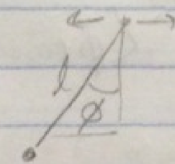
Problem 1:

(a) horizontally, with $x = x_0 \cos\omega t$



$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgl(1-\cos\phi)$$

$x = l\sin\phi + x_0\cos\omega t$

$\dot{x} = l\dot{\phi}\cos\phi - x_0\omega\sin\omega t$

$y = l(1-\cos\phi)$

$\dot{y} = l\dot{\phi}\sin\phi$

$$= \frac{1}{2}m\left[(l\dot{\phi}\cos\phi - x_0\omega\sin\omega t)^2 + l^2\dot{\phi}^2\sin^2\phi\right] - mgl(1-\cos\phi)$$

$$L = \frac{1}{2}m\left[l^2\dot{\phi}^2 - x_0^2\omega^2\sin^2\omega t - 2lx_0\omega\dot{\phi}\cos\phi\sin\omega t\right] - mgl(1-\cos\phi)$$

EOM $\rightarrow$ $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{\partial L}{\partial\phi}$

$$\frac{d}{dt}\left(ml^2\dot{\phi} - mlx_0\omega\sin\omega t\cos\phi\right) = -mgl\sin\phi + mlx_0\omega\dot{\phi}\sin\phi\sin\omega t$$

$$l^2\ddot{\phi} = lx_0\omega\left(\omega\cos\omega t\cos\phi - \cancel{\dot{\phi}\sin\omega t\sin\phi}\right) - gl\sin\phi + \cancel{lx_0\omega\dot{\phi}\sin\phi\sin\omega t}$$

$$\ddot{\phi} = -\frac{g}{l}\sin\phi + \frac{x_0\omega^2}{l}\cos\omega t\cos\phi \leftarrow \text{EOM}$$

let $\phi = \bar{\phi} + \tilde{\phi} \leftarrow$ fast part

$$\ddot{\bar{\phi}} + \ddot{\tilde{\phi}} = -\frac{g}{l}\sin[\bar{\phi} + \tilde{\phi}] + \frac{x_0\omega^2}{l}\cos\omega t\cos[\bar{\phi} + \tilde{\phi}]$$

Expand

$$\underbrace{\ddot{\bar{\phi}}}_{S} + \underbrace{\ddot{\tilde{\phi}}}_{F} = \underbrace{-\frac{g}{l}\sin\bar{\phi}}_{S} - \underbrace{\frac{g}{l}\tilde{\phi}\cos\bar{\phi}}_{F} + \underbrace{\frac{x_0\omega^2}{l}\overset{\cos\bar{\phi}}{\cos\omega t}}_{F} - \underbrace{\frac{x_0\omega^2}{l}\tilde{\phi}\cos\omega t\sin\bar{\phi}}_{S}$$

Average

$$\ddot{\bar{\phi}} = -\frac{g}{l}\sin\bar{\phi} - \frac{x_0\omega^2}{l}\sin\bar{\phi}\langle\tilde{\phi}\cos\omega t\rangle \leftarrow \text{need to find } \tilde{\phi}$$

Fast Parts

Forget this part because $\omega^2 \gg g/l$

$$\ddot{\tilde{\phi}} = \cancel{-\frac{g}{l}\tilde{\phi}\cos\bar{\phi}} + \frac{x_0\omega^2}{l}\cos\omega t\cos\bar{\phi}$$

$$\tilde{\phi} = a(\bar{\phi})\cos\omega t \rightarrow -\omega^2 a(\bar{\phi})\cos\omega t = \frac{x_0\omega^2}{l}\cos\bar{\phi}\cos\omega t$$

$$a(\bar{\phi}) = -\frac{x_0}{l}\cos\bar{\phi}$$

Plug in for $\tilde{\phi}$ $\qquad \langle \cos^2 \omega t \rangle = \frac{1}{2}$

then

$$\langle \tilde{\phi} \cos\omega t \rangle = \langle -\frac{x_0}{\ell} \cos\bar{\phi} \cos^2\omega t \rangle$$

$$= -\frac{x_0}{2\ell} \cos\bar{\phi}$$

Back to Averaged Eq.

$$\ddot{\bar{\phi}} = -\frac{g}{\ell}\sin\bar{\phi} - \frac{x_0^2\omega^2}{2\ell^2}\cos\bar{\phi}\sin\bar{\phi}$$

$$\ddot{\bar{\phi}} = -\frac{g}{\ell}\sin\bar{\phi} - \frac{x_0^2\omega^2}{4\ell^2}\frac{d}{d\bar{\phi}}\left(\sin^2\bar{\phi}\right)$$

AND

$$\ddot{\bar{\phi}} = -\frac{d}{d\bar{\phi}}\Big[\underbrace{-\frac{g}{\ell}\cos\bar{\phi} + \frac{x_0^2\omega^2}{4\ell^2}\sin^2\bar{\phi}}\Big]$$

$$U_{\text{effective}} \rightarrow -\frac{g}{\ell}\cos\bar{\phi} + \frac{x_0^2\omega^2}{4\ell^2}\sin^2\bar{\phi}$$

$$\searrow \left(\frac{1}{2} - \frac{1}{2}\cos 2\bar{\phi}\right)$$

$$\frac{dU}{d\bar{\phi}} = \frac{g}{\ell}\sin\bar{\phi} + \frac{x_0^2\omega^2}{4\ell^2}\sin 2\bar{\phi} = 0$$

$$\boxed{\text{Extrema for } \bar{\phi} = 0, \pi}$$

For Stability

look @ $$\frac{d^2U}{d\bar{\phi}^2} = \frac{g}{\ell}\cos\bar{\phi} + \frac{x_0^2\omega^2}{2\ell^2}\cos(2\bar{\phi}) > 0 \quad \text{for } \bar{\phi} = \pi$$

$$-\frac{g}{\ell} + \frac{x_0^2\omega^2}{2\ell^2} > 0$$

$$\boxed{\frac{\omega^2 x_0^2}{2\ell^2} > \frac{g}{\ell} \quad \text{for stable equil. @ } \bar{\phi} = \pi}$$

(b) In a circle: $x_1 = r_0\cos\omega t$ $y_1 = r_0\sin\omega t$

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$x = l\sin\phi + r_0\cos\omega t$$
$$\dot{x} = l\dot{\phi}\cos\phi - r_0\omega\sin\omega t$$
$$y = l(1-\cos\phi) + r_0\sin\omega t$$
$$\dot{y} = l\dot{\phi}\sin\phi + r_0\omega\cos\omega t$$

$$L = \frac{1}{2}m\Big(l^2\dot{\phi}^2 + r_0^2\omega^2\sin^2\omega t - 2r_0\omega l\dot{\phi}\cos\phi\sin\omega t$$
$$+ r_0^2\omega^2\cos^2\omega t + 2r_0\omega l\dot{\phi}\sin\phi\cos\omega t\Big) - mgl(1-\cos\phi) - \cancel{mgr_0\sin\omega t}$$

EOM: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$

$$ml^2\ddot{\phi} + mr_0\omega l\frac{d}{dt}\left[\sin\phi\cos\omega t - \cos\phi\sin\omega t\right] = -mgl\sin\phi + mr_0\omega l\left[\dot{\phi}\sin\phi\sin\omega t + \dot{\phi}\cos\phi\cos\omega t\right]$$

$$l^2\ddot{\phi} + r_0\omega l\left[\cancel{\dot{\phi}\cos\phi\cos\omega t} - \omega\sin\phi\sin\omega t + \cancel{\dot{\phi}\sin\phi\sin\omega t} - \omega\cos\phi\cos\omega t\right] = -mgl\sin\phi$$
$$+ r_0\omega l\left[\cancel{\dot{\phi}\sin\phi\sin\omega t} + \cancel{\dot{\phi}\cos\phi\cos\omega t}\right]$$

$$\ddot{\phi} = \frac{r_0\omega^2}{l}\left[\sin\phi\sin\omega t + \cos\phi\cos\omega t\right] - \frac{g}{l}\sin\phi$$

we can write $\phi = \bar{\phi} + \tilde{\phi}$ made of a slowly varying and fast varying parts

$$\rightarrow \ddot{\bar{\phi}} + \ddot{\tilde{\phi}} = \frac{r_0\omega^2}{l}\left[\sin(\bar{\phi}+\tilde{\phi})\sin\omega t + \cos(\bar{\phi}+\tilde{\phi})\cos\omega t\right] - \frac{g}{l}\sin\left[\bar{\phi}+\tilde{\phi}\right]$$

Expand

$$\underset{S}{\ddot{\bar{\phi}}} + \underset{F}{\ddot{\tilde{\phi}}} = \underset{F}{\frac{r_0\omega^2}{l}\sin\bar{\phi}\sin\omega t} + \underset{S}{\frac{r_0\omega^2}{l}\tilde{\phi}\cos\bar{\phi}\sin\omega t} + \underset{F}{\frac{r_0\omega^2}{l}\cos\bar{\phi}\cos\omega t} - \underset{S}{\frac{r_0\omega^2}{l}\tilde{\phi}\sin\bar{\phi}\cos\omega t}$$
$$- \underset{S}{\frac{g}{l}\sin\phi} - \underset{F}{\frac{g}{l}\tilde{\phi}\cos\bar{\phi}}$$

Take Time Average so only slowly varying terms survive

$$\ddot{\bar{\phi}} = -\frac{g}{l}\sin\phi + \frac{r_0\omega^2}{l}\cos\bar{\phi}\langle\tilde{\phi}\sin\omega t\rangle - \frac{r_0\omega^2}{l}\sin\bar{\phi}\langle\tilde{\phi}\cos\omega t\rangle$$

To find $\tilde{\phi}$, we can just look at the fast varying parts

$$\ddot{\tilde{\phi}} = \frac{r_0\omega^2}{l}\sin\bar{\phi}\sin\omega t + \frac{r_0\omega^2}{l}\cos\bar{\phi}\cos\omega t - \cancel{\frac{g}{l}\tilde{\phi}\cos\bar{\phi}}$$

Because $\omega^2 \gg \frac{g}{l}$

$$\tilde{\phi} = a(\bar{\phi})\sin\omega t + b(\bar{\phi})\cos\omega t$$

$$\hookrightarrow -\omega^2 a(\bar{\phi})\sin\omega t - \omega^2 b(\bar{\phi})\cos\omega t = \frac{r_0\omega^2}{l}\sin\bar{\phi}\sin\omega t + \frac{r_0\omega^2}{l}\cos\bar{\phi}\cos\omega t$$

$$\rightarrow a(\bar{\phi}) = -\frac{r_0}{l}\sin\bar{\phi} \qquad b(\bar{\phi}) = -\frac{r_0}{l}\cos\bar{\phi}$$

Now Back to Average Equation

$$\ddot{\bar{\phi}} = -\frac{g}{l}\sin\bar{\phi} + \frac{r_0\omega^2}{l}\cos\bar{\phi}\langle\tilde{\phi}\sin\omega t\rangle - \frac{r_0\omega^2}{l}\sin\bar{\phi}\langle\tilde{\phi}\cos\omega t\rangle$$

$$\ddot{\bar{\phi}} = -\frac{g}{l}\sin\bar{\phi} + \frac{r_0^2\omega^2}{l^2}\Big[-\sin\bar{\phi}\cos\bar{\phi}\langle\sin^2\omega t\rangle - \cos^2\bar{\phi}\langle\cancel{\sin\omega t\cos\omega t}\rangle + \sin^2\bar{\phi}\langle\cancel{\sin\omega t\cos\omega t}\rangle + \cos\bar{\phi}\sin\bar{\phi}\langle\cos^2\omega t\rangle\Big]$$

$$\ddot{\bar{\phi}} = \frac{-g}{l}\sin\bar{\phi} - \cancel{\frac{r_0^2\omega^2}{2l^2}\sin\bar{\phi}\cos\bar{\phi}} + \cancel{\frac{r_0^2\omega^2}{2l^2}\sin\bar{\phi}\cos\bar{\phi}}$$

$$\ddot{\bar{\phi}} = \frac{-g}{l}\sin\bar{\phi}$$ ← Back to Regular old pendulum

Stable Equilibrium

Only stable orbit is @ $\theta = 0$
Circular motion destroys to ponderamotive force which creates a stable equil.
@ $\phi = \pi$

2. 

$\uparrow\downarrow\ y = y_0 \cos \omega t$

$\omega_0 = \sqrt{g/l} \quad \omega = 2\omega_0 + \varepsilon$

$y' = l(1-\cos\theta)$

$x = l\sin\theta$

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + (\dot{y}' + \dot{y})^2\right) - mg(y' + y)$$

$$= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{y}'^2 + 2\dot{y}\dot{y}'\right) + mg(y' + y)$$

$$= \frac{1}{2}m\left(l^2\dot{\theta}^2 + \omega^2 y_0^2 \sin^2\omega t + 2y_0 l\omega\dot{\theta}\sin\theta\sin\omega t\right) + mg(l\cos\theta - y_0\cos\omega t)$$

$$\Rightarrow m\left(l^2\ddot{\theta} + y_0 l\omega\cos\theta\sin\omega t\,\dot{\theta} + y_0 l\omega^2\sin\theta\cos\omega t\right) = my_0 l\omega\dot{\theta}\cos\theta\sin\omega t - mgl\sin\theta$$

$$\Rightarrow l^2\ddot{\theta} + y_0 l\omega^2\sin\theta\cos\omega t = -gl\sin\theta$$

$$l^2\ddot{\theta} + y_0 l\omega^2\theta\cos\omega t \approx -gl\theta$$

$$\ddot{\theta} + \frac{y_0\omega^2\theta}{l}\cos\omega t = -\frac{g}{l}\theta$$

$$\ddot{\theta} + \left(\frac{y_0\omega^2\cos\omega t + g}{l}\right)\theta = 0$$

$$\ddot{\theta} + \omega_0^2\theta + \frac{y_0}{l}\omega^2\cos\omega t\,\theta = 0$$

$$\ddot{\theta} + \omega_0^2\theta + \frac{y_0}{l}(4\omega_0^2 + 4\omega_0\varepsilon)\cos\omega t\,\theta = 0$$

$$\ddot{\theta} + \omega_0^2\theta + \frac{y_0}{l}(4\omega_0^2 + \cancelto{0}{4\omega_0\varepsilon})\cos(2\omega_0+\varepsilon)\theta = 0$$

$$\ddot{\theta} + \omega_0^2\left(1 + \frac{4y_0}{l}\cos(2\omega_0+\varepsilon)\right)\theta = 0$$ ← Mathieu's

let $\theta = a(t)\cos[(\omega_0+\varepsilon/2)t] + b(t)\sin[(\omega_0+\varepsilon/2)t]$

$$\dot\theta = \dot a\cos(\tfrac{\omega}{2}t) - a(\omega_0+\varepsilon/2)\sin(\tfrac{\omega}{2}t) + \dot b\sin(\omega/2\, t) + b\,\omega/2\cos\omega/2\, t$$

$$\ddot\theta = \ddot a\cos - \dot a\frac{\omega}{2}\sin - \dot a\,\omega/2\sin - a\frac{\omega^2}{4}\cos + \ddot b\sin + \dot b\frac{\omega}{2}\cos + \dot b\frac{\omega}{2}\cos$$
$$- b\frac{\omega^2}{4}\sin$$

$$\ddot\theta = \ddot a\cos(\omega_0+\varepsilon/2)t - 2\dot a(\omega_0+\tfrac{\varepsilon}{2})\sin(\omega_0+\tfrac{\varepsilon}{2})t - a(\omega_0+\tfrac{\varepsilon}{2})^2\cos(\omega_0+\tfrac{\varepsilon}{2})t$$
$$+ \ddot b\sin(\omega_0+\varepsilon/2)t + 2\dot b(\omega_0+\varepsilon/2)\cos(\omega_0+\varepsilon/2)t - b(\omega_0+\varepsilon/2)^2\sin(\omega_0+\varepsilon/2)t$$

neglect $\ddot a, \ddot b$

$$\ddot\theta = -2\dot a\omega_0\sin(\omega_0+\tfrac{\varepsilon}{2})t - a(\omega_0^2+\omega_0\varepsilon)\cos(\omega_0+\varepsilon/2)t$$
$$+ 2\dot b\omega_0\cos(\omega_0+\varepsilon/2)t - b(\omega_0^2+\omega_0\varepsilon)\sin(\omega_0+\varepsilon/2)t$$

$$\ddot\theta + \omega_0^2\left(1+\frac{4y_0}{l}\cos(2\omega_0+\varepsilon)t\right)\theta = 0$$

$$\rightarrow -2\dot a\omega_0\sin(\omega_0+\tfrac{\varepsilon}{2})t - a\omega_0\varepsilon\cos(\omega_0+\varepsilon/2)t$$
$$+ 2\dot b\omega_0\cos(\omega_0+\varepsilon/2)t - b\omega_0\varepsilon\sin(\omega_0+\varepsilon/2)t$$
$$+ \frac{4y_0\omega_0}{l}\cos(2\omega_0+\varepsilon)t\,\left(a\cos(\omega_0+\varepsilon/2)t + b\sin(\omega_0+\varepsilon/2)t\right)$$

define $\frac{4y_0}{l} = h$ simplify $\cos(2\cdots)\cos$ and $\cos(2\cdots)\sin$

$$\rightarrow \sin(\omega_0+\varepsilon/2)(-2\omega_0\dot a - b\omega_0\varepsilon - \omega_0^2 hb/2)$$
$$+ \cos(\omega_0+\varepsilon/2)(2\omega_0\dot b - a\varepsilon\omega_0 + \omega_0^2 ha/2) = 0$$

$\rightarrow a(t) = a_0e^{st}$ $\qquad s a_0 + (\varepsilon/2 + \omega_0 h/4)b_0 = 0$

$b(t) = b_0e^{st}$ $\qquad (-\varepsilon/2 + \omega_0 h/4)a_0 + s b_0 = 0$

$$s^2 = \frac{\omega_0^2h^2}{16} - \frac{\varepsilon^2}{4} > 0 \quad \Big| \quad \varepsilon^2 < \frac{\omega_0^2 h}{4} = \frac{\omega_0^2 h}{l}$$

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3. Starting from Mathieu's equation, add a linear damping term:

$$\ddot{\phi} + 2\lambda\dot{\phi} + \omega_0^2\left[1 + h\cos(2\omega_0+\epsilon)t\right]\phi = 0$$

Since we only need to calculate the range of ϵ that will cause instability, and not the growth rate of said instability, it is sufficient to consider the endpoints of the range, where the growth rate is zero and the amplitudes a and b are constants.

Trying $\phi = a\cos(\omega_0+\frac{\epsilon}{2})t + b\sin(\omega_0+\frac{\epsilon}{2})t$, and using the shorthand $() \equiv [(\omega_0+\frac{\epsilon}{2})t]$,

$$(*) \quad -a(\omega_0+\tfrac{\epsilon}{2})^2\cos() - b(\omega_0+\tfrac{\epsilon}{2})^2\sin() - 2\lambda a(\omega_0+\tfrac{\epsilon}{2})\sin() + 2\lambda b(\omega_0+\tfrac{\epsilon}{2})\cos()$$
$$+ a\omega_0^2\cos() + a\omega_0^2 h\cos(2\omega_0+\epsilon)t\cos() + b\omega_0^2\sin() + b\omega_0^2 h\cos(2\omega_0+\epsilon)t\sin() = 0$$

As in the previous problem, we use the product-to-sum trigonometric identities and omit the higher frequency terms

$$\cos(2\omega_0+\epsilon)t\cdot\cos(\omega_0+\tfrac{\epsilon}{2})t = \tfrac{1}{2}\cos(\omega_0+\tfrac{\epsilon}{2})t + \tfrac{1}{2}\cos 3(\omega_0+\tfrac{\epsilon}{2})t$$
$$\cos(2\omega_0+\epsilon)t\cdot\sin(\omega_0+\tfrac{\epsilon}{2})t = -\tfrac{1}{2}\sin(\omega_0+\tfrac{\epsilon}{2})t + \tfrac{1}{2}\sin 3(\omega_0+\tfrac{\epsilon}{2})t$$

Assuming that λ is a small number, of order ϵ, we retain terms in $(*)$ to first order in ϵ to get

$$(-\cancel{a\omega_0^2} - a\omega_0\epsilon)\cos() + (-\cancel{b\omega_0^2} - b\omega_0\epsilon)\sin() - 2\lambda a\omega_0\sin() + 2\lambda b\omega_0\cos()$$
$$+ \cancel{a\omega_0^2\cos()} + \tfrac{1}{2}a\omega_0^2 h\cos() + \cancel{b\omega_0^2\sin()} - \tfrac{1}{2}b\omega_0^2 h\sin() = 0$$

$$\Longrightarrow (-a\omega_0\epsilon + 2\lambda b\omega_0 + \tfrac{1}{2}a\omega_0^2 h)\cos() + (-b\omega_0\epsilon - 2\lambda a\omega_0 - \tfrac{1}{2}b\omega_0^2 h)\sin() = 0$$

The coefficients of the sine and cosine must both be zero, giving

$$-a\epsilon + 2\lambda b + \tfrac{1}{2}a\omega_0 h = 0 \quad\Longrightarrow\quad (\tfrac{1}{2}\omega_0 h - \epsilon)a + 2\lambda b = 0$$
$$b\epsilon + 2\lambda a + \tfrac{1}{2}b\omega_0 h = 0 \quad\Longrightarrow\quad 2\lambda a + (\tfrac{1}{2}\omega_0 h + \epsilon)b = 0$$

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This has nontrivial solutions if the determinant of the matrix of coefficients of a and b is zero, i.e.

$$\left(\tfrac{1}{2}\omega_0 h - \epsilon\right)\left(\tfrac{1}{2}\omega_0 h + \epsilon\right) - (2\lambda)^2 = 0$$
$$\Rightarrow \left(\tfrac{1}{2}\omega_0 h\right)^2 - \epsilon^2 - 4\lambda^2 = 0$$
$$\Rightarrow \epsilon^2 = \left(\tfrac{1}{2}\omega_0 h\right)^2 - 4\lambda^2$$

Recalling from problem 2 that $h = 4y_0/l$ for this pendulum with the vertically oscillating support, we find that instability occurs when

$$\epsilon^2 < \left(\frac{2y_0\omega_0}{l}\right)^2 - 4\lambda^2$$

Specifically, in the presence of linear frictional damping, there is a threshold amplitude $y_0^{(MIN)} = \lambda l/\omega_0$, below which parametric instability will never set in, even for $\epsilon \to 0$.

200B Problem Set I, Solution to No.4

Emily Nardoni

1/21/14

Problem Statement:

Let $H(q,p,t) = H_0(q,p) + V(q)\frac{d^2A}{dt^2}$, for $A(t)$ periodic with period $\tau \ll T$, and T the period of the motion governed by H_0.

(a) Derive the mean field (i.e. short time averaged) equations for this system.

Solution:

First, we derive the Lagrangian from the given Hamiltonian, and find the Euler-Lagrange equations of motion:

$$
\begin{aligned}
H &= \frac{p^2}{2m} + V_0(q) + V(q)\frac{d^2A}{dt^2} \\
\mathcal{L} &= p\dot{q} - H, \qquad \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \;\Rightarrow p = m\dot{q} \\
\mathcal{L} &= \frac{1}{2}m\dot{q}^2 - V_0(q) - V(q)\frac{d^2A}{dt^2}
\end{aligned}
$$

By the Euler-Lagrange equations,

$$
\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} \quad \Rightarrow \quad -\frac{\partial V_0(q)}{\partial q} - \frac{\partial V(q)}{\partial q}\frac{d^2A}{dt^2} = m\ddot{q}
$$

Now, we separate the motion into a centroid part $y(t)$ and a quiver motion $\epsilon(t)$:

$$
q(t) = y(t) + \epsilon(t), \qquad \epsilon \backsim \cos\omega t, \;\; \omega = \frac{2\pi}{\tau}
$$

We plug this into the Euler-Lagrange equations, expand (assuming ϵ is small), and pick out the slow and fast parts:

$$
-\frac{\partial}{\partial q}V_0(y+\epsilon) - \frac{\partial}{\partial q}V(y+\epsilon)\frac{d^2A(t)}{dt^2} = m\ddot{y} + m\ddot{\epsilon}
$$

$$
-\frac{\partial V_0(y)}{\partial y}\bigg|_{\text{slow}} - \epsilon\frac{\partial^2 V_0(y)}{\partial y^2}\bigg|_{\text{fast}} - \left\{\frac{\partial V(y)}{\partial y}\bigg|_{\text{fast}} + \epsilon\frac{\partial^2 V(y)}{\partial y^2}\bigg|_{\text{slow}}\right\}\frac{d^2A(t)}{dt^2} = m\ddot{y}\bigg|_{\text{slow}} + m\ddot{\epsilon}\bigg|_{\text{fast}}
$$

We take the short time average of the previous equation, where $<>= \frac{1}{\tau}\int_0^\tau dt$, and note that $\langle\ddot{\epsilon}\rangle = 0,\ \langle\epsilon\rangle = 0$: then we are left with the "slow", or mean field, equation

$$m\ddot{y} = -\frac{\partial V_0(y)}{\partial y} - \frac{\partial^2 V(y)}{\partial y^2}\left\langle \epsilon\frac{d^2A}{dt^2}\right\rangle$$

The "fast" equation is given by

$$m\ddot{\epsilon} = -\epsilon\frac{\partial^2 V_0(y)}{\partial q^2} - \frac{\partial V}{\partial y}\frac{d^2A}{dt^2}$$

Note that since $\epsilon \backsim \cos\omega t$,

$$\begin{aligned} m\ddot{\epsilon} &\backsim m\omega^2\epsilon \\ \epsilon\frac{\partial^2 V_0(y)}{\partial y^2} &\backsim m\Omega_0^2\epsilon, \qquad \omega^2 \gg \Omega_0^2 \end{aligned}$$

so that we can drop the above term in our fast equation relative to the other terms, and we write our fast equation as

$$m\ddot{\epsilon} = -\frac{\partial V}{\partial y}\frac{d^2A(t)}{dt^2}$$

The idea is to solve the fast equation and plug into the slow equation. Since V is independent of time, we find

$$\begin{aligned} \epsilon = -\frac{1}{m}\frac{\partial V}{\partial q}A \qquad \Rightarrow \qquad -\frac{\partial^2 V(y)}{\partial y^2}\left\langle \epsilon\frac{d^2A}{dt^2}\right\rangle &= \frac{\partial^2 V(y)}{\partial y^2}\left\langle \frac{1}{m}\frac{\partial V}{\partial y}A(t)\frac{d^2A(t)}{dt^2}\right\rangle \\ &= \frac{1}{m}\frac{\partial^2 V(y)}{\partial y^2}\frac{\partial V}{\partial y}\left\langle A(t)\frac{d^2A(t)}{dt^2}\right\rangle \end{aligned}$$

Note that we can rewrite

$$\left\langle A\frac{d^2A}{dt^2}\right\rangle = \left\langle \frac{d}{dt}\left(A\frac{dA}{dt}\right)\right\rangle - \left\langle \left(\frac{dA}{dt}\right)^2\right\rangle = -\left\langle \left(\frac{dA}{dt}\right)^2\right\rangle$$

since

$$A \backsim \sin\omega t \quad \to \quad \frac{d}{dt}\left\langle(A\dot{A})\right\rangle \backsim \frac{d}{dt}\left\langle(\sin\omega t\cos\omega t)\right\rangle = 0$$

Thus we plug this result back in to our slow equation, and (finally) find our mean field equation:

$$\boxed{m\ddot{y} = -\frac{\partial V_0(y)}{\partial y} - \frac{1}{m}\frac{\partial^2 V}{\partial y^2}\frac{\partial V}{\partial y}\left\langle \left(\frac{dA}{dt}\right)^2\right\rangle}$$

(b) Show that these mean field equations may be obtained from the effective Hamiltonian

$$K(p,q) = H_0(p,q) + \frac{1}{4m}\left\langle \left(\frac{dA}{dt}\right)^2 \right\rangle \left(\frac{\partial V(q)}{\partial q}\right)^2$$

where here <> means a short time average, and we may assume $H_0 = \frac{p^2}{2m} + V_0(q)$.

Solution:

Again, we derive the Lagrangian and Euler-Lagrange equations of motion, but this time for our Hamiltonian being the given effective Hamiltonian:

$$\mathcal{L} = p\dot{q} - H, \quad \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \;\rightarrow\; p = m\dot{q}$$

$$\mathcal{L} = \frac{1}{2}m\dot{q}^2 - V_0(q) - \frac{1}{4m}\left\langle \left(\frac{dA}{dt}\right)^2 \right\rangle \left(\frac{\partial V(q)}{\partial q}\right)^2$$

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} \;\Rightarrow\; \boxed{-\frac{\partial V_0(q)}{\partial q} - \frac{1}{2m}\frac{\partial V}{\partial q}\frac{\partial^2 V}{\partial q^2}\left\langle \left(\frac{dA}{dt}\right)^2 \right\rangle = m\ddot{q}}$$

Thus we derive the same mean field equation as before (with $m \rightarrow 2m$, which I believe is a mistake in the problem statement).

5. a). Euler's Eqns: without external torque.

$$\begin{cases} I_1 \frac{d\Omega_1}{dt} = \Omega_2\Omega_3(I_2-I_3) & ① \\ I_2 \frac{d\Omega_2}{dt} = \Omega_3\Omega_1(I_3-I_1) & ② \\ I_3 \frac{d\Omega_3}{dt} = \Omega_1\Omega_2(I_1-I_2). & ③ \end{cases}$$

Derivation: $\frac{d\vec{L}}{dt} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega}\times\vec{L}$.

b). Let $\Omega_2 = \Omega_0 + \eta_2$. $\Omega_1 = \eta_1$. $\Omega_3 = \eta_3$. where $\eta_1 . \eta_2 . \eta_3$ are small.

Then:
1st order:
$$\begin{cases} I_1 \dot{\eta}_1 = \Omega_0 \eta_3 (I_2 - I_3) + O(\eta_2\eta_3) & ④ \\ I_2 \dot{\eta}_2 = O(\eta_3\eta_1) & ⑤ \\ I_3 \dot{\eta}_3 = \Omega_0 \eta_1 (I_1 - I_2) + O(\eta_1\eta_2). & ⑥ \end{cases}$$

Combine ④⑥:
$$\ddot{\eta}_3 = \Omega_0^2 \cdot \frac{(I_1-I_2)(I_2-I_3)}{I_3 I_1} \eta_3.$$

Let $A_3 = \Omega_0^2 \cdot \frac{(I_1-I_2)(I_2-I_3)}{I_3 I_1}$. Then $\ddot{\eta}_3 = A\eta_3$.

$\because I_1 < I_2 < I_3$ $\therefore A > 0$. $\therefore$ instability

Similarly, change $1 \to 2 \to 3$,

we can get $A_1 = \Omega_0^2 \cdot \frac{(I_2-I_3)(I_3-I_1)}{I_1 I_2} < 0$.

$A_2 = \Omega_0^2 \cdot \frac{(I_3-I_1)(I_1-I_2)}{I_2 I_3} < 0$.

$\therefore$ In these two situations, stable.

c). ① Angular momentum.

$$L^2 = L_1^2 + L_2^2 + L_3^2 = I_1^2\Omega_1^2 + I_2^2\Omega_2^2 + I_3^2\Omega_3^2.$$

②. Energy.

$$E = T = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2).$$

(6) We seek the lowest-order nontrivial solution to

$$\ddot{x} + \omega_0^2 x = -\alpha x^2 - \beta x^3. \qquad (1)$$

To begin, let us write x and ω as a series of approximations:

$$x = x^{(1)} + x^{(2)} + x^{(3)} + \dots,$$
$$\omega = \omega_0 + \omega^{(1)} + \omega^{(2)} + \dots,$$

where we can immediately see that

$$x^{(1)} = a\cos\omega t.$$

Rewriting Eq. (1) as

$$\frac{\omega_0^2}{\omega^2}\ddot{x} + \omega_0^2 x = -\alpha x^2 - \beta x^3 - \left(1 - \frac{\omega_0^2}{\omega^2}\right)\ddot{x}, \quad (2)$$

so that the LHS is zero for $x = x^{(1)}$, we now plug in $x = x^{(1)} + x^{(2)}$, $\omega = \omega_0 + \omega^{(1)}$ to obtain

$$\frac{\omega_0^2}{\omega^2}\ddot{x}^{(2)} + \omega_0^2 x^{(2)} = -\alpha\left(a\cos\omega t + x^{(2)}\right)^2 - \beta\left(a\cos\omega t + x^{(2)}\right)^3 - \left(1 - \frac{\omega_0^2}{\omega^2}\right)\left(-a\omega^2\cos\omega t + \ddot{x}^{(2)}\right)$$

$$= -\alpha a^2\cos^2\omega t + a\omega^2\cos\omega t - \ddot{x}^{(2)} - a\omega_0^2\cos\omega t + \frac{\omega_0^2}{\omega^2}\ddot{x}^{(2)}$$

$$\Longrightarrow \ddot{x}^{(2)} + \omega_0^2 x^{(2)} = -\alpha a^2\cos^2\omega t + a\left(\omega_0^2 + 2\omega_0\omega^{(1)}\right)\cos\omega t - a\omega_0^2\cos\omega t$$

$$= -\frac{\alpha}{2}a^2 - \frac{\alpha}{2}a^2\cos 2\omega t + 2\omega_0\omega^{(1)}\cos\omega t.$$

Therefore, to second order in a, the resonant term vanishes for $\omega^{(1)} = 0$. The solution of the equation in this case is then

$$x^{(2)} = -\frac{\alpha a^2}{2\omega_0^2} + \frac{\alpha a^2}{6\omega_0^2}\cos 2\omega t.$$

For the NL frequency shift, however, we need to expand further by plugging $x = x^{(1)} + x^{(2)} + x^{(3)}$, $\omega = \omega_0 + \omega^{(2)}$ into Eq. (2). This yields

$$\frac{\omega_0^2}{\omega^2}\ddot{x}^{(3)} + \omega_0^2 x^{(3)} = -2\alpha x^{(1)}x^{(2)} - \beta x^{(1)3} - \left(1 - \frac{\omega_0^2}{\omega^2}\right)\ddot{x}^{(3)} + 2\omega_0\omega^{(2)}x^{(1)}$$

$$\Longrightarrow \ddot{x}^{(3)} + \omega_0^2 x^{(3)} = -2\alpha(a\cos\omega t)\left(-\frac{\alpha a^2}{2\omega_0^2} + \frac{\alpha a^2}{6\omega_0^2}\cos 2\omega t\right) - \beta a^3\cos^3\omega t + 2\omega_0\omega^{(2)}a\cos\omega t$$

$$= \cos 3\omega t\left[-\frac{\beta}{4}a^3 - \frac{\alpha^2}{6\omega_0^2}a^3\right] + \cos\omega t\left[-\frac{3\beta}{4}a^3 + \frac{5\alpha^2}{6\omega_0^2}a^3 + 2\omega_0\omega^{(2)}a\right].$$

For the resonant term to disappear, we require

$$\frac{3\beta}{4}a^3 - \frac{5\alpha^2}{6\omega_0^2}a^3 = 2\omega_0\omega^{(2)}a$$

$$\Longrightarrow \boxed{\omega^{(2)} = \left(\frac{3\beta}{8} - \frac{5\alpha^2}{12\omega_0^3}\right)a^2.}$$

Hence the $O(a^3)$ approximation of the solution has

$$\boxed{x^{(3)} = \frac{a^3}{16\omega_0^2}\left(\frac{\beta}{2} + \frac{\alpha^2}{3\omega_0^2}\right)\cos 3\omega t.}$$