12.31

$E_T = E_1 + E_2 + \cdots;\quad p_T = p_1 + p_2 + \cdots;\quad \bar{p}_T = \gamma(p_T - \beta E_T/c) = 0 \Rightarrow \beta = v/c = p_T c/E_T.$

$v = c^2 p_T/E_T = \boxed{c^2(p_1 + p_2 + \cdots)/(E_1 + E_2 + \cdots).}$

12.32

$$E_\mu = \frac{(m_\pi^2 + m_\mu^2)}{2m_\pi}c^2 = \gamma m_\mu c^2 \Rightarrow \gamma = \frac{(m_\pi^2 + m_\mu^2)}{2m_\pi m_\mu} = \frac{1}{\sqrt{1 - v^2/c^2}};\ 1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2};$$

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2} = 1 - \frac{4m_\pi^2 m_\mu^2}{(m_\pi^2 + m_\mu^2)^2} = \frac{m_\pi^4 + 2m_\pi^2 m_\mu^2 + m_\mu^4 - 4m_\pi^2 m_\mu^2}{(m_\pi^2 + m_\mu^2)^2} = \frac{(m_\pi^2 - m_\mu^2)^2}{(m_\pi^2 + m_\mu^2)^2};\ v = \boxed{\left(\frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}\right)c.}$$

12.34

First calculate pion's energy: $E^2 = p^2c^2 + m^2c^4 = \frac{9}{16}m^2c^4 + m^2c^4 = \frac{25}{16}m^2c^4 \Rightarrow E = \frac{5}{4}mc^2.$

Conservation of energy: $\frac{5}{4}mc^2 = E_A + E_B$

Conservation of momentum: $\frac{3}{4}mc^2 = p_A + p_B = \frac{E_A}{c} - \frac{E_B}{c} \Rightarrow \frac{3}{4}mc^2 = E_A - E_B$

$\Big\}\ 2E_A = 2mc^2.$

$\Rightarrow \boxed{E_A = mc^2;}\quad \boxed{E_B = \frac{1}{4}mc^2.}$

12.35

Classically, $E = \frac{1}{2}mv^2$. In a colliding beam experiment, the relative velocity (classically) is *twice* the velocity of either one, so the relative energy is $4E$.

Let $\bar{\mathcal{S}}$ be the system in which ① is at rest. Its speed v, relative to $\mathcal{S}$, is just the speed of ① in $\mathcal{S}$.

$\bar{p}^0 = \gamma(p^0 - \beta p^1) \Rightarrow \frac{\bar{E}}{c} = \gamma\left(\frac{E}{c} - \beta p\right)$, where p is the momentum of ② in $\mathcal{S}$.

$E = \gamma Mc^2$, so $\gamma = \frac{E}{Mc^2}$; $p = -\gamma Mv = -\gamma M\beta c;\ \bar{E} = \gamma\left(\frac{E}{c} + \beta\gamma M\beta c\right)c = \gamma(E + \gamma Mc^2\beta^2).$

$\gamma^2 = \frac{1}{1-\beta^2} \Rightarrow 1 - \beta^2 = \frac{1}{\gamma^2} \Rightarrow \beta^2 = 1 - \frac{1}{\gamma^2} = \frac{\gamma^2-1}{\gamma^2};\ \bar{E} = \frac{E}{Mc^2}E + \left[\left(\frac{E}{Mc^2}\right)^2 - 1\right]Mc^2.$

$\bar{E} = \frac{E^2}{Mc^2} + \frac{E^2}{Mc^2} - Mc^2;\ \boxed{\bar{E} = \frac{2E^2}{Mc^2} - Mc^2.}$

For $E = 30$ GeV and $Mc^2 = 1$ GeV, we have $\bar{E} = \frac{(2)(900)}{1} - 1 = 1800 - 1 = \boxed{1799\text{ GeV}} = \boxed{60E.}$

12.37

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}\frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}} = m\left\{\frac{\frac{d\mathbf{u}}{dt}}{\sqrt{1 - u^2/c^2}} + \mathbf{u}\left(-\frac{1}{2}\right)\frac{-\frac{1}{c^2}2\mathbf{u}\cdot\frac{d\mathbf{u}}{dt}}{(1 - u^2/c^2)^{3/2}}\right\}$$

$$= \frac{m}{\sqrt{1 - u^2/c^2}}\left\{\mathbf{a} + \frac{\mathbf{u}(\mathbf{u}\cdot\mathbf{a})}{(c^2 - u^2)}\right\}.\quad \text{qed}$$

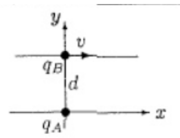
12.41

$$\mathbf{F} = \frac{m}{\sqrt{1-u^2/c^2}}\left[\mathbf{a} + \frac{\mathbf{u}(\mathbf{u}\cdot\mathbf{a})}{c^2-u^2}\right] = q(\mathbf{E}+\mathbf{u}\times\mathbf{B}) \Rightarrow \mathbf{a} + \frac{\mathbf{u}(\mathbf{u}\cdot\mathbf{a})}{(c^2-u^2)} = \frac{q}{m}\sqrt{1-u^2/c^2}(\mathbf{E}+\mathbf{u}\times\mathbf{B}).$$

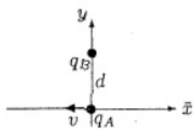
Dot in $\mathbf{u}$: $(\mathbf{u}\cdot\mathbf{a}) + \frac{u^2(\mathbf{u}\cdot\mathbf{a})}{c^2(1-u^2/c^2)} = \frac{\mathbf{u}\cdot\mathbf{a}}{(1-u^2/c^2)} = \frac{q}{m}\sqrt{1-u^2/c^2}\big[\mathbf{u}\cdot\mathbf{E} + \underbrace{\mathbf{u}\cdot(\mathbf{u}\times\mathbf{B})}_{=0}\big];$

$$\therefore \frac{\mathbf{u}(\mathbf{u}\cdot\mathbf{a})}{(c^2-u^2)} = \frac{q}{m}\sqrt{1-u^2/c^2}\,\frac{\mathbf{u}(\mathbf{u}\cdot\mathbf{E})}{c^2}. \quad \text{So } \mathbf{a} = \frac{q}{m}\sqrt{1-u^2/c^2}\left[\mathbf{E}+\mathbf{u}\times\mathbf{B} - \frac{1}{c^2}\mathbf{u}(\mathbf{u}\cdot\mathbf{E})\right]. \quad \text{qed}$$

12.45



(a) Fields of A at B: $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q_A}{d^2}\hat{\mathbf{y}}$; $\mathbf{B} = 0$. So force on q_B is $\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{d^2}\hat{\mathbf{y}}.}$



(b) (i) From Eq. 12.68: $\boxed{\bar{\mathbf{F}} = \frac{\gamma}{4\pi\epsilon_0}\frac{q_A q_B}{d^2}\hat{\mathbf{y}}.}$ (*Note:* here the particle is at rest in $\bar{S}$.)

(ii) From Eq. 12.92, with $\theta = 90°$: $\bar{E} = \frac{1}{4\pi\epsilon_0}\frac{q_A(1-v^2/c^2)}{(1-v^2/c^2)^{3/2}}\frac{1}{d^2}\hat{\mathbf{y}} = \frac{\gamma}{4\pi\epsilon_0}\frac{q_A}{d^2}\hat{\mathbf{y}}$ (this also follows from Eq. 12.108).

$\bar{\mathbf{B}} \neq 0$, but since $v_B = 0$ in $\bar{S}$, there is no magnetic force anyway, and $\boxed{\bar{\mathbf{F}} = \frac{\gamma}{4\pi\epsilon_0}\frac{q_A q_B}{d^2}\hat{\mathbf{y}}}$ (as before).

12.48

(a) Making the appropriate modifications in Eq. 9.48 (and picking $\delta = 0$ for convenience),

$$\boxed{\mathbf{E}(x,y,z,t) = E_0\cos(kx-\omega t)\,\hat{\mathbf{y}}, \quad \mathbf{B}(x,y,z,t) = \frac{E_0}{c}\cos(kx-\omega t)\,\hat{\mathbf{z}}, \quad \text{where } k \equiv \frac{\omega}{c}.}$$

(b) Using Eq. 12.108 to transform the fields:

$$\bar{E}_x = \bar{E}_z = 0, \quad \bar{E}_y = \gamma(E_y - vB_z) = \gamma E_0\left[\cos(kx-\omega t) - \frac{v}{c}\cos(kx-\omega t)\right] = \alpha E_0\cos(kx-\omega t),$$

$$\bar{B}_x = \bar{B}_y = 0, \quad \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y) = \gamma E_0\left[\frac{1}{c}\cos(kx-\omega t) - \frac{v}{c^2}\cos(kx-\omega t)\right] = \alpha\frac{E_0}{c}\cos(kx-\omega t),$$

where $\boxed{\alpha \equiv \gamma\left(1-\frac{v}{c}\right) = \sqrt{\frac{1-v/c}{1+v/c}}.}$

Now the inverse Lorentz transformations (Eq. 12.19) $\Rightarrow x = \gamma(\bar{x} + v\bar{t})$ and $t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right)$, so

$$kx - \omega t = \gamma\left[k(\bar{x} + v\bar{t}) - \omega\left(\bar{t} + \frac{v}{c^2}\bar{x}\right)\right] = \gamma\left[\left(k - \frac{\omega v}{c^2}\right)\bar{x} - (\omega - kv)\bar{t}\right] = \bar{k}\bar{x} - \bar{\omega}\bar{t},$$

where (recalling that $k = \omega/c$): $\bar{k} \equiv \gamma\left(k - \frac{\omega v}{c^2}\right) = \gamma k(1 - v/c) = \alpha k$ and $\bar{\omega} \equiv \gamma\omega(1 - v/c) = \alpha\omega$.

Conclusion:

$$\boxed{\begin{gathered}\bar{\mathbf{E}}(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \bar{E}_0 \cos(\bar{k}\bar{x} - \bar{\omega}\bar{t})\,\hat{\mathbf{y}}, \quad \bar{\mathbf{B}}(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \frac{\bar{E}_0}{c}\cos(\bar{k}\bar{x} - \bar{\omega}\bar{t})\,\hat{\mathbf{z}}, \\ \text{where } \bar{E}_0 = \alpha E_0, \quad \bar{k} = \alpha k, \quad \bar{\omega} = \alpha\omega, \quad \text{and } \alpha \equiv \sqrt{\frac{1 - v/c}{1 + v/c}}.\end{gathered}}$$

(c) $\boxed{\bar{\omega} = \omega\sqrt{\frac{1 - v/c}{1 + v/c}}.}$ This is the $\boxed{\text{Doppler shift}}$ for light. $\bar{\lambda} = \frac{2\pi}{\bar{k}} = \frac{2\pi}{\alpha k} = \frac{\lambda}{\alpha}$. The velocity of the wave in $\bar{S}$ is $\bar{v} = \frac{\bar{\omega}}{2\pi}\bar{\lambda} = \frac{\omega}{\lambda} = \boxed{c.}$ $\boxed{\text{Yup,}}$ this is exactly what I expected (the velocity of a light wave is the same in any inertial system).

(d) Since intensity goes like E^2, the ratio is $\frac{\bar{I}}{I} = \frac{\bar{E}_0^2}{E_0^2} = \alpha^2 = \boxed{\frac{1 - v/c}{1 + v/c}.}$

Dear Al,

The amplitude, frequency, and intensity of the light wave will all $\boxed{\text{decrease to zero}}$ as you run faster and faster. It'll get so faint you won't be able to see it, and so red-shifted even your night-vision goggles won't help. But it'll still be going 3×10^8 m/s relative to you. Sorry about that.

Sincerely,

David

12.51

$$\begin{aligned}F^{\mu\nu}F_{\mu\nu} &= F^{00}F^{00} - F^{01}F^{01} - F^{02}F^{02} - F^{03}F^{03} - F^{10}F^{10} - F^{20}F^{20} - F^{30}F^{30} \\ &\quad + F^{11}F^{11} + F^{12}F^{12} + F^{13}F^{13} + F^{21}F^{21} + F^{22}F^{22} + F^{23}F^{23} + F^{31}F^{31} + F^{32}F^{32} + F^{33}F^{33} \\ &= -(E_x/c)^2 - (E_y/c)^2 - (E_z/c)^2 - (E_x/c)^2 - (E_y/c)^2 - (E_z/c)^2 + B_z^2 + B_y^2 + B_z^2 + B_x^2 + B_y^2 + B_x^2 \\ &= 2B^2 - 2E^2/c^2 = \boxed{2\left(B^2 - \frac{E^2}{c^2}\right),}\end{aligned}$$

which, apart from the constant factor $-\frac{2}{c^2}$, is the invariant we found in Prob. 12.46(b).

$\boxed{G^{\mu\nu}G_{\mu\nu} = 2(E^2/c^2 - B^2)}$ (the same invariant).

$$\begin{aligned}F^{\mu\nu}G_{\mu\nu} &= -2\left(F^{01}G^{01} + F^{02}G^{02} + F^{03}G^{03}\right) + 2\left(F^{12}G^{12} + F^{13}G^{13} + F^{23}G^{23}\right) \\ &= -2\left(\frac{1}{c}E_xB_x + \frac{1}{c}E_yB_y + \frac{1}{c}E_zB_z\right) 2\left[B_z(-E_z/c) + (-B_y)(E_y/c) + B_x(-E_x/c)\right] \\ &= -\frac{2}{c}(\mathbf{E}\cdot\mathbf{B}) - \frac{2}{c}(\mathbf{E}\cdot\mathbf{B}) = \boxed{-\frac{4}{c}(\mathbf{E}\cdot\mathbf{B}),}\end{aligned}$$

which, apart from the factor $-4/c$, is the invariant of Prob. 12.46(a). [These are, incidentally, the *only* fundamental invariants you can construct from $\mathbf{E}$ and $\mathbf{B}$.]

12.53

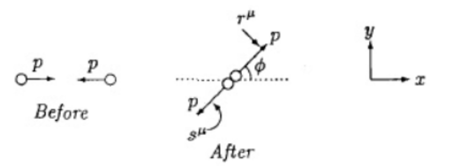
$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$. Differentiate: $\partial_\mu \partial_\nu F^{\mu\nu} = \mu_0 \partial_\mu J^\mu$.

But $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$ (the combination is *symmetric*) while $F^{\nu\mu} = -F^{\mu\nu}$ (*anti*symmetric).

$\therefore \partial_\mu \partial_\nu F^{\mu\nu} = 0$. [Why? Well, these indices are both summed from $0 \to 3$, so it doesn't matter which we call μ, which ν: $\partial_\mu \partial_\nu F^{\mu\nu} = \partial_\nu \partial_\mu F^{\nu\mu} = \partial_\mu \partial_\nu (-F^{\mu\nu}) = -\partial_\mu \partial_\nu F^{\mu\nu}$. But if a quantity is equal to minus itself, it must be zero.] *Conclusion:* $\partial_\mu J^\mu = 0$. qed

12.61

In CM:



(p = magnitude of 3-momentum in CM, ϕ = CM scattering angle)

Outgoing 4-momenta: $r^\mu = \left(\frac{E}{c}, p\cos\phi, p\sin\phi, 0\right)$; $s^\mu = \left(\frac{E}{c}, -p\cos\phi, -p\sin\phi, 0\right)$.

In Lab:

Problem: calculate θ, in terms of p, ϕ.

Lorentz transformation: $\bar{r}_x = \gamma(r_x - \beta r^0)$; $\bar{r}_y = r_y$; $\bar{s}_x = \gamma(s_x - \beta s^0)$; $\bar{s}_y = s_y$.

Now $E = \gamma mc^2$; $p = -\gamma m v$ (v here is to the *left*); $E^2 - p^2c^2 = m^2c^4$, so $\beta = -\frac{pc}{E}$.

$\therefore \bar{r}_x = \gamma\left(p\cos\phi + \frac{pc}{E}\frac{E}{c}\right) = \gamma p(1+\cos\phi)$; $\bar{r}_y = p\sin\phi$; $\bar{s}_x = \gamma p(1-\cos\phi)$; $\bar{s}_y = -p\sin\phi$.

$$\cos\theta = \frac{\bar{\mathbf{r}}\cdot\bar{\mathbf{s}}}{\bar{r}\bar{s}} = \frac{\gamma^2p^2(1-\cos^2\phi) - p^2\sin^2\phi}{\sqrt{\left[\gamma^2p^2(1+\cos\phi)^2 + p^2\sin^2\phi\right]\left[\gamma^2p^2(1-\cos\phi)^2 + p^2\sin^2\phi\right]}}$$

$$= \frac{(\gamma^2-1)\sin^2\phi}{\sqrt{\left[\gamma^2(1+\cos\phi)^2+\sin^2\phi\right]\left[\gamma^2(1-\cos\phi)^2+\sin^2\phi\right]}}$$

$$= \frac{(\gamma^2-1)}{\sqrt{\left[\gamma^2\left(\frac{1+\cos\phi}{\sin\phi}\right)^2+1\right]\left[\gamma^2\left(\frac{1-\cos\phi}{\sin\phi}\right)^2+1\right]}} = \frac{(\gamma^2-1)}{\sqrt{\left(\gamma^2\cot^2\frac{\phi}{2}+1\right)\left(\gamma^2\tan^2\frac{\phi}{2}+1\right)}}$$

$$\cos\theta = \frac{\omega}{\sqrt{\left(1+\cot^2\frac{\phi}{2}+\omega\cot^2\frac{\phi}{2}\right)\left(1+\tan^2\frac{\phi}{2}+\omega\tan^2\frac{\phi}{2}\right)}} \quad (\text{where } \omega \equiv \gamma^2-1)$$

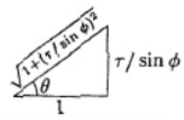
$$= \frac{\omega}{\sqrt{\left(\csc^2\frac{\phi}{2}+\omega\cot^2\frac{\phi}{2}\right)\left(\sec^2\frac{\phi}{2}+\omega\tan^2\frac{\phi}{2}\right)}} = \frac{\omega\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{\sqrt{\left(1+\omega\cos^2\frac{\phi}{2}\right)\left(1+\omega\sin^2\frac{\phi}{2}\right)}}$$

$$= \frac{\frac{1}{2}\omega\sin\phi}{\sqrt{\left[1+\frac{1}{2}\omega(1+\cos\phi)\right]\left[1+\frac{1}{2}\omega(1-\cos\phi)\right]}} = \frac{\sin\phi}{\sqrt{\left[\left(\frac{2}{\omega}+1\right)+\cos\phi\right]\left[\left(\frac{2}{\omega}+1\right)-\cos\phi\right]}}$$

$$= \frac{\sin\phi}{\sqrt{\left(\frac{2}{\omega}+1\right)^2-\cos^2\phi}} = \frac{\sin\phi}{\sqrt{\frac{4}{\omega^2}+\frac{4}{\omega}+\sin^2\phi}} = \frac{1}{\sqrt{1+(\tau/\sin\phi)^2}}, \text{ where } \tau^2 = \frac{4}{\omega^2}+\frac{4}{\omega}.$$

$\sin\theta = \frac{\tau}{\sin\phi}$. $\tau^2 = \frac{4}{\omega^2}(1+\omega) = \frac{4}{(\gamma^2-1)^2}\gamma^2$, so $\tan\theta = \frac{2\gamma}{(\gamma^2-1)\sin\phi}$.

Or, since $(\gamma^2-1) = \gamma^2\left(1-\frac{1}{\gamma^2}\right) = \gamma^2\frac{v^2}{c^2}$, $\boxed{\tan\theta = \frac{2c^2}{\gamma v^2\sin\phi}}$.



12.64

(a) $A^\mu = (V/c, A_x, A_y, A_z)$ is a 4-vector (like $x^\mu = (ct, x, y, z)$), so (using Eq. 12.19): $V = \gamma(\bar{V} + v\bar{A}_x)$. But $\bar{V} = 0$, and

$$\bar{A}_x = \frac{\mu_0}{4\pi}\frac{(\mathbf{m}\times\bar{\mathbf{r}})_x}{\bar{r}^3}.$$

Now $(\mathbf{m}\times\bar{\mathbf{r}})_x = m_y\bar{z} - m_z\bar{y} = m_y z - m_z y$. So

$$V = \gamma v\frac{\mu_0}{4\pi}\frac{(m_y z - m_z y)}{\bar{r}^3}.$$

Now $\bar{x} = \gamma(x - vt) = \gamma R_x$, $\bar{y} = y = R_y$, $\bar{z} = z = R_z$, where $\mathbf{R}$ is the vector (in $\mathcal{S}$) from the (instantaneous) location of the dipole to the point of observation. Thus

$$\bar{r}^2 = \gamma^2 R_x^2 + R_y^2 + R_z^2 = \gamma^2(R_x^2 + R_y^2 + R_z^2) + (1-\gamma^2)(R_y^2 + R_z^2) = \gamma^2\left(R^2 - \frac{v^2}{c^2}R^2\sin^2\theta\right)$$

(where θ is the angle between $\mathbf{R}$ and the x axis, so that $R_y^2 + R_z^2 = R^2\sin^2\theta$).

$$\therefore V = \frac{\mu_0}{4\pi}\frac{v\gamma(m_y R_z - m_z R_y)}{\gamma^3 R^3\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}};\quad \text{but } \mathbf{v}\cdot(\mathbf{m}\times\mathbf{R}) = v(\mathbf{m}\times\mathbf{R})_x = v(m_y R_z - m_z R_y),\quad \text{so}$$

$$\boxed{V = \frac{\mu_0}{4\pi}\frac{\mathbf{v}\cdot(\mathbf{m}\times\mathbf{R})\left(1 - \frac{v^2}{c^2}\right)}{R^3\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}},}$$

or, using $\mu_0 = \frac{1}{\epsilon_0 c^2}$ and $\mathbf{v}\cdot(\mathbf{m}\times\mathbf{R}) = \mathbf{R}\cdot(\mathbf{v}\times\mathbf{m})$: $V = \dfrac{1}{4\pi\epsilon_0}\dfrac{\hat{\mathbf{R}}\cdot(\mathbf{v}\times\mathbf{m})\left(1 - \frac{v^2}{c^2}\right)}{c^2R^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}$.

(b) In the nonrelativistic limit ($v^2 \ll c^2$):

$$V = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{R}}\cdot(\mathbf{v}\times\mathbf{m})}{c^2R^2} = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{R}}\cdot\mathbf{p}}{R^2},\quad \text{with } \mathbf{p} = \frac{\mathbf{v}\times\mathbf{m}}{c^2},$$

which is the potential of an *electric* dipole.