The zeros of $h_l^{(1)}$ $(l=1,2)$ are then

$$x_{11} = -i$$
$$x_{21} = \frac{\sqrt{3}}{2} - \frac{3i}{2}, \qquad x_{22} = -\frac{\sqrt{3}}{2} - \frac{3i}{2}$$

while the zeros of $[\zeta h_l^{(1)}(\zeta)]'$ $(l=1,2)$ are

$$y_{11} = \frac{\sqrt{3}}{2} - \frac{i}{2}, \qquad y_{12} = -\frac{\sqrt{3}}{2} - \frac{i}{2}$$
$$y_{21} \approx -1.596i, \qquad y_{22} \approx 1.807 - 0.702i, \qquad y_{23} \approx -1.807 - 0.702i$$

Since the complex frequencies are given by these zeros multiplied by c/a, we end up with

Mode$_{nlm}$	λ/a	$\tau/(a/c)$
TE$_{11m}$	∞	$1/2$
TE$_{12m}$	$4\pi/\sqrt{3}$	$1/3$
TM$_{11m}$	$4\pi/\sqrt{3}$	1
TM$_{12m}$	∞	0.313
TM$_{22m}$	3.476	0.712

where the wavelength λ and the energy decay time τ is given by

$$\omega = \frac{2\pi c}{\lambda} - \frac{i}{2\tau}$$

10.1 *a*) Show that for arbitrary initial polarization, the scattering cross section of a perfectly conducting sphere of radius a, summed over outgoing polarizations, is given in the long-wavelength limit by

$$\frac{d\sigma}{d\Omega}(\vec{\epsilon}_0, \hat{n}_0, \hat{n}) = k^4 a^6 \left[\frac{5}{4} - |\vec{\epsilon}_0 \cdot \hat{n}|^2 - \frac{1}{4}|\hat{n} \cdot (\hat{n}_0 \times \vec{\epsilon}_0)|^2 - \hat{n}_0 \cdot \hat{n}\right]$$

where $\hat{n}_0$ and $\hat{n}$ are the directions of the incident and scattered radiations, respectively, while $\vec{\epsilon}_0$ is the (perhaps complex) unit polarization vector of the incident radiation ($\vec{\epsilon}_0{}^* \cdot \vec{\epsilon}_0 = 1$; $\hat{n}_0 \cdot \vec{\epsilon}_0 = 0$).

If all polarizations are specified, the conducting sphere scattering cross section is given by

$$\frac{d\sigma}{d\Omega}(\hat{n}, \vec{\epsilon}; \hat{n}_0, \vec{\epsilon}_0) = k^4 a^6 |\vec{\epsilon}^* \cdot \vec{\epsilon}_0 - \tfrac{1}{2}(\hat{n} \times \vec{\epsilon}^*) \cdot (\hat{n}_0 \times \vec{\epsilon}_0)|^2 \qquad (11)$$

What we would like to do is to sum this over both orthogonal outgoing polarizations. One way to do this is to introduce a linear polarization basis transverse to the outgoing direction $\hat{n}$. To do so, we first assume the scattering is not in the forward direction. Then the incoming direction $\hat{n}_0$ may be used to define orthogonal polarizations

$$\vec{\epsilon}^{\,1} = \frac{\hat{n}\times\hat{n}_0}{\sin\theta}, \qquad \vec{\epsilon}^{\,2} = \hat{n}\times\vec{\epsilon}^{\,1} = \frac{\hat{n}(\hat{n}\cdot\hat{n}_0)-\hat{n}_0}{\sin\theta}$$

where θ is the angle between $\hat{n}$ and $\hat{n}_0$. In particular, we may write $\sin^2\theta = 1-(\hat{n}\cdot\hat{n}_0)^2$. In this case, the cross section summed over outgoing polarizations becomes

$$\begin{aligned}
\frac{d\sigma}{d\Omega}(\hat{n};\hat{n}_0,\vec{\epsilon}_0) &= \frac{k^4a^6}{1-(\hat{n}\cdot\hat{n}_0)^2}\big[|(\hat{n}\times\hat{n}_0)\cdot\vec{\epsilon}_0 - \tfrac{1}{2}(\hat{n}\times(\hat{n}\times\hat{n}_0))\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2 \\
&\qquad + |(\hat{n}(\hat{n}\cdot\hat{n}_0)-\hat{n}_0)\cdot\vec{\epsilon}_0 - \tfrac{1}{2}(\hat{n}\times(\hat{n}(\hat{n}\cdot\hat{n}_0)-\hat{n}_0))\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2\big] \\
&= \frac{k^4a^6}{1-(\hat{n}\cdot\hat{n}_0)^2}\big[|(\hat{n}\times\hat{n}_0)\cdot\vec{\epsilon}_0 - \tfrac{1}{2}(\hat{n}(\hat{n}\cdot\hat{n}_0)-\hat{n}_0)\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2 \\
&\qquad + |(\hat{n}\cdot\hat{n}_0)(\hat{n}\cdot\vec{\epsilon}_0) - \tfrac{1}{2}(\hat{n}_0\times\hat{n})\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2\big] \\
&= \frac{k^4a^6}{1-(\hat{n}\cdot\hat{n}_0)^2}\big[|\hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0) - \tfrac{1}{2}(\hat{n}\cdot\hat{n}_0)\hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2 \\
&\qquad + |(\hat{n}\cdot\hat{n}_0)(\hat{n}\cdot\vec{\epsilon}_0) - \tfrac{1}{2}(\hat{n}\cdot\vec{\epsilon}_0)|^2\big] \\
&= \frac{k^4a^6}{1-(\hat{n}\cdot\hat{n}_0)^2}\big[|\hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2(1-\tfrac{1}{2}(\hat{n}\cdot\hat{n}_0))^2 \\
&\qquad + |\hat{n}\cdot\vec{\epsilon}_0|^2(\tfrac{1}{2}-(\hat{n}\cdot\hat{n}_0))^2\big]
\end{aligned}$$

Note that we have used transversality of the initial polarization, $\hat{n}_0\cdot\vec{\epsilon}_0 = 0$. To proceed, we expand the squares and rewrite the above as

$$\begin{aligned}
\frac{d\sigma}{d\Omega}(\hat{n};\hat{n}_0,\vec{\epsilon}_0) = \frac{k^4a^6}{1-(\hat{n}\cdot\hat{n}_0)^2}\big[&(\tfrac{5}{4}-(\hat{n}\cdot\hat{n}_0))(|\hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2 + |\hat{n}\cdot\vec{\epsilon}_0|^2) \\
&- (1-(\hat{n}\cdot\hat{n}_0)^2)(\tfrac{1}{4}|\hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2 + |\hat{n}\cdot\vec{\epsilon}_0|^2\big]
\end{aligned} \tag{12}$$

The second line cancels the denominator. However the first line needs a bit of work. We now use the fact that ϵ_0 is a unit polarization vector orthogonal to $\hat{n}_0$. As a result, the three vectors

$$\hat{n}_0, \qquad \vec{\epsilon}_0, \qquad \hat{n}_0\times\vec{\epsilon}_0 \tag{13}$$

form a normalized right-handed coordinate basis spanning the three-dimensional space. (There is a slight subtlety if $\vec{\epsilon}_0$ is complex, although the end result is okay, provided we are careful with magnitude squares.) The components of $\hat{n}$ expanded in this basis are

$$\hat{n}\cdot\hat{n}_0, \qquad \hat{n}\cdot\vec{\epsilon}_0, \qquad \hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0)$$

and since $\hat{n}$ is a unit vector, the sum of the squares of these components must be one. In other words

$$(\hat{n}\cdot\hat{n}_0)^2 + |\hat{n}\cdot\vec{\epsilon}_0|^2 + |\hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2 = 1$$

where we have been careful about complex quantities. Using this result, we see that the denominator in (12) can be completely eliminated, resulting in

$$\frac{d\sigma}{d\Omega}(\hat{n};\hat{n}_0,\vec{\epsilon}_0) = k^4a^6[\tfrac{5}{4} - (\hat{n}\cdot\hat{n}_0) - \tfrac{1}{4}|\hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0)|^2 - |\hat{n}\cdot\vec{\epsilon}_0|^2] \tag{14}$$

b) If the incident radiation is linearly polarized, show that the cross section is

$$\frac{d\sigma}{d\Omega}(\vec{\epsilon}_0,\hat{n}_0,\hat{n}) = k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos 2\phi\right]$$

where $\hat{n}\cdot\hat{n}_0 = \cos\theta$ and the azimuthal angle ϕ is measured from the direction of the linear polarization.

As stated, the scattering angle θ is given by $\hat{n}\cdot\hat{n}_0 = \cos\theta$. The azimuthal angle ϕ is the one between $\hat{n}$ and $\vec{\epsilon}_0$, measured in the plan perpendicular to $\hat{n}_0$. What this means is that, using the basis vectors (13) with $\vec{\epsilon}_0$ real, the components of $\hat{n}$ can be written as

$$\hat{n} = \hat{n}_0\cos\theta + \vec{\epsilon}_0\sin\theta\cos\phi + (\hat{n}_0\times\vec{\epsilon}_0)\sin\theta\sin\phi$$

or alternatively

$$\hat{n}\cdot\hat{n}_0 = \cos\theta, \qquad \hat{n}\cdot\vec{\epsilon}_0 = \sin\theta\cos\phi, \qquad \hat{n}\cdot(\hat{n}_0\times\vec{\epsilon}_0) = \sin\theta\sin\phi$$

Substituting this into (14) gives

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\theta,\phi) &= k^4a^6[\tfrac{5}{4} - \cos\theta - \tfrac{1}{4}\sin^2\theta\sin^2\phi - \sin^2\theta\cos^2\phi] \\ &= k^4a^6[\tfrac{5}{4} - \cos\theta - \tfrac{1}{8}\sin^2\theta(1-\cos 2\phi) - \tfrac{1}{2}\sin^2\theta(1+\cos 2\phi)] \\ &= k^4a^6[\tfrac{5}{8}(1+\cos^2\theta) - \cos\theta - \tfrac{3}{8}\sin^2\theta\cos 2\phi]\end{aligned}$$

c) What is the ratio of scattered intensities at $\theta = \pi/2$, $\phi = 0$ and $\theta = \pi/2$, $\phi = \pi/2$? Explain physically in terms of the induced multipoles and their radiation patterns.

At $\theta = \pi/2$, we have

$$\frac{d\sigma}{d\Omega}(\pi/2,\phi) = k^4a^6[\tfrac{5}{8} - \tfrac{3}{8}\cos 2\phi]$$

Hence

$$\frac{d\sigma}{d\Omega}(\pi/2,0) = \tfrac{1}{4}k^4a^6, \qquad \frac{d\sigma}{d\Omega}(\pi/2,\pi/2) = k^4a^6$$

Scattering at 90° is fairly easy to understand physically. For $\phi = 0$, the scattered wave is lined up with the incident polarization ϵ_0. Since the polarization is given by the electric field vector, this indicates that the induced electric dipole of the sphere is lined up with the direction of the scattered wave. Since the radiation must be transverse, no dipole radiation can be emitted on axis, and in this case the scattering must be purely magnetic dipole in nature. On the other hand, for $\phi = \pi/2$, the scattered wave is lined up with the incident magnetic field, and hence the scattering must be purely electric dipole in nature. This demonstrates that the maximum strength of magnetic dipole scattering is a quarter that of electric dipole scattering. This is in fact evident by the factor of 1/2 in the magnetic dipole term in the cross section expression (11).

To lowest order in δ, this is simply

$$\sigma_{\text{abs}} \approx 3\pi(k\delta)a^2$$

On the other hand, for $\delta = a$, we find

$$\sigma_{\text{abs}} \approx 3\pi(k\delta)a^2 \times \left(\frac{2}{5}\right)$$

Hence the true value of the absorption cross section for $\delta = a$ is $2/5$ as large as the simple first order approximation. (This is all done in the long wavelength approximation, of course. Note furthermore that when $\delta = a$, the skin depth is comparable to the size of the sphere. In this case, we can hardly expect to trust the analysis of Section 8.1.)

10.9 In the scattering of light by a gas very near the critical point the scattered light is observed to be "whiter" (i.e., its spectrum is less predominantly peaked toward the blue) than far from the critical point. Show that this can be understood by the fact that the volumes of the density fluctuations become large enough that Rayleigh's law fails to hold. In particular, consider the lowest order approximation to the scattering by a uniform dielectric sphere of radius a whose dielectric constant ϵ_r differs only slightly from unity.

a) Show that for $ka \gg 1$, the differential cross section is sharply peaked in the forward direction and the total scattering cross section is approximately

$$\sigma \approx \frac{\pi}{2}(ka)^2|\epsilon_r - 1|^2 a^2$$

with a k^2, rather than k^4, dependence on frequency.

Since ϵ_r differs only slightly from unity, we may use the first Born approximation. The scattering amplitude then has the form

$$\frac{\vec{\epsilon}^* \cdot \vec{A}_{\text{sc}}^{(1)}}{D_0} = \frac{k^2}{4\pi}\int e^{i\vec{q}\cdot\vec{x}}\left[\vec{\epsilon}^* \cdot \vec{\epsilon}_0 \frac{\delta\epsilon}{\epsilon_0} + (\hat{n}\times\vec{\epsilon}^*)\cdot(\hat{n}_0\times\vec{\epsilon}_0)\frac{\delta\mu}{\mu_0}\right] d^3x$$

where $\vec{q} = k(\hat{n}_0 - \hat{n})$, so that

$$q^2 = k^2(2 - 2\cos\theta) = (2k)^2 \sin^2\frac{\theta}{2} \tag{10}$$

Here θ is the angle between $\hat{n}$ and $\hat{n}_0$ (ie the incident and scattered waves). For the dielectric sphere, we set $\delta\mu = 0$. Noting that

$$\frac{\delta\epsilon}{\epsilon_0} = \begin{cases} \epsilon_r - 1 & r < a \\ 0 & r > a \end{cases}$$

we end up with

$$\frac{\vec{\epsilon}^* \cdot \vec{A}_{\rm sc}^{(1)}}{D_0} = \frac{k^2}{4\pi}(\epsilon_r - 1)(\vec{\epsilon}^* \cdot \vec{\epsilon}_0) \int_{r<a} e^{i\vec{q}\cdot\vec{x}} d^3x$$

The integral can be performed in spherical coordinates

$$\begin{aligned}
\int_{r<a} e^{i\vec{q}\cdot\vec{x}} d^3x &= \int_{r<a} e^{iqr\cos\gamma} r^2 dr d\cos\gamma d\phi \\
&= 2\pi \int_0^a dr \int_{-1}^1 d\cos\gamma\, r^2 e^{iqr\cos\gamma} \\
&= \frac{4\pi}{q} \int_0^a r\sin(qr) dr = \frac{4\pi}{q^3}[\sin(qa) - qa\cos(qa)]
\end{aligned}$$

As a result

$$\begin{aligned}
\frac{\vec{\epsilon}^* \cdot \vec{A}_{\rm sc}^{(1)}}{D_0} &= \frac{(ka)^2}{q}(\epsilon_r - 1)(\vec{\epsilon}^* \cdot \vec{\epsilon}_0)\frac{\sin(qa) - qa\cos(qa)}{(qa)^2} \\
&= \frac{(ka)^2}{q}(\epsilon_r - 1)(\vec{\epsilon}^* \cdot \vec{\epsilon}_0) j_1(qa)
\end{aligned}$$

where j_1 is the $l = 1$ spherical Bessel function

$$j_1(\zeta) = \frac{\sin\zeta}{\zeta^2} - \frac{\cos\zeta}{\zeta}$$

The differential cross section is then

$$\frac{d\sigma}{d\Omega} = \left|\frac{\vec{\epsilon}^* \cdot \vec{A}_{\rm sc}^{(1)}}{D_0}\right|^2 = k^4 a^6 |\epsilon_r - 1|^2 \left(\frac{j_1(qa)}{qa}\right)^2 |\vec{\epsilon}^* \cdot \vec{\epsilon}_0|^2$$

where q is given by (10). The unpolarized cross section is

$$\frac{d\sigma}{d\Omega} = k^4 a^6 |\epsilon_r - 1|^2 \left(\frac{j_1(qa)}{qa}\right)^2 \frac{1 + \cos^2\theta}{2} \qquad (11)$$

Note that in the long wavelength limit ($ka \ll 1$) we also have $qa \ll 1$. In this case, we use the small argument expansion of the spherical Bessel function

$$j_1(\zeta) \approx \frac{\zeta}{3} - \cdots \qquad (\zeta \to 0)$$

to obtain

$$\frac{d\sigma}{d\Omega} \approx \frac{a^2}{9}(ka)^4 |\epsilon_r - 1|^2 \frac{1 + \cos^2\theta}{2}$$

which agrees with the long wavelength dipole approximation when ϵ_r is close to unity.

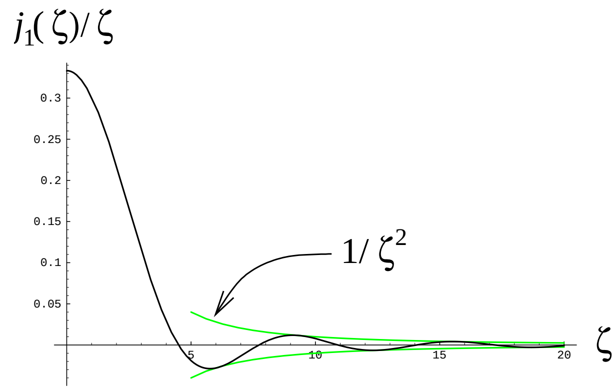
We are, of course, more interested in the short wavelength limit $ka \gg 1$. In this case, we note that the argument of the spherical Bessel function is

$$qa = 2ka\sin\frac{\theta}{2} \tag{12}$$

This quantity vanishes in the forward direction ($\theta = 0$), but otherwise is very large when $ka \gg 1$. In fact, the behavior of $j_1(\zeta)/\zeta$ is as follows

$$\frac{j_1(\zeta)}{\zeta} \sim \begin{cases} 1/3 & \zeta \ll 1 \\ -\cos\zeta/\zeta^2 & \zeta \gg 1 \end{cases}$$

This is peaked when $\zeta \approx 0$



As a result, the cross section (11) falls off as

$$\frac{d\sigma}{d\Omega} \sim \frac{1}{(qa)^4} = \frac{1}{[2ka\sin(\theta/2)]^4} \qquad (ka \gg 1)$$

away from the forward direction. Looking at the figure, we see that the cross section is large for $qa \lesssim 2$ but rapidly falls off for $qa \gtrsim 2$. From (12), we see that this forward peak corresponds to a cone with

$$\theta \lesssim \frac{2}{ka} \ll 1$$

With this in mind, we may make a rough estimate of the total cross section

$$\begin{aligned} \sigma &= k^4 a^6 |\epsilon_r - 1|^2 \int \left(\frac{j_1(qa)}{qa}\right)^2 \frac{1+\cos^2\theta}{2} d\Omega \\ &\approx k^4 a^6 |\epsilon_r - 1|^2 \left(\frac{1}{3}\right)^2 \times (\pi\theta^2)\Big|_{\theta = 2/ka} \\ &= \frac{4\pi}{9} k^2 a^4 |\epsilon_r - 1|^2 \end{aligned}$$

We can make a better estimate by approximating the integral more carefully. Since the integrand is highly peaked at $\theta \approx 0$, we take

$$\begin{aligned}
\int \left(\frac{j_1(qa)}{qa}\right)^2 \frac{1+\cos^2\theta}{2} d\Omega &\approx 2\pi \int_0^\pi \left(\frac{j_1(qa)}{qa}\right)^2 \sin\theta\, d\theta \\
&\approx 2\pi \int_0^\pi \left(\frac{j_1(ka\theta)}{ka\theta}\right)^2 \theta\, d\theta \\
&\approx \frac{2\pi}{(ka)^2} \int_0^\infty j_1(\zeta)^2 \frac{d\zeta}{\zeta} = \frac{\pi}{2(ka)^2}
\end{aligned}$$

This gives an approximate value of the total cross section

$$\sigma \approx \frac{\pi a^2}{2}(ka)^2 |\epsilon_r - 1|^2$$

This results in the asymptotic forms of the coefficients (8)

$$B_l \sim x' \cot(x' - \frac{l\pi}{2}) - 1 \to \infty$$
$$B'_l \sim \frac{1}{\sqrt{\epsilon_r}} x \cot(x' - \frac{l\pi}{2}) - 1 \to -1$$

Substituting this into (7) gives

$$\tan\delta_l = \frac{j_l(x)}{n_l(x)}, \qquad \tan\delta'_l = \frac{xj'_l(x) + j_l(x)}{xn'_l(x) + n_l(x)} = \frac{\frac{d}{dx} x j_l(x)}{\frac{d}{dx} x n_l(x)}$$

which reproduce exactly the perfectly conducting sphere phase shifts.

10.10 The aperture or apertures in a perfectly conducting plane screen can be viewed as the location of effective sources that produce radiation (the diffracted fields). An aperture whose dimensions are small compared with a wavelength acts as a source of dipole radiation with the contributions of other multipoles being negligible.

a) Beginning with (10.101) show that the effective electric and magnetic dipole moments can be expressed in terms of integrals of the tangential electric field in the aperture as follows:

$$\vec{p} = \epsilon\hat{n} \int (\vec{x} \cdot \vec{E}_{\text{tan}})\, da$$
$$\vec{m} = \frac{2}{i\omega\mu} \int (\hat{n} \times \vec{E}_{\text{tan}})\, da$$

where $\vec{E}_{\text{tan}}$ is the *exact* tangential electric field in the aperture, $\hat{n}$ is the normal to the plane screen, directed into the region of interest, and the integration is over the area of the openings.

The diffraction result (10.101) states

$$\vec{E}(\vec{x}\,) = \frac{1}{2\pi} \vec{\nabla} \times \int_{\text{apertures}} (\hat{n}' \times \vec{E}) \frac{e^{ikR}}{R} da' \tag{9}$$

In the radiation zone, we may take

$$\frac{e^{ikR}}{R} \approx \frac{e^{ikr}}{r} e^{-i\vec{k}\cdot\vec{x}'}$$

Furthermore, for a small aperture (long wavelength limit), we may expand the second exponential

$$\frac{e^{ikR}}{R} \approx \frac{e^{ikr}}{r}(1 - i\vec{k}\cdot\vec{x}')$$

Inserting this into (9) and noting that we may use the replacement $\vec{\nabla} \to i\vec{k}$ in the radiation zone, we obtain the expansion

$$\vec{E} = \frac{i}{2\pi} \frac{e^{ikr}}{r} \vec{k} \times \int (\hat{n}' \times \vec{E})(1 - i\vec{k} \cdot \vec{x}'\,)\, da' \tag{10}$$

We start with the first term in the expansion

$$\vec{E}_1 = \frac{i}{2\pi} \frac{e^{ikr}}{r} \vec{k} \times \int \hat{n}' \times \vec{E}\, da'$$

which may be compared with the electric field of magnetic dipole radiation (in the radiation zone)

$$\vec{E} = -\frac{Z_0}{4\pi} k^2 \frac{e^{ikr}}{r} \hat{k} \times \vec{m}$$

This allows us to read off the effective magnetic dipole moment

$$\vec{m} = \frac{2}{ikZ_0} \int \hat{n}' \times \vec{E}\, da' = \frac{2}{i\omega\mu_0} \int \hat{n}' \times \vec{E}\, da' \tag{11}$$

The effective electric dipole moment is somewhat trickier to extract. It is related to the second term in (10), which we write as

$$\vec{E}_2 = \frac{1}{2\pi} \frac{e^{ikr}}{r} \vec{k} \times \int (\hat{n}' \times \vec{E})(\vec{k} \cdot \vec{x}\,')\, da' \tag{12}$$

Since we have a flat screen, the normal vector $\hat{n}'$ is constant. Furthermore, the outgoing momentum vector $\vec{k}$ is unrelated to the integration coordinates (which line on the screen). Thus these two vectors may be pulled out of the integral. This means, we need to evaluate the integral (given in components)

$$\int E_i x'_j\, da'$$

where the indices i and j only lie in the screen directions (ie $i, j = 1, 2$ if we take $\hat{n}' = \hat{z}$). We now show that

$$\int E_i x'_j\, da' = \tfrac{1}{2}\delta_{ij} \int \vec{E} \cdot \vec{x}\,'\, da' \tag{13}$$

where we reemphasize that i and j lie in the screen directions only. Perhaps the most direct way to prove this is to write $E_i x'_j$ in tensor form

$$\begin{aligned} \vec{E} \otimes \vec{x}\,' = \begin{pmatrix} E_1 x'_1 & E_1 x'_2 \\ E_2 x'_1 & E_2 x'_2 \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} E_1 x'_1 + E_2 x'_2 & 0 \\ 0 & E_1 x'_1 + E_2 x'_2 \end{pmatrix} \\ &\quad + \frac{1}{2} \begin{pmatrix} E_1 x'_1 - E_2 x'_2 & 2E_1 x'_2 \\ 2E_2 x'_1 & -E_1 x'_1 + E_2 x'_2 \end{pmatrix} \\ &= \tfrac{1}{2}\delta_{ij}(\vec{E} \cdot \vec{x}\,') + \tfrac{1}{2}[E_i x'_j - (\hat{n}' \times \vec{x}\,')_i (\hat{n}' \times \vec{E})_j] \end{aligned} \tag{14}$$

The second term vanishes when integrated over the openings. This is because we may use $\vec{\nabla}\times\vec{E}=0$ in a source-free region. Then

$$
\begin{aligned}
0 &= \int x_i' x_j' \hat{n}'\cdot(\vec{\nabla}'\times\vec{E})\,da' = \epsilon_{klm}\hat{n}_k'\int x_i'x_j'\partial_l E_m\,da' \\
&= -\epsilon_{klm}\hat{n}_k'\int \partial_l(x_i'x_j')E_m\,da' = \hat{n}_k'\int(\epsilon_{ikm}x_j'+\epsilon_{jkm}x_i')E_m\,da' \\
&= \int [x_i'(\hat{n}'\times\vec{E})_j + x_j'(\hat{n}'\times\vec{E})_i]\,da'
\end{aligned}
\tag{15}
$$

Note that the surface term arising from the integration by parts vanishes because it is proportional to $E_{\|}$, which must vanish on the boundaries of the openings. Substituting in explicit components $ij=11$, 12, and 22 then proves that the integral of E_2x_1', $E_1x_1'-E_2x_2'$, and E_1x_2' vanish, as needed to remove the second term from (14). This can also be seen directly by taking a cross product of (15) with $\hat{n}'$ in the ith component to get

$$
\int[(\hat{n}'\times\vec{x}\,')_i(\hat{n}'\times\vec{E})_j - x_j'E_i]\,da' = 0
$$

In any case, the result is simply (13), which may be substituted into (12) to obtain

$$
\begin{aligned}
\vec{E}_2 &= \frac{1}{4\pi}\frac{e^{ikr}}{r}\vec{k}\times(\hat{n}'\times\vec{k})\int\vec{x}\,'\cdot\vec{E}\,da' \\
&= -\frac{1}{4\pi}\frac{e^{ikr}}{r}\vec{k}\times(\vec{k}\times\hat{n}')\int\vec{x}\,'\cdot\vec{E}\,da'
\end{aligned}
$$

Comparing this with the radiation patter for electric dipole radiation

$$
\vec{E} = -\frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\hat{k}\times(\hat{k}\times\vec{p})
$$

gives an effective electric dipole moment

$$
\vec{p} = \epsilon_0\hat{n}'\int\vec{x}\,'\cdot\vec{E}\,da'
$$

Note the curious fact that the magnetic dipole term comes from the lowest order in the expansion of (9), while the electric dipole term comes from the next order. This is 'backwards' from what happens for a conventional source given by a specified current density.

b) Show that the expression for the magnetic moment can be transformed into

$$
\vec{m} = \frac{2}{\mu}\int\vec{x}(\hat{n}\cdot\vec{B})\,da
$$

Be careful about possible contributions from the edge of the aperture where some components of the fields are singular if the screen is infinitesimally thick.

To relate the electric field to the magnetic field, we may use Faraday's equation for harmonic fields $\vec{\nabla} \times \vec{E} - i\omega\vec{B} = 0$ to write

$$\hat{n}' \cdot (\vec{\nabla}' \times \vec{E}) = i\omega(\hat{n}' \cdot \vec{B})$$

Multiplying this by a vector $\vec{x}\,'$ and integrating gives

$$\begin{aligned} i\omega \int \vec{x}\,'(\hat{n}' \cdot \vec{B})\, da' &= \int \vec{x}\,'[\hat{n}' \cdot (\vec{\nabla}' \times \vec{E})]\, da' \\ &= \int \vec{x}\,'\epsilon_{ijk}\hat{n}'_i\partial_j E_k\, da' \\ &= -\int \partial_j(\vec{x}\,')\epsilon_{ijk}\hat{n}'_i E_k\, da' = \int \hat{n}' \times \vec{E}\, da' \end{aligned}$$

Note that for integration by parts, we use the fact that $\hat{n}'$ is a constant surface normal vector and that $E_{||}$ vanishes at the edges of the aperture. More precisely, the generalization of Stokes' theorem indicates that the surface term is of the form

$$\oint \vec{x}\,'(\vec{E} \cdot d\vec{l}\,)$$

so the electric field contribution indeed arises only from the parallel component to the edge of the aperture. Finally, substituting this integrated relation between $\vec{E}$ and $\vec{B}$ into (11) gives

$$\vec{m} = \frac{2}{i\omega\mu_0} \int \hat{n}' \times \vec{E}\, da' = \frac{2}{\mu_0} \int \vec{x}\,'(\hat{n}' \cdot \vec{B})\, da'$$

1. *Jackson* 10.20. A suspension of transparent fibers is modeled as a collection of scatters, each being a right circular cylinder of radius a and length L of a uniform dielectric material whose dielectric constant differs from the surrounding medium by a fractional amount $\delta\epsilon/\epsilon^{(0)}$.

   (a) We want to calculate the differential scattering cross section per scatterer in the first Born approximation, for unpolarized incident light. The general result is

$$\frac{d\sigma}{d\Omega} = \frac{|\boldsymbol{\epsilon}^* \cdot \mathbf{A}_{sc}|^2}{\mathbf{D}^{(0)}}, \tag{1}$$

   where the scattering amplitude in the first Born approximation is

$$\frac{\boldsymbol{\epsilon}^* \cdot \mathbf{A}_{sc}}{D_0} = \frac{k^2}{4\pi}\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0 \int d^3x\, e^{i\mathbf{q}\cdot\mathbf{x}} \frac{\delta\epsilon(\mathbf{x})}{\epsilon^{(0)}}. \tag{2}$$

   For the cylinder,

$$\begin{aligned} \int d^3x\, e^{i\mathbf{q}\cdot\mathbf{x}} \frac{\delta\epsilon(\mathbf{x})}{\epsilon^{(0)}} &= \frac{\delta\epsilon}{\epsilon^{(0)}} \int_{\text{cylinder}} d^3x\, e^{i\mathbf{q}_\perp\cdot\mathbf{x}_\perp + iq_\parallel z} \\ &= \frac{\delta\epsilon}{\epsilon^{(0)}} \int_0^a \rho\, d\rho \int_0^{2\pi} d\phi \int_{-L/2}^{L/2} dz\, e^{iq_\perp \rho\cos\phi + iq_\parallel z} \\ &= \frac{\delta\epsilon}{\epsilon^{(0)}} \pi a^2 L \frac{2J_1(q_\perp a)}{q_\perp a} \frac{\sin(q_\parallel L/2)}{q_\parallel L/2}. \end{aligned} \tag{3}$$

   Assuming that the incident light is unpolarized, we have $\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0 = (1+\cos^2\theta)/2$. Pulling all of these results together,

$$\frac{d\sigma}{d\Omega} = \left|\frac{\delta\epsilon}{\epsilon^{(0)}}\right|^2 \frac{k^4 a^4 L^2}{32}(1+\cos^2\theta) \left[\frac{2J_1(q_\perp a)}{q_\perp a} \frac{\sin(q_\parallel L/2)}{q_\parallel L/2}\right]^2. \tag{4}$$

   (b) Next, assume that the cylinders in the suspension are slender, in the sense that $ka \ll 1$. Then $2J_1(q_\perp a)/q_\perp a \approx 1$, and the cross section is

$$\frac{d\sigma}{d\Omega} = \left|\frac{\delta\epsilon}{\epsilon^{(0)}}\right|^2 \frac{k^4 a^4 L^2}{32}(1+\cos^2\theta) \left[\frac{\sin(q_\parallel L/2)}{q_\parallel L/2}\right]^2. \tag{5}$$

   If we have an ensemble of cylinders with random orientations, we can average over the orientations; this is equivalent to averaging over $q_\parallel = \sqrt{q^2 - q_\perp^2}$, which ranges between 0 and q. Therefore,

$$\left\langle \left[\frac{\sin(q_\parallel L/2)}{q_\parallel L/2}\right]^2 \right\rangle = \frac{1}{q}\int_0^q dq_\parallel \left[\frac{\sin(q_\parallel L/2)}{q_\parallel L/2}\right]^2 = \frac{4}{qL}\int_0^{qL} du\, \frac{\sin^2(u/2)}{u^2} = \frac{2}{qL}\mathrm{Si}(qL) - \left[\frac{\sin(qL/2)}{qL/2}\right]^2, \tag{6}$$

where the last line was obtained by integrating by parts, and $\mathrm{Si}(x)$ is the sine integral,

$$\mathrm{Si}(x) = \int_0^x \frac{\sin u}{u}\, du. \tag{7}$$

Therefore,

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle = \left|\frac{\delta\epsilon}{\epsilon^{(0)}}\right|^2 \frac{k^4 a^4 L^2}{32}(1+\cos^2\theta)F(qL), \tag{8}$$

where

$$F(x) = \frac{2}{x}\mathrm{Si}(x) - \left[\frac{\sin(x/2)}{x/2}\right]^2 = \begin{cases} 1 - x^2/36 + O(x^4) & x \ll 1 \\ \pi/x + O(x^{-2}) & x \gg 1 \end{cases} \tag{9}$$

and $q = k\sqrt{2(1-\cos\theta)}$.

(c) In the limit $kL \ll 1$, the ensemble averaged differential cross section is

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle = \left|\frac{\delta\epsilon}{\epsilon^{(0)}}\right|^2 \frac{k^4 a^4 L^2}{32}(1+\cos^2\theta), \tag{10}$$

which when integrated over solid angles gives

$$\sigma = \frac{\pi^2}{6}\left|\frac{\delta\epsilon}{\epsilon^{(0)}}\right|^2 k^4 a^4 L^2. \tag{11}$$

The k^4 dependence is characteristic of Rayleigh scattering.
In the limit $kL \gg 1$ (but $ka \ll 1$) we have

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle = \left|\frac{\delta\epsilon}{\epsilon^{(0)}}\right|^2 \frac{k^4 a^4 L^2}{32}(1+\cos^2\theta)\frac{\pi}{kL\sqrt{2(1-\cos\theta)}}. \tag{12}$$

Integrating over solid angles, we have

$$\sigma = \left|\frac{\delta\epsilon}{\epsilon^{(0)}}\right|^2 \frac{\pi k^3 a^4 L}{32} 2\pi \int_{-1}^{1} d(\cos\theta)\frac{1+\cos^2\theta}{\sqrt{2(1-\cos\theta)}}. \tag{13}$$

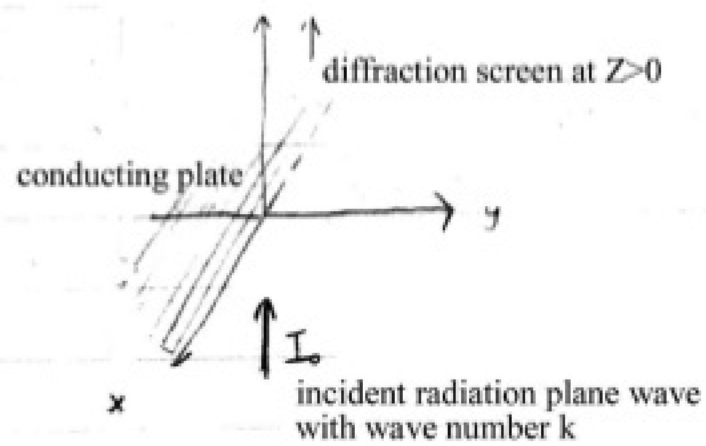
The integral has the value 44/15, so the final result is

$$\sigma = \frac{11\pi^2}{60} \left| \frac{\delta\epsilon}{\epsilon^{(0)}} \right|^2 k^3 a^4 L. \tag{14}$$

Note that in this frequency regime there is less scattering at short wavelengths than the Rayleigh case (k^3 vs. k^4).

Problem 10.11

A perfectly conducting flat screen occupies half of the x-y plane (i.e., $x < 0$). A plane wave of intensity I_0 and wave number k is incident along the z axis from the region $z < 0$. discuss the values of the diffracted fields in the plane parallel to the x-y plane defined by $z = Z > 0$. Let the coordinates of the observation point by $(X,0,Z)$.



a. Show that, for the usual scalar Kirchoff approximation and in the limit $Z >> X$ and $\sqrt{kZ} >> 1$, the diffracted field is

$$\Psi = \sqrt{I_0}\left(\frac{1+i}{2i}\right)e^{ikZ-i\omega t}\sqrt{\frac{2}{\pi}}\int_{-\Xi}^{\infty} e^{iu^2}du$$

where $\Xi = X(\frac{k}{2Z})^{\frac{1}{2}}$.

$$\Psi(r_0) = \frac{k}{2\pi i}\sqrt{I_0}\int_{Aperture}\frac{e^{ikr_p}}{r_p}dA'$$

r_0 is the observation point, and $r_p = \sqrt{(x'-X)^2+(y'-Y)^2+(z'-Z)^2}$ is the distance from the area point at the aperture to the observation point. The small letters denote the aperture values while the large letters denote values at the observation point. $dA' = dx'dy'$ in this case because the screen is in the xy plane.

I proceed first by evaluating the integral over the y coordinate.

$$I_1 = \int_{\infty}^{\infty}\frac{e^{ikr_p}}{r_p}dy'$$

I exploit the symmetry of the integral about $y = 0$, and replace $\rho^2 = (x' - X)^2 + (z' - Z)^2$.

$$I_1 = 2\int_0^\infty \frac{e^{ik\sqrt{(y'-Y)^2+\rho^2}}}{\sqrt{(y'-Y)^2+\rho^2}}dy'$$

Substitute $\nu = \sqrt{(y'-Y)^2+\rho^2}$.

$$I_1 = 2\int_\rho^\infty \frac{e^{ik\nu}}{\sqrt{\nu^2-\rho^2}}d\nu$$

Remember from basic calculus,

$$\int_0^\infty \frac{\sin(Ax)}{\sqrt{x^2-1}}dx = \frac{\pi}{2}J_0(A)$$

$$\int_0^\infty \frac{\cos(Ax)}{\sqrt{x^2-1}}dx = -\frac{\pi}{2}N_0(A)$$

J_0 is a Bessel function and N_0 is a Neumann function. I will use these to reduce the integral to a more tractable form. By Euler's handy formula, $e^{ix} = \cos x + i\sin x$. so we can write

$$I_1 = 2\int_\rho^\infty \frac{e^{ik\nu}}{\sqrt{\nu^2-\rho^2}}d\nu = 2\int_\rho^\infty \frac{[\cos(k\nu) + i\sin(k\nu)]}{\sqrt{\nu^2-\rho^2}}d\nu$$

Let $\xi = \nu/\rho$ and $d\xi = \frac{1}{\rho}d\nu$.

$$I_1 = 2\int_1^\infty \frac{[\cos(k\rho\xi) + i\sin(k\rho\xi)]}{\sqrt{\xi^2-1}}d\xi = 2[-\frac{\pi}{2}N_0(k\rho) + i\frac{\pi}{2}J_0(k\rho)]$$

And so the first part of the surface integral is done.
Now, I will attempt to integrate over dx'. Don't forget ρ is a function of x'.

$$I_2 = \int_0^\infty -\pi N_0(k\rho) + i\pi J_0(k\rho)dx' = i\pi\int J_0(k\rho) + iN_0(k\rho)dx'$$

In the limit $\sqrt{kZ} >> 1 \rightarrow kZ >> 1$ and $\rho k >> 1$, the Bessel function and its friend can be approximated by the following:

$$J_0(A) \simeq \sqrt{\frac{2}{\pi A}}\cos(A - \frac{\pi}{4})$$

$$N_0(A) \simeq -\sqrt{\frac{2}{\pi A}}\sin(A - \frac{\pi}{4})$$

And the integral reduces to

$$I_2 = \int_0^\infty i\pi\sqrt{\frac{2}{\pi k\rho}}[\cos(k\rho - \frac{\pi}{4}) + i\sin(k\rho - \frac{\pi}{4})]dx'$$

Which easily reduces to

$$I_2 = \int_0^\infty i\sqrt{\frac{2\pi}{k\rho}}e^{i(\rho k - \frac{\pi}{4})}dx' = i\sqrt{2\pi}\int_0^\infty e^{i(\rho k - \frac{\pi}{4})}\sqrt{\frac{1}{k\rho}}dx'$$

Lest I loose track of all the coefficients, I'll rewrite Ψ.

$$\Psi = \frac{k}{2\pi i}\sqrt{I_0}i\sqrt{2\pi}\int_0^\infty \frac{e^{i(\rho k - \frac{\pi}{4})}}{\sqrt{\rho k}}dx' = k\sqrt{\frac{I_0}{2\pi}}e^{-i\frac{\pi}{4}}\int_0^\infty \frac{e^{[ik\sqrt{(x'-X)^2+(z'-Z)^2}]}}{\sqrt{k\sqrt{(x'-X)^2+(z'-Z)^2}}}dx'$$

I have written ρ in explicitly to remind us that ρ depends on x'. Now, I label the integral as I_3 and tackle this integration.

$$I_3 = \int_0^\infty \frac{e^{[ik\sqrt{(x'-X)^2+(z'-Z)^2}]}}{\sqrt{k\sqrt{(x'-X)^2+(z'-Z)^2}}}dx'$$

So far, I haven't make use of the fact that $z' = 0$. I'll do that now.
If $(x' - X) << Z$, we can expand $\sqrt{(x'-X)^2 + Z^2} \simeq Z + \frac{(x'-X)^2}{2Z}$. So

$$I_3 = \frac{e^{ikZ}}{\sqrt{kZ}}\int_{-\Xi}^\infty \sqrt{\frac{2Z}{k}}e^{iu^2}du$$

where $u = \sqrt{\frac{k}{2Z}}(x' - X)$, and the limits of integration have been changed accordingly, $\Xi = X\sqrt{k/(2Z)}$. This gives the result:

$$\Psi = k\sqrt{\frac{I_0}{2\pi}}e^{-i\frac{\pi}{4}}\frac{e^{ikZ}}{\sqrt{kZ}}\sqrt{\frac{2Z}{k}}\int_{-\Xi}^\infty e^{iu^2}du$$

A little work with an Argand diagram should convince you that

$$e^{-i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} = \sqrt{2}\left(\frac{1+i}{2i}\right)$$

and then, Ψ reduces to Jackson's result.

$$\Psi = \sqrt{I_0}\left(\frac{1+i}{2i}\right)e^{ikZ-i\omega t}\sqrt{\frac{2}{\pi}}\int_{-\Xi}^{\infty} e^{iu^2}du$$

where $\Xi = X(\frac{k}{2Z})^{\frac{1}{2}}$. Note: I didn't assume time dependence from the start, but if I did the derivation would be the same. I would have simply factored the $e^{-i\omega t}$ out from the start. So I just put it back here.

b. Find the intensity. Determine the asymptotic behavior of I for ξ large and positive (illuminated region) and ξ large and negative (shadow region). what is the value of I at $X = 0$? Make a sketch of I as a function of X for fixed Z.

We need to rewrite I_4 is a suggestive way.

$$I_4 = \int_{-\Xi}^{\infty} e^{iu^2}du$$

Everybody should know the friendly Fresnel Integrals:

$$C(\lambda) = \int_0^{\lambda} \cos(\frac{\pi x^2}{2})dx$$

$$S(\lambda) = \int_0^{\lambda} \sin(\frac{\pi x^2}{2})dx$$

And using Euler's handy relationship,

$$\int_0^{\lambda} e^{i\pi\frac{x^2}{2}}dx = C(\lambda) + iS(\lambda)$$

In our case.

$$\int_{-\Xi}^{\infty} e^{iu^2}du = \sqrt{\frac{\pi}{2}}[C(\infty) + iS(\infty) - C(-\Xi) - iS(-\Xi)]$$

I will use the symmetry of $C(x)$ and $S(x)$, namely, $C(x) = -C(-x)$ and $S(x) = -S(-x)$ to get rid of all the unwanted minus signs.

$$\int_{-\Xi}^{\infty} e^{iu^2}du = \sqrt{\frac{\pi}{2}}\left[C(\infty) + iS(\infty) + C(\Xi) + iS(\Xi)\right]$$

To find $C(x)$ and $S(x)$ at infinity, we need $\lim_{t\to\pm\infty} C(t) = \pm\frac{1}{2}$ and $\lim_{t\to\pm\infty} S(t) = \pm\frac{1}{2}$. I_4 is evidently representable by $\frac{1}{2}(1+i) + C(\Xi) + iS(\Xi)$. The intensity is given by $|\Psi|^2$ so

$$\mathcal{I} = |\sqrt{\frac{2}{\pi}}\sqrt{I_0}\left(\frac{1+i}{2i}\right)e^{ikZ-i\omega t}|^2\left[\frac{1+i}{2} + C(\Xi) + iS(\Xi)\right]^2 =$$

$$\left(\frac{2}{\pi}I_0\right)\frac{\pi}{2}\left[\left(C(\Xi)+\frac{1}{2}\right)^2 + \left(S(\Xi)+\frac{1}{2}\right)^2\right]$$

And finally, we have what Jackson wants.

$$\mathcal{I} = \frac{I_0}{2}\left[\left(C(\Xi)+\frac{1}{2}\right)^2 + \left(S(\Xi)+\frac{1}{2}\right)^2\right]$$

As $\Xi \to \infty+$, $\mathcal{I} \to I_0$, and we have a bright spot. As $\Xi \to \infty-$, $\mathcal{I} \to 0$, and we have a shadow. At $X = 0$, $\Xi = 0$, and $\mathcal{I} = \frac{I_0}{4}$.
The graph is coming soon!