu)^\nu}{\partial x^\rho}M^\rho - \dfrac{\partial M^\nu}{\partial x^\rho}(E_\mu)^\rho = -\dfrac{\partial M^\nu}{\partial x^\mu}$

so $\tilde\omega(\mathcal{L}_{\vec M}(\vec E_\mu)) = -\omega_\nu M^\nu{}_{,\mu}$

$$\Rightarrow (\mathcal{L}_{\vec M}(\tilde\omega))_\mu = \omega_{\mu,\nu}M^\nu + M^\nu{}_{,\mu}\omega_\nu$$

Exercise: Show

$$\mathcal{L}_{\vec{M}} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell} = M^\sigma \partial_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}$$

$$- (\partial_\lambda M^{\mu_1}) T^{\lambda \mu_2 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}$$

$$- (\partial_\lambda M^{\mu_2}) T^{\mu_1 \lambda \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}$$

$$\vdots$$

$$+ (\partial_{\nu_1} M^\lambda) T^{\mu_1 \cdots \mu_k}{}_{\lambda \nu_2 \cdots \nu_\ell}$$

$$+ (\partial_{\nu_2} M^\lambda) T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \lambda \cdots \nu_\ell}$$

In particular

$$\mathcal{L}_{\vec{M}} g_{\mu\nu} = M^\sigma \partial_\sigma g_{\mu\nu} + \partial_\mu M^\lambda g_{\lambda\nu} + \partial_\nu M^\lambda g_{\mu\lambda}$$

Since these are <u>tensors</u> equations, we can replace ∂ by ∇.

$$\Rightarrow \quad \mathcal{L}_M g_{\mu\nu} = \cancel{M} \; M^\lambda{}_{;\mu} g_{\lambda\nu} + M^\lambda{}_{;\nu} g_{\mu\lambda} = M_{\nu;\mu} + M_{\mu;\nu}$$

or

$$\boxed{\mathcal{L}_M g_{\mu\nu} = 2 M_{(\mu;\nu)}}$$

~~Easy~~ This is useful stuff. We will use it for symmetries later, but ~~the~~ here is a simple application. Assume the action for GR breaks down into

$$S = S_G(g_{\mu\nu}) + S_M(g_{\mu\nu}, \psi) \qquad (\star)$$

ψ = matter fields

S_G = "Hilbert" action (gives Einstein's eqs... we'll see this later in course).

~~Consider~~ This theory is "diffeomorphism invariant": ~~then~~ given $\phi: M \to M$

$$(M, g_{\mu\nu}, \psi) \quad \text{and} \quad (M, \phi^* g_{\mu\nu}, \phi^* \psi)$$

represent the same physics. The change in S_M under a diffeomorphism

$$\delta S_M = \int d^n x \, \frac{\delta S_M}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \int d^n x \, \frac{\delta S_M}{\delta \psi} \delta \psi$$

Since we could have set $\psi = 0$, δS_G can be considered separately (it is invariant by itself; here is where the separation assumption in $(\star)$ comes in).

But $\frac{\delta S_M}{\delta \psi} = 0$ for <u>any</u> variation. So while here we look only at variations from diffeomorphisms, that term vanishes separately for any variation. Left with first term, we consider diffeomorphisms generated by a vector field U^μ:

$$\delta g_{\mu\nu} = \mathcal{L}_U g_{\mu\nu} = 2 U_{(\mu;\nu)}$$

$$\Rightarrow \quad \delta S_M = 0 = \int d^n x \, \frac{\delta S_M}{\delta g_{\mu\nu}} 2 U_{(\mu;\nu)} = 4 \int d^n x \, \frac{\delta S_M}{\delta g_{\mu\nu}} U_{\mu;\nu}$$

or

$$= 4 \int d^n x \left(\left[\frac{\delta S_M}{\delta g_{\mu\nu}} U_\mu \right]_{;\nu} - U_\mu \left(\frac{\delta S_M}{\delta g_{\mu\nu}} \right)_{;\nu} \right)$$

Dropping the surface term and multiplying by $\frac{\sqrt{-g}}{\sqrt{-g}}$ we have

$$\int dV \; U_\mu \nabla_\nu \left[\frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}} \right] = 0$$

~~If~~ Since this holds for arbitrary U_μ (diffeomorphisms generated by arbitrary vector fields) it must be that

$$\nabla_\nu \left(\frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}} \right) = 0$$

But $\quad T^{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}} \quad$ is the energy-momentum tensor.