Solutions to Homework Problems 5

8.6. (a) Solutions are as given in Jackson Section 8.7.

TM modes given by $\psi(\rho,\phi) = E_0 J_m(\gamma_{mn}\rho) e^{\pm im\phi}$, $\gamma_{mn} = \frac{x_{mn}}{R}$ (1)

x_{mn} are zeroes of Bessel functions J_m

By Eqn (8.79) of book, resonant frequencies are given by

(TM) $$\omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}}\left[\gamma_{mn}^2 + \left(\frac{p\pi}{L}\right)^2\right]^{\frac{1}{2}} = (\text{by Eq(1)}) = \frac{1}{\sqrt{\mu\epsilon}\,R}\left[x_{mn}^2 + \left(\frac{p\pi R}{L}\right)^2\right]^{\frac{1}{2}} \quad (2)$$

$(p \geq 0)$

x_{mn} are ~~given~~ tabulated in Jackson after Eq. (3.92)

$x_{01} = 2.405$, $x_{11} = 3.832$, $x_{21} = 5.136$, $x_{02} = 5.520$

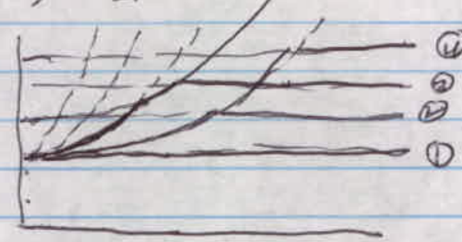
$p = 0$ modes are independent of R/L from Eq. (2)

(TE) $$\omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}}\frac{1}{R}\left[x'^2_{mn} + \left(p\pi\frac{R}{L}\right)^2\right]^{\frac{1}{2}} \quad (3)$$

$x'_{11} = 1.841$, $x'_{21} = 3.054$, $x'_{01} = 3.832$, $x'_{31} = 4.201$

Lowest 4 frequencies

TM



Similarly for TE.

(b) Lowest TM mode is (010) mode. For this mode, stored energy/unit length

Eq (8.92) of book → $$U = \frac{L\epsilon}{4}\int |\psi|^2 da \times \left[1 + (p\pi/\gamma_{010}L)^2\right]$$ (ψ given by Eq. (1) $(\psi = E_z)$) (3)

From Eq. (8.93) of book,

$$P_{loss} = \frac{1}{2\sigma\delta}\left[\oint_C dl \int_0^L dz\, |\vec{n}\times\vec{H}_0|_{sides} + 2\int_A da\, |\vec{n}\times\vec{H}|^2_{ends}\right]$$

by Eq (8.94) $$= \frac{\epsilon}{\sigma\delta\mu}\left[1 + \left(\frac{p\pi}{\gamma_{010}L}\right)^2\right]\left(1 + 2\xi_{010}\frac{CL}{4A}\right)\int_A |\psi|^2 da$$

C = Circumference of cavity $= 2\pi R$, A = Cross-sectional area $= \pi R^2$; $\xi_{010} \sim 1$ (4)

From (3) & (4), $$Q = \omega_{010}\frac{U}{P} = \frac{\omega_{010} L \frac{\epsilon}{2}\int_A |\psi|^2 da \left[1 + \left(\frac{p\pi}{\gamma_{010}L}\right)^2\right]}{\frac{\epsilon}{\sigma\delta\mu}\left[1 + \left(\frac{p\pi}{\gamma_{010}L}\right)^2\right]\left(1 + 2\xi_{010}\frac{L}{2R}\right)\int_A |\psi|^2 da}$$

$$= \frac{\omega_{010}\,\sigma\delta\mu L}{2(1 + \xi_{010}\frac{L}{R})} \rightarrow \left(\text{use } \delta = \left(\frac{2}{\mu_c\sigma\omega_{010}}\right)^{\frac{1}{2}}\right) = \frac{\mu}{\mu_c}\frac{L}{\delta}\frac{1}{(1 + \xi_{010}L/R)}$$

For copper take $\mu_c = \mu_0$, $\epsilon = \epsilon_0$
$= \mu$

$$Q = \frac{1}{\delta}\left[\frac{L}{(1+\xi_{010} L/R)}\right] \quad \text{Take } \xi_{010} = 1 \quad = \frac{1}{\delta}\left[\frac{3}{2.5}\right] \text{cm}$$

$$= \frac{1.2 \times 10^{-2}\,\text{m}}{\delta}$$

Calculate $\delta = \left(\frac{2}{\mu_c \sigma \omega_{010}}\right)^{\frac{1}{2}}$

$$\omega_{010} = \frac{x_{01} c}{R} = \frac{2.405\, c}{2\times 10^{-2}\,\text{m}} = \frac{2.405 \times 3 \times 10^8}{2 \times 10^{-2}}\, s^{-1} = 3.61 \times 10^{10}\, s^{-1}$$

which gives $\delta = 8.6 \times 10^{-7}$ m

$\therefore \quad Q = 1.4 \times 10^4$

8.9. (a) $$k^2 = \frac{\int_V \vec{E}^* \cdot [\nabla \times (\nabla \times \vec{E})]\, d^3x}{\int_V (\vec{E}^* \cdot \vec{E})\, d^3x}$$

$E \to E + \delta E$

$$\delta k^2 = \frac{\int_V \vec{E}^* \cdot [\nabla \times \nabla \times (\vec{E} + \delta\vec{E})]\, d^3x}{\int_V \vec{E}^* \cdot \vec{E}\, d^3x} - \frac{\int_V \vec{E}^* \cdot [\nabla\times\nabla\times\vec{E}]\, d^3x \int_V \vec{E}^* \cdot (\vec{E} + \delta\vec{E})\, d^3x}{\left(\int_V \vec{E}^* \cdot \vec{E}\, d^3x\right)^2}$$

$$= \frac{\int_V \vec{E}^* \cdot [\nabla\times\nabla\times\vec{E} + \nabla\times\nabla\times\delta\vec{E}]\, d^3x}{\int_V (\vec{E}^* \cdot \vec{E})\, d^3x} - \frac{\int_V \vec{E}^* \cdot (\nabla\times\nabla\times\vec{E})\, d^3x \cancel{\int_V \vec{E}^* \cdot \vec{E}\, d^3x}}{\left(\int \vec{E}^* \cdot \vec{E}\, d^3x\right)^{\cancel{2}}}$$

$$- \frac{\int_V \vec{E}^* \cdot [\nabla\times\nabla\times\vec{E}]\, d^3x \int_V \vec{E}^* \cdot \delta\vec{E}\, d^3x}{\left(\int_V (\vec{E}^* \cdot \vec{E})\, d^3x\right)^2}$$

$$= \frac{\int_V \vec{E}^* \cdot [\nabla\times\nabla\times\delta\vec{E}]\, d^3x}{\int_V (\vec{E} \cdot \vec{E}^*)\, d^3x} - \underbrace{\frac{\int_V \vec{E}^* \cdot [\nabla\times\nabla\times\vec{E}]\, d^3x}{\int_V (\vec{E}^* \cdot \vec{E})\, d^3x}}_{k^2} \frac{\int_V \vec{E}^* \cdot \delta\vec{E}\, d^3x}{\int_V (\vec{E}^* \cdot \vec{E})\, d^3x}$$

$$= \frac{\int_V \vec{E}^* \cdot [\nabla\times\nabla\times\delta\vec{E}]\, d^3x - k^2 \int_V (\vec{E}^* \cdot \delta\vec{E})\, d^3x}{\int_V (\vec{E}^* \cdot \vec{E})\, d^3x}$$

③

Now $\int_V \vec{E}^*\cdot\left[\nabla\times\nabla\times\delta\vec{E}\right] d^3x = \int_V \vec{E}^*\cdot\left[\nabla(\cancelto{0}{\nabla\cdot\delta\vec{E}}) - \nabla^2\delta\vec{E}\right] d^3x$

$$= -\int_V \left(\vec{E}^*\cdot\nabla^2\delta\vec{E}\right) = -\int \delta\vec{E}\cdot\nabla^2\vec{E}^*$$ (using Green's Identity + Boundary Conditions)

$$= \int_V \delta\vec{E}\cdot\left(\nabla\times\nabla\times\vec{E}^*\right) d^3x$$

$$\therefore\ \delta k^2 = \frac{\int_V \delta\vec{E}\cdot\left(\nabla\times\nabla\times\vec{E}^*\right) d^3x}{\int_V \left(\vec{E}^*\cdot\vec{E}\right) d^3x} - k^2\frac{\int_V\left(\vec{E}^*\cdot\delta\vec{E}\right) d^3x}{\int_V \left(\vec{E}^*\cdot\vec{E}\right) d^3x}$$

From Helmholtz Eqn $\nabla\times\nabla\times E^* = k^2\vec{E}$

$$\therefore\ \delta k^2 = \frac{k^2\int_V \delta\vec{E}\cdot E^*\, d^3x - k^2\int\left(\vec{E}^*\cdot\delta\vec{E}\right)}{\int_V \left(\vec{E}^*\cdot\vec{E}\right) d^3x} = 0$$ to 1^{st} order in δE.

(b) $\vec{E} = E_0\cos\left(\pi\rho/2R\right)\hat{z}$

Put this in above expression for k^2 + see what we get

$$\nabla\times\left(\nabla\times\vec{E}\right) = \nabla(\underbrace{\nabla\cdot\vec{E}}_{\text{vanishes}}) - \nabla^2\vec{E} = -\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\vec{E}}{\partial\rho}\right) + \cancelto{0}{\frac{1}{\rho^2}\frac{\partial^2\vec{E}}{\partial\phi^2}} + \cancelto{0}{\frac{\partial^2\vec{E}}{\partial z^2}}\right)$$

(in cylindrical coordinates)

$$\therefore\ k^2 = \frac{\int_0^R \left[\frac{\pi}{2R\rho}\cos\left(\frac{\pi\rho}{2R}\right)\sin\left(\frac{\pi\rho}{2R}\right) + \frac{\pi^2}{4R^2}\cos^2\left(\frac{\pi\rho}{2R}\right)\right]\rho\, d\rho}{\int_0^R \cos^2\left(\frac{\pi\rho}{2R}\right)\rho\, d\rho}$$

$$= \frac{\frac{1}{16}\left(4+\pi^2\right)}{\frac{1}{4\pi^2}R^2\left(-4+\pi^2\right)} = \frac{\pi^2}{4R^2}\,\frac{\pi^2+4}{\pi^2-4}$$

or $kR = \frac{\pi}{2}\sqrt{\frac{\pi^2+4}{\pi^2-4}} = 2.4146$ Actually first root of $J_0(x) = 2.4048$.

Pretty good approximation for lowest eigenvalue!

(c) $\vec{E} = E_0\left[1+\alpha\left(\rho/R\right)^2 - (1+\alpha)\left(\rho/R\right)^4\right]\hat{z}$

$$\nabla\times\left(\nabla\times\vec{E}\right) = -\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\vec{E}}{\partial\rho}\right) + \cancelto{0}{\frac{1}{\rho^2}\frac{\partial^2\vec{E}}{\partial\phi^2}} + \cancelto{0}{\frac{\partial^2\vec{E}}{\partial z^2}}\right)$$

$$= E_0\frac{4}{R^4}\left(\alpha R^2 - 4\rho^2 - 4\rho^2\alpha\right)\hat{z}$$

Putting this into expression for k^2:

$$k^2 = \frac{\int_0^R \left[1+\alpha(\rho/R)^2-(1+\alpha)(\rho/R)^4\right]\left[\frac{4}{R^4}(\alpha R^2-4\rho^2-4\rho^2\alpha)\right]\rho\, d\rho}{\int_0^R \left[1+\alpha(\rho/R)^2-(1+\alpha)(\rho/R)^4\right]^2 \rho\, d\rho}$$

$$= \frac{2+\frac{4}{3}a+\frac{1}{3}a^2}{\frac{R^2}{60}(16+7a+a^2)} = \frac{20}{R^2}\frac{(6+4a+a^2)}{(16+7a+a^2)}$$

Differentiate with respect to variational parameter α,

$$\frac{dk^2}{da} = \frac{20}{R^2}\left[\frac{(16+7a+a^2)(4+2a)-(6+4a+a^2)(7+2a)}{(16+7a+a^2)^2}\right] = 0$$

gives roots $\alpha_\pm = -\frac{1}{3}(10 \pm \sqrt{34})$

α_+ gives maximum; α_- gives minimum, so

$$k^2 = \frac{1}{R^2}(80)\left(\frac{17-2\sqrt{34}}{68+\sqrt{34}}\right)$$

so $kR = 2.405$ → very close to exact eigenvalue!

8.14. (a) Solution given is $\alpha x = \sinh^{-1}[\sinh(\alpha x_{max})\sin(\alpha z)]$ (1)

$$\rightarrow \sinh(\alpha x) = \sinh(\alpha x_{max})\sin(\alpha z) \quad (2)$$

Differentiate both sides with respect to z:

$$\alpha \cosh(\alpha x)\frac{dx}{dz} = \alpha \sinh(\alpha x_{max})\cos \alpha z$$

$$\rightarrow \left(\frac{dx}{dz}\right) = \frac{\sinh(\alpha x_{max})\cos(\alpha z)}{\cosh(\alpha x)}$$

$$\rightarrow \left(\frac{dx}{dz}\right)^2 = \frac{\sinh^2(\alpha x_{max})\cos^2(\alpha z)}{\cosh^2(\alpha x)} = \frac{\sinh^2(\alpha x_{max})\cos^2(\alpha z)}{1+\sinh^2 \alpha x}$$

$$= (\text{by Eq.(2)}) \quad \frac{\sinh^2(\alpha x_{max})\cos^2(\alpha z)}{1+\sinh^2(\alpha x_{max})\sin^2(\alpha z)}$$

Now $\bar{n} = n(0)\,\mathrm{sech}(\alpha x_{max})$

So $$\bar{n}^2\left(\frac{dx}{dz}\right)^2 + \bar{n}^2 = \frac{n(0)^2\mathrm{sech}^2(\alpha x_{max})\left[\sinh^2(\alpha x_{max})\cos^2(\alpha z)+\sinh^2(\alpha x_{max})\sin^2(\alpha z)+1\right]}{1+\sinh^2(\alpha x_{max})\sin^2(\alpha z)}$$

$$= \frac{n(0)^2\mathrm{sech}^2(\alpha x_{max})\left[\sinh^2(\alpha x_{max})+1\right]}{1+\sinh^2(\alpha x_{max})\sin^2(\alpha z)}$$

Using Eq.(2) again,

$$\bar{n}^2\left(\frac{dx}{dz}\right)^2+\bar{n}^2 = n_0^2\,\mathrm{sech}^2(\alpha x_{max})\frac{\left[1+\sinh^2(\alpha x_{max})\right]}{\left[1+\sinh^2(\alpha x)\right]} = n(0)^2\mathrm{sech}^2(\alpha x)$$

$$\rightarrow \quad \bar{n}^2 \left(\frac{dx}{dz}\right)^2 = n^2(x) - \bar{n}^2$$

i.e. (2) is a solution of Eikonal Equation

(a) Now $\bar{n} = n(x_{max}) = n(0)\,\text{sech}(\alpha x_{max}) = n(0)\cos\theta(0)$

$$\therefore \quad \text{sech}(\alpha x_{max}) = \cos\theta(0)$$

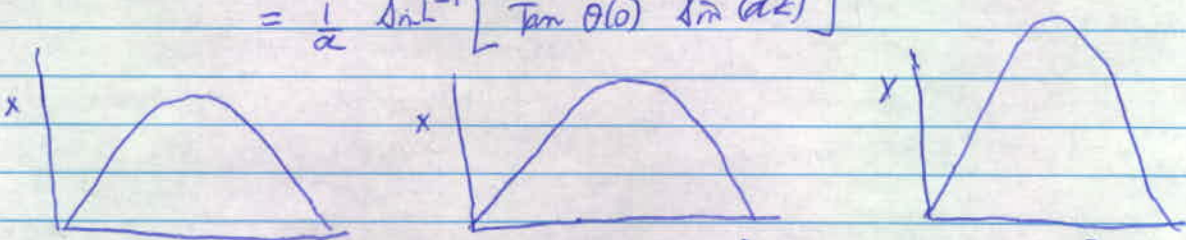
$$\cosh(\alpha x_{max}) = \frac{1}{\cos\theta(0)}$$

$$x_{max} = \frac{1}{\alpha}\cosh^{-1}\left[\frac{1}{\cos\theta(0)}\right]$$

Now rays are given by

$$x = \frac{1}{\alpha}\sinh^{-1}\left[\sinh\left(\cosh^{-1}\frac{1}{\cos\theta(0)}\right)\sin\alpha z\right]$$

$$= \frac{1}{\alpha}\sinh^{-1}\left[\tan\theta(0)\,\sin(\alpha z)\right]$$



$\theta(0) = \pi/6$ $\qquad$ $\theta(0) = \frac{\pi}{4}$ $\qquad$ $\theta(0) = \frac{\pi}{3}$

(b) to find Half-Period $Z = 2z(x_{max})$, use Eq (2)

$$z(x) = \frac{1}{\alpha}\sin^{-1}\left(\frac{\sinh(\alpha x)}{\sinh(\alpha x_{max})}\right)$$

$$\therefore \; Z = 2z(x_{max}) = \frac{2}{\alpha}\sin^{-1}(1) = \frac{2}{\alpha}\frac{\pi}{2} = \frac{\pi}{\alpha}.$$

(c) From Eq. (8.119) of Jackson,

$$L_{opt} = 2\int_0^{x_{max}} \frac{n^2(x)\,dx}{\sqrt{n^2(x)-\bar{n}^2}} \qquad \left[\text{But } \frac{dx}{dz} = \frac{1}{\bar{n}}\sqrt{n^2(x)-\bar{n}^2} \quad \therefore \frac{dx}{\sqrt{n^2(x)-\bar{n}^2}} = \frac{dz}{\bar{n}}\right]$$

$$= 2\int_0^{x_{max}} \frac{n^2(x)\,dz}{\bar{n}}$$

$$= 2\int_0^{Z/2} \frac{n(0)^2\,\text{sech}^2(\alpha x)}{n(0)\,\text{sech}(\alpha x_{max})}\,dz = 2\,n(0)\cosh(\alpha x_{max})\int_0^{Z/2}\frac{dz}{1+\sinh^2(\alpha x)}$$

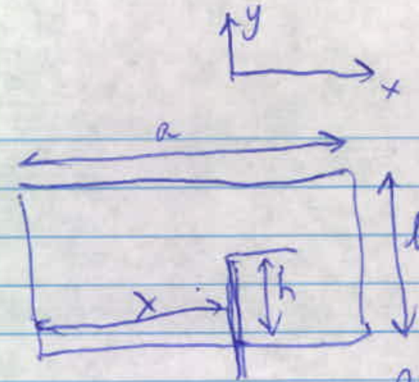
$$= 2\,n(0)\cosh(\alpha x_{max})\int_0^{\frac{Z}{2}} \frac{dz}{1+\sinh^2(\alpha x_{max})\sin^2(\alpha z)} \quad \text{using Eq (2).}$$

$$= 2\,n(0)\cosh(\alpha x_{max})\int_0^{\pi/2\alpha}\frac{dz}{1+\sinh^2(\alpha x_{max})\sin^2(\alpha z)}$$

$$= 2\,n(0)\cosh(\alpha x_{max})\frac{\pi}{2\alpha\sqrt{1+\sinh^2(\alpha x_{max})}} = \frac{n(0)\,\pi}{\alpha} = n(0)\,Z$$

8.19

(a)



$$I(y) = I_0 \sin\left[\tfrac{\omega}{c}(h-y)\right]$$

Only TE_{10} mode can propagate in guide

(a) TM Mode solutions $\quad E_z = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \quad (1)$

$$\vec{E}_t = -\frac{i k_{mn}}{\gamma_{mn}^2}\vec{\nabla}_t E_z \qquad E_y = \frac{-iE_0 k_{mn}}{\gamma_{mn}^2}\frac{n\pi}{b}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \quad (2)$$

(wave propagating in $-z$ direction)

$$k_{mn}^2 = \frac{\omega^2}{c^2} - \gamma_{mn}^2 \;;\; \gamma_{mn}^2 = \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

Now normalization condition 6 (for TM modes) $\int E_{z,\lambda}E_{z,\mu}\,da = \frac{\gamma_\lambda^2}{k_\lambda^2}\delta_{\lambda\mu}$

$$\Rightarrow \int (E_{z,mn})^2 da = -\frac{\gamma_{mn}^2}{k_{mn}^2} \rightarrow \text{from (1)} \quad \frac{1}{4}ab E_0^2 = -\frac{\gamma_{mn}^2}{k_{mn}^2}$$

$$E_0 = \frac{2i\gamma_{mn}}{k_{mn}\sqrt{ab}} \quad (3)$$

TE mode solutions $\quad H_z = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$

$$\vec{E}_t = \frac{i\mu_0\omega}{\gamma_{mn}^2}\hat{z}\times\vec{\nabla}_t H_z \longrightarrow E_y = \frac{i\mu_0\omega}{\gamma_{mn}^2}\left(\frac{m\pi}{a}\right)H_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \quad (4)$$

Normalization cond^n gives $\frac{1}{4}abH_0^2 = -\frac{\gamma_{mn}^2}{\mu_0^2\omega^2} \longrightarrow H_0 = \frac{2i\gamma_{mn}}{\mu_0\omega\sqrt{ab}} \quad (5)$

(for $m=0$, $a \to 2a$; for $n=0$, $b \to 2b$ } in square root.

Excitation amplitudes $A_{mn}^{\pm} = -\frac{Z_{mn}}{2}\int \vec{J}\cdot\vec{E}_{mn}^{(\mp)}\,d^3x \quad (6)$

$Z_{mn} = \frac{k_{mn}}{\epsilon_0\omega}$ for TM modes $+$ $Z_{mn} = \frac{\mu_0\omega}{k_{mn}}$ for TE modes

Now $\vec{J} = \hat{y}\, I_0 \sin\left(\frac{\omega}{c}(h-y)\right)\delta(x-X)\,\delta(z)\,\Theta(h-y) \qquad \Theta(x) = 1 \;\; x>0, \; = 0 \;\; x<0$

$$\Rightarrow A_{mn}^{\pm} = -\frac{Z_{mn}}{2} I_0 \int_0^h \sin\left[\frac{\omega}{c}(h-y)\right] E_{y,mn}^{(\mp)}(X,y,0)\,dy$$

where $E_{y,mn}$ is given by (2) for the TM modes + (4) for the TE modes. Since we only need the electric field at $z=0$, this expression will be independent of whether we choose a left-moving or right-moving mode. So the (+) (−) modes will be equally excited

For TM modes given by Eq (2),

$$A^{\pm}_{mn} = \frac{E_0 Z_{mn}}{2} I_0 \frac{i k_{mn}}{\gamma_{mn}^2}\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\int_0^h \sin\left[\frac{\omega}{c}(h-y)\right]\cos\left(\frac{n\pi y}{b}\right)dy$$

$$= \frac{E_0 k_{mn}}{2\epsilon_0\omega} I_0\left(\frac{n\pi}{b}\right)\frac{i k_{mn}}{\gamma_{mn}^2}\sin\left(\frac{m\pi x}{a}\right)\int_0^h \sin\left[\frac{\omega}{c}(h-y)\right]\cos\left(\frac{n\pi y}{b}\right)dy \quad (8)$$

writing $\sin\left[\frac{\omega}{c}(h-y)\right]\cos\left(\frac{n\pi y}{b}\right)$ as $\frac{1}{2}\left[\sin\left[\frac{\omega}{c}h - \left(\frac{\omega}{c}-\frac{n\pi}{b}\right)y\right] + \sin\left[\frac{\omega}{c}h - \left(\frac{\omega}{c}+\frac{n\pi}{b}\right)y\right]\right]$

we can evaluate the integral as

$$\int_0^h \sin\left[\frac{\omega}{c}(h-y)\right]\cos\left(\frac{n\pi y}{b}\right)dy = \frac{\omega}{c}\frac{1}{\left(\frac{\omega}{c}\right)^2-\left(\frac{n\pi}{b}\right)^2}\left[\cos\left(\frac{n\pi h}{b}\right)-\cos\left(\frac{\omega h}{c}\right)\right] \quad (9)$$

so inserting in Eq (8) & using Eq. (3)

$$A^{\pm}_{mn} = \frac{2i\gamma_{mn}}{k_{mn}\sqrt{ab}}\frac{k_{mn}}{2\epsilon_0\omega} I_0\left(\frac{n\pi}{b}\right)\frac{i k_{mn}}{\gamma_{mn}^2}\sin\left(\frac{m\pi x}{a}\right)\left(\frac{\omega}{c}\right)\frac{1}{\left(\frac{\omega}{c}\right)^2-\left(\frac{n\pi}{b}\right)^2}\left[\cos\left(\frac{n\pi h}{b}\right)-\cos\left(\frac{\omega h}{c}\right)\right]$$

$$= -\frac{k_{mn}}{\gamma_{mn}\epsilon_0 c\sqrt{ab}}\left(\frac{n\pi}{b}\right) I_0 \sin\left(\frac{m\pi x}{a}\right)\frac{1}{\left(\frac{\omega}{c}\right)^2-\left(\frac{n\pi}{b}\right)^2}\left[\cos\left(\frac{n\pi h}{b}\right)-\cos\left(\frac{\omega h}{c}\right)\right]$$

$$\frac{k_{mn}}{\gamma_{mn}} = \frac{\sqrt{\left(\frac{\omega}{c}\right)^2-\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}}{\sqrt{\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}}$$

For $m, n \gg 1$ $\quad \frac{k_{mn}}{\gamma_{mn}} \to i$

$$\frac{\frac{n\pi}{b}}{\left(\frac{\omega}{c}\right)^2-\left(\frac{n\pi}{b}\right)^2} \to -\frac{1}{n}\left(\frac{b}{\pi}\right)$$

so $A^{\pm}_{mn} \sim \frac{1}{n}$ for $m\,n \gg 1$

For TE modes, similarly

$$A^{\pm}_{mn} = \frac{\mu_0\omega^2}{c k_{mn}\gamma_{mn}}\left(\frac{m\pi}{a}\right) I_0 \sin\left(\frac{m\pi x}{a}\right)\frac{1}{\left(\frac{\omega}{c}\right)^2-\left(\frac{n\pi}{b}\right)^2}\left(\cos\left(\frac{n\pi h}{b}\right)-\cos\left(\frac{\omega h}{c}\right)\right)$$

For $m, n \gg 1$ $\quad A^{\pm}_{mn} \sim \frac{m}{n^2}\frac{1}{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2} \sim \frac{1}{N^3}$ $\quad N \sim m, n \gg 1.$

For propagating TE_{10} mode,

$$A^{\pm}_{10} = \frac{\mu_0 c}{k_{10}\sqrt{2ab}} I_0 \sin\left(\frac{\pi x}{a}\right)\left(1-\cos\left(\frac{\omega h}{c}\right)\right) = \frac{\sqrt{2}\mu_0 c}{k_{10}\sqrt{ab}} I_0 \sin\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{\omega h}{2c}\right)$$

(b) Radiated power in +Z direction for mode mn

$$P_{mn} = \frac{1}{2Z_{mn}} |A^{+}_{mn}|^2$$

for TE_{10} mode, $$P = \frac{k_{10}}{2\mu_0 \omega} |A^{(+)}_{10}|^2 = \frac{\mu_0 c^2}{\omega k_{10} ab} I_0^2 \sin^2\left(\frac{\pi x}{a}\right) \sin^4\left(\frac{\omega h}{2c}\right)$$

(c) For a perfectly conducting surface at $z = L$ (taken positive).
then the right-moving wave will be perfectly reflected at this surface.
So wave flowing out of left end will be a linear superposition of the left-moving wave generated by the inserted wire and the reflected wave of surface at $z = L$

Left-moving wave due to source $\vec{E}^{(-)} = A^{(-)}_{10} \vec{E}_{t,10} e^{-ikz}$

Right moving wave due to source $\vec{E}^{(+)} = A^{(+)}_{10} \vec{E}_{t,10} e^{ikz}$

so reflected wave must be $\vec{E}^{r} = -A^{(+)}_{10} \vec{E}_{t,10} e^{ik(2L-z)}$

These superimpose to give $\vec{E} = 0$ at $z = L$ (Boundary condition)

so for $z < 0$, total left-moving wave is

$$\vec{E} = \vec{E}^{-} + \vec{E}^{r} = (A^{-}_{10} - A^{+}_{10} e^{2ikL}) \vec{E}_{t,10} e^{-ikz}$$
$$= A_{10} (1 - e^{2ikL}) E_{t,10} e^{-ikz}$$

Maximum amplitude when $kL = (n + \frac{1}{2})\pi$, $\vec{E}$ field is twice $\vec{E}^{-}$

so power is 4 times power given in part (b).

$$P = \frac{4\mu_0 c^2}{\omega k_{10} ab} I_0^2 \sin^2\left(\frac{\pi x}{a}\right) \sin^4\left(\frac{\omega h}{2c}\right)$$

If we put this $= \frac{1}{2} I_0^2 R_{rad}$ (R_{rad} = radiation resistance of probe)

then $$R_{rad} = \frac{8\mu_0 c^2}{\omega k_{10} ab} \sin^2\left(\frac{\pi x}{a}\right) \sin^4\left(\frac{\omega h}{2c}\right).$$