All Possible Symmetries of the S Matrix*

SIDNEY COLEMAN† AND JEFFREY MANDULA‡
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts
(Received 16 March 1967)

We prove a new theorem on the impossibility of combining space-time and internal symmetries in any but a trivial way. The theorem is an improvement on known results in that it is applicable to infinite-parameter groups, instead of just to Lie groups. This improvement is gained by using information about the S matrix; previous investigations used only information about the single-particle spectrum. We define a symmetry group of the S matrix as a group of unitary operators which turn one-particle states into one-particle states, transform many-particle states as if they were tensor products, and commute with the S matrix. Let G be a connected symmetry group of the S matrix, and let the following five conditions hold: (1) G contains a subgroup locally isomorphic to the Poincaré group. (2) For any $M>0$, there are only a finite number of one-particle states with mass less than M. (3) Elastic scattering amplitudes are analytic functions of s and t, in some neighborhood of the physical region. (4) The S matrix is nontrivial in the sense that any two one-particle momentum eigenstates scatter (into something), except perhaps at isolated values of s. (5) The generators of G, written as integral operators in momentum space, have distributions for their kernels. Then, we show that G is necessarily locally isomorphic to the direct product of an internal symmetry group and the Poincaré group.

I. INTRODUCTION

UNTIL a few years ago, most physicists believed that the exact or approximate symmetry groups of the world were (locally) isomorphic to direct products of the Poincaré group and compact Lie groups. This world-view changed drastically with the publication of the first papers on $SU(6)$[1]; these raised the dazzling possibility of a relativistic symmetry group which was not simply such a direct product. Unfortunately, all attempts to find such a group came to disastrous ends, and the situation was finally settled by the discovery of a set of theorems[2] which showed that, for a wide class of Lie groups, any group which contained the Poincaré group and admitted supermultiplets containing finite numbers of particles was necessarily a direct product.

However, although these theorems served their polemic purposes, they are in many ways displeasing: Their statements involve many unnatural and artificial assumptions, typically concerning the normality of the translation subgroup. Even worse, they are restricted to Lie groups—this despite the fact that infinite-parameter groups have been proposed in the literature. The theories based on these groups were destroyed not by general theorems but by particular arguments. Typically, these arguments showed that these groups do not allow scattering except in the forward and backward directions.[3] Thus, if one accepts the usual dogma on the analyticity of scattering amplitudes, they allow no scattering at all.

The purpose of this paper is to tie up these loose ends. We prove the following theorem: Let G be a connected symmetry group of the S matrix, which contains the Poincaré group and which puts a finite number of particles in a supermultiplet. Let the S matrix be nontrivial and let elastic scattering amplitudes be analytic functions of s and t in some neighborhood of the physical region. Finally, let the generators of G be representable as integral operators in momentum space, with kernels that are distributions. Then G is locally isomorphic to the direct product of the Poincaré group and an internal symmetry group. (This is a loose statement of the theorem; a more precise one follows below.)

We believe that all of the assumptions in this theorem are physical, except for the last one, which, although weak, is ugly. We hope that it can be eliminated with sufficiently careful analysis; to date we have been unable to do so.

We emphasize that our theorem has application only to groups which are symmetries of the S matrix. Therefore it has nothing to say about symmetry groups arising from the saturation of current commutators; these groups generate symmetries of form factors only.

The remainder of this section contains a precise statement of the theorem and some remarks about its implications. Section II contains the proof. Although at times this attains mathematical levels of obscurity, we make no claim for corresponding standards of rigor.

A. Statement of the Theorem

We begin by briefly reviewing some of the fundamental definitions of scattering theory. The Hilbert space of scattering theory, $\mathcal{H}$, is an infinite direct sum of subspaces,

$$\mathcal{H}=\mathcal{H}^{(1)}\oplus\mathcal{H}^{(2)}\oplus\cdots. \qquad (1)$$

$\mathcal{H}^{(n)}$ is called the n-particle subspace. It is a subspace (determined by the generalized exclusion principle) of the direct product of n Hilbert spaces, each isomorphic to $\mathcal{H}^{(1)}$. The S matrix S is a unitary operator on $\mathcal{H}$. A unitary operator U on $\mathcal{H}$ is said to be a *symmetry*

* Work supported in part by the U. S. Air Force Office of Scientific Research under Contract AF49(638)–1380.
† Alfred P. Sloan Research Fellow.
‡ National Science Foundation Postdoctoral Fellow.
[1] B. Sakita, Phys. Rev. **136**, B1756 (1964); F. Gürsey and L. Radicati, Phys. Rev. Letters **13**, 299 (1964).
[2] S. Coleman, Phys. Rev. **138**, B1262 (1965); S. Weinberg, *ibid.* **139**, B597 (1965); L. Michel and B. Sakita, Ann. Inst. Henri Poincaré **2**, 167 (1965).
[3] T. F. Jordan, Phys. Rev. **140**, B766 (1965).

transformation (of the S matrix) if: (1) U turns one-particle states into one-particle states; (2) U acts on many-particle states as if they were tensor products of one-particle states; and (3) U commutes with S. A group of such transformations is said to be a *symmetry group* (of S). S is said to be Lorentz-invariant if it possesses a symmetry group locally isomorphic to the Poincaré group P. In this case we may introduce a basis for $\mathcal{H}^{(1)}$, consisting of the plane-wave states $|\alpha,\lambda,p\rangle$, where p is the four-momentum of the state, λ is the spin index, and α labels the particle type (technically, the irreducible subspaces of $\mathcal{H}^{(1)}$ under the action of P).[4] The number of irreducible representations of P contained in $\mathcal{H}^{(1)}$ is called the number of particle types.

A symmetry transformation is said to be an *internal symmetry transformation* if it commutes with P. This implies that it acts only on particle-type indices, and has no matrix elements between particles of different four-momentum or different spin. A group composed of such transformations is called an *internal symmetry group*.

The T matrix is defined in the usual way:

$$S=1-i(2\pi)^4\delta(P_\mu-P_\mu{}')T. \tag{2}$$

Equation (2), of course, defines T only between states of the same four-momentum. The scattering amplitudes are the matrix elements of T.

It will be convenient for some of our subsequent arguments to define a subset of $\mathcal{H}^{(1)}$, called $\mathfrak{D}$. $\mathfrak{D}$ is the set of all single-particle states whose momentum-space wave functions are test functions—that is to say, infinitely differentiable functions with compact support.

We are now in a position to state the main result of our investigation:

Theorem: Let G be a connected[5] symmetry group of the S matrix, and let the following five conditions hold:

1. (Lorentz invariance.) G contains a subgroup locally isomorphic to P.
2. (Particle-finiteness.) All particle types correspond to positive-energy representations of P. For any finite M, there are only a finite number of particle types with mass less than M.
3. (Weak elastic analyticity.) Elastic-scattering amplitudes are analytic functions of center-of-mass energy, s, and invariant momentum transfer, t, in some neighborhood of the physical region, except at normal thresholds.
4. (Occurrence of scattering.) Let $|p\rangle$ and $|p'\rangle$ be any two one-particle momentum eigenstates, and let $|p,p'\rangle$ be the two-particle state made from these. Then

$$T|p,p'\rangle\neq 0, \tag{3}$$

except perhaps for certain isolated values of s. Phrased briefly, at almost all energies, any two plane waves scatter.

5. (An ugly technical assumption.) The generators of G, considered as integral operators in momentum space, have distributions for their kernels. More precisely: There is a neighborhood of the identity in G such that every element of G in this neighborhood lies on some one-parameter group $g(t)$. Further if x and y are any two states in $\mathfrak{D}$, then

$$\frac{1}{i}\frac{d}{dt}(x,g(t)y)=(x,Ay), \tag{4}$$

exists at $t=0$, and defines a continuous[6] function of x and y, linear in y and antilinear in x.[7]

Then, G is locally isomorphic to the direct product of an internal symmetry group and the Poincaré group.

B. Remarks

1. Note that we do not assume that G is a finite-parameter group. Note also that the theorem possesses a trivial corollary a result reminiscent of the famous theorem of O'Raifeartaigh[8]: All the particles in a G supermultiplet have the same mass. This is surprising, for it is known that there exist infinite-parameter groups which do not obey O'Raifeartaigh's theorem.[9] We are able to reject these groups because our assumptions are stronger than those of O'Raifeartaigh; our analysis shows that these groups can not be symmetry group of a nontrivial S matrix.

2. Lorentz invariance is a necessary condition for there are many examples of Galilean-invariant spin-independent theories,[1,10] for which the corresponding theorem does not hold.

3. Particle-finiteness is also necessary, in order to exclude the well-known infinite-supermultiplet theories, most intensively investigated by Fronsdal and collaborators.[11]

4. The analyticity assumption is somewhat surprising in this group-theoretical context. However, it is something that most physicists believe to be a property of the real world, and we will use it continually in our proof. If it is eliminated completely, the theorem is not true; groups are known which are not direct products, but which do allow scattering, although only in the forward and backward directions.[3]

[4] Spin bases for different moment are connected by the usual conventions of E. P. Wigner, Ann. Math. **40**, 149 (1939).

[5] "Connected" means arcwise connected in the weak operator topology.

[6] "Continuous" means continuous in the usual (Schwartz) topology for test functions.

[7] Alternatively, A may be thought of as a linear function from $\mathfrak{D}$ to its dual space.

[8] L. O'Raifeartaigh, Phys. Rev. Letters **14**, 575 (1965); Phys. Rev. **139**, B1052 (1965); R. Jost, Helv. Phys. Acta **39**, 369 (1966); I. Segal, J. Functional Anal. (to be published).

[9] M. Nakamura, Progr. Theoret. Phys. (Kyoto) **37**, 195 (1967).

[10] E. P. Wigner, Phys. Rev. **51**, 106 (1937).

[11] A survey is contained in the contribution of P. Budini and C. Fronsdal, *High Energy Physics and Elementary Particles* (International Atomic Energy Agency, Vienna, 1966).

5. Some form of nontriviality is certainly required, for if $S=1$, the group of all unitary transformations on $\mathcal{H}^{(1)}$ satisfies all the other conditions of the theorem. However, assumption 4 is much stronger than $S\neq 1$. We believe it is possible to derive our assumption from much weaker assumptions, essentially equivalent to $S\neq 1$, with the aid of somewhat stronger analyticity conditions (in particular, the crossing relations); but we have not yet completed this investigation.

6. Our fifth assumption is, as we have said, both technical and ugly. It is necessary for our proof because we are physicists, and accustomed to the manipulation of infinitesimal generators. Since there is, in general, no analytic-vector theorem for infinite-parameter groups, a special assumption is needed to justify such manipulations. We feel that the assumption we have chosen is a weak one; for example, it does not imply that $\mathfrak{D}$ is in the domain of all the group generators. However, we have no doubt that more competent analysts will be able to weaken it further, and perhaps even eliminate it altogether.

7. It has sometimes been suggested that nonunitary bounded linear operators which commute with the S matrix might be of physical interest.[12] These operators, of course, would not represent symmetries in the usual sense, but they still might restrict scattering amplitudes in interesting ways. The proof for the unitary case also works here, with the addition of a few minor supplementary arguments. For the reader interested in this case, we give these arguments, where they are required, as footnotes.

8. The theorem is not true if, in the conclusion, local isomorphism is replaced by global isomorphism. For example, it is possible to construct theories in which only particles of half-integral spin have half-integral hypercharge. The symmetry groups of these theories are locally, but not globally, isomorphic to $SU(2)\otimes P$.

9. Finally, if there are only a finite number of particle types, the theorem implies that the internal symmetry group has compact closure. However, if there are an infinite number of particle types, spread out along the mass spectrum, this need not be so. For example, every particle type could have associated with it an independent particle-number convervation law. The internal symmetry group would then be a direct product of an infinite number of factors, each isomorphic to $U(1)$. This is not a compact group.

II. PROOF OF THE THEOREM

We begin with some trivial remarks about the generators of G, defined by Eq. (4). We may readily extend our definition of the generators to two-particle states; indeed, if $x_1\otimes x_2$ and $y_1\otimes y_2$ are two-particle states in $\mathfrak{D}\otimes\mathfrak{D}$, it is easy to show that

$$\frac{1}{i}\frac{d}{dt}(y_1\otimes y_2,\, g(t)[x_1\otimes x_2])=(y_1\otimes y_2,\, A[x_1\otimes x_2])$$
$$=(y_1,Ax_1)(y_2,x_2)+(y_1,x_1)(y_2,Ax_2)\,. \quad (5a)$$

The fact that G commutes with S implies that[13]

$$(S[y_1\otimes y_2],AS[x_1\otimes x_2])=(y_1\otimes y_2,A[x_1\otimes x_2])\,. \quad (5b)$$

Lorentz invariance implies that if A is any distribution obeying Eqs. (5), then so is $U(\Lambda,a)^+AU(\Lambda,a)$ where $U(\Lambda,a)$ is, as usual, the unitary operator representing the element (Λ,a) of P. Likewise, any convergent sum or integral of distributions obeying (5) will also obey (5). We denote by $\mathfrak{A}$ the set of all distributions obeying Eq. (5).[14]

As any reader of Dirac knows, it is sometimes convenient to speak of a distribution as if it were a function. We will follow this practice, and define

$$A_{\alpha'\lambda'\alpha\lambda}(p',p)=\langle\alpha'\lambda'p'|A|\alpha\lambda p\rangle\,, \quad (6)$$

a distribution in momentum space. Sometimes we will suppress the indices, and speak of $\mathbf{A}(p',p)$, a matrix-valued distribution.

We are now ready for the first stage of the proof. Let f be a test function, with support in a region R in p space, and let $\tilde{f}$ be the Fourier transform of f. It is easy to show that the integral

$$\int d^4a\; U^+(1,a)AU(1,a)\tilde{f}(a)=f\cdot A \quad (7)$$

converges, and defines a distribution in $\mathfrak{A}$. Since

$$U(1,a)|\alpha,\lambda,p\rangle=e^{-ip\cdot a}|\alpha,\lambda,p\rangle\,, \quad (8)$$

then

$$f\cdot\mathbf{A}(p',p)=f(p-p')\mathbf{A}(p',p)\,. \quad (9)$$

Thus $f\cdot A$ only has matrix elements between states whose supports in p space are such that they may be connected by a vector in R.

Now, by our particle-finiteness assumption, the momentum support of one-particle states is restricted to a countable set of mass hyperboloids. It requires only trivial algebra (which we leave to the reader) to show that if R is sufficiently small, and does not contain the zero vector, there will be regions on these hyperboloids such that, if any vector in R is added to any vector in these regions, a vector is obtained which is on none of the hyperboloids. Thus, any state in $\mathfrak{D}$ whose support lies within these regions must be annihilated by $f\cdot A$.

[12] S. Coleman (Ref. 2); H. Lipkin, Phys. Rev. **139**, B1633 (1965). It has also been suggested that invariance under an algebra, with the representation of the algebra such that it can not be exponentiated to form a representation of the corresponding group, might be of physical interest. [M. Flato and D. Sternheimer, Phys. Rev. Letters **15**, 934 (1965); **16**, 1185 (1966).] Our theorem also holds in this case, with the obvious recasting of our assumptions into purely algebraic language.

[13] The analyticity of the scattering amplitude guarantees that the left-hand side of Eq. (5b) is well defined.

[14] Note that we do not assert that $\mathfrak{A}$ is a Lie algebra, nor that every element of $\mathfrak{A}$ is obtained by differentiating a one-parameter group.

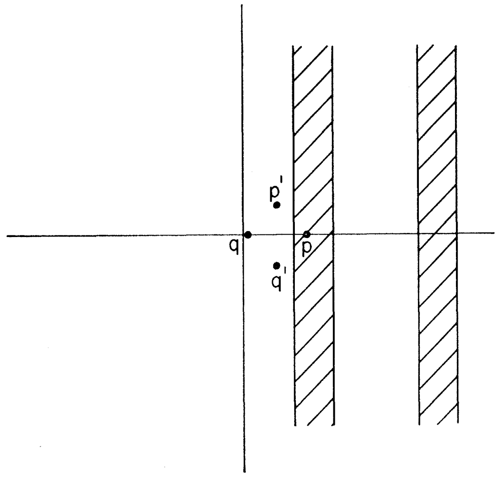

FIG. 1. The x-y plane in p space for particles of lowest mass. The operator $f\cdot A$ annihilates all states whose momentum supports are outside the shaded bands.

Figure 1 depicts the hyperboloid of lowest mass for a particular situation. R has been chosen to be a small sphere surrounding a vector lying in the x direction. One-particle states whose supports lie outside the shaded bands are annihilated by $f\cdot A$. There is one band for every mass hyperboloid. For the sake of clarity, we will construct all our subsequent arguments for this situation; they may easily be extended to the general case.

Let x be a state in $\mathcal{H}^{(1)}$ orthogonal to all states annihilated by $f\cdot A$. Let p be a momentum in the support of x. By the arguments above, p must lie in one of the shaded bands. Let q, p', q', be points as shown, chosen such that

$$p+q=p'+q'. \tag{10}$$

Let us define, as usual

$$s=(p+q)^2 \quad \text{and} \quad t=(p-p')^2, \tag{11}$$

and let s be chosen so that it lies below the threshold for production of pairs of the next-heaviest particle.

Now, let us construct an initial two-particle state by putting one particle in the state x and another particle in an arbitrary state with momentum support localized about q. Likewise, let us construct a final two-particle state by putting one particle in an arbitrary state with support localized about p' and the other in an arbitrary state with support localized about q'. The S-matrix element between these two states must be zero, because $f\cdot A$ obeys Eqs. (5). This implies that the elastic scattering amplitude vanishes for s and t as defined by Eq. (11), if the particle with momentum p is in a state determined by the wave function of x evaluated at p. However, by making a rotation in the rest frame of p—that is to say, by transforming q, q', and p' in accordance with such a rotation—and realizing that the whole argument goes through without change in this case, we can eliminate this last restriction, and deduce that for at least one particle type α, $\alpha\beta$ scattering is zero at these values of s and t for any particle β.

But now we can choose a slightly different configuration, and change the values of s and t continuously. Thus we deduce that scattering vanishes for a range of s and t. By analyticity, this means that $\alpha\beta$ scattering vanishes everywhere. This is in contradiction with the nontriviality of the S matrix. Thus, $x=0$, and $f\cdot A$ annihilates all states with support on the lowest mass hyperboloid. Now we can go on, inductively, to the higher-mass hyperboloids. By simple repetition of the arguments above, we finally conclude that

$$f\cdot A=0. \tag{12}$$

Since f is an arbitrary test function whose support does not include the zero vector, this implies:

Lemma 1: The support of $\mathbf{A}(p',p)$ is restricted to the set $p=p'$.

In particular, this lemma implies that A can not connect states on different mass hyperboloids; therefore, neither can G. We thus have a generalization of O'Raifeartaigh's theorem[15] for symmetry groups of the S matrix.

It is a well-known result of distribution theory that a distribution whose support is a point is a finite sum of derivatives of δ-functions. In our case, this means that on each mass hyperboloid, A may be written as a differential operator. Of course, since A acts only on functions with support on the hyperboloid, this differential operator does not involve differentiation with respect to all four components of p, but with respect to the three components tangent to the hyperboloid. That is to say, on each mass hyperboloid, A is a polynomial in the tangential differential operator:

$$\nabla_\mu=\partial/\partial p_\mu - m^{-2}p_\mu p_\nu \partial/\partial p_\nu. \tag{13}$$

(We adopt the summation convention for Greek indices.) The next lemma is a restatement of these trivial consequences of lemma 1 in terms of equations.

Lemma 2: Any element of $\mathfrak{A}$ may be written in the form

$$A=\sum_{n=0}^{N} \mathbf{A}^{(n)}(p)_{\mu_1\cdots\mu_n}\frac{\partial}{\partial p_{\mu_1}}\cdots\frac{\partial}{\partial p_{\mu_n}}, \tag{14}$$

where the $\mathbf{A}^{(n)}$ are matrix-valued distributions. Also,

[15] In this connection, it is worth noting that Lemma 1 holds if the particle-finiteness condition is replaced by the following (much weaker) condition: For any $M>0$ there is an $m>M$, such that m is not a limit point of the one-particle mass spectrum. Thus, even if we allow an infinite number of particles with a given mass, G still cannot connect particles of different mass, unless the one-particle mass spectrum contains all sufficiently large masses. A closely related result has been obtained by H. J. Bernstein and F. Halpern [Phys. Rev. **151**, 1226 (1966)] under the assumptions essentially equivalent to ours, plus the additional assumption that the generators of the group commute, on $\mathcal{H}^{(1)}$, with the velocity operators $\mathbf{P}/E$. One of us (S.C.) would like to thank Dr. Bernstein for a discussion of this work. See also J. Formánek, Czech. J. Phys. **B17**, 99 (1967).

since A is a polynomial in ∇_μ,

$$[A,p^\mu p_\mu]=0\,, \tag{15}$$

acting on any state in $\mathfrak{D}$.

We now lay aside this line of argument and initiate an independent one. We will not return to Lemma 2 until almost the end of the proof. We begin with a preliminary lemma, and a volley of definitions.

Assumption 4 states that Eq. (3) is valid except for certain isolated values of s. We call pairs of momenta corresponding to such values *null pairs*.

Lemma 3: Let $|p\rangle$ and $|q\rangle$ be any two single-particle momentum eigenstates, and let $|p,q\rangle$ be the two-particle state made from these. Let $U(\theta)$ be the unitary transformation corresponding to space rotations by an angle θ about an arbitrary axis in the rest frame of the two-particle system. Then if (p,q) is not a null pair, there exists a δ such that

$$\langle p,q|U(\theta)T|p,q\rangle\neq 0\,, \tag{16}$$

for $\theta<\delta$.

For, by unitary and Eq. (3), the left-hand side of Eq. (16) is not zero for $\theta=0$, and, by the analyticity assumption, it is a continuous function of θ.

Let $\mathfrak{z}$ be the subset of $\mathfrak{A}$ consisting of all Hermitian distributions which commute with the space-time translations.[16] If B is any element of $\mathfrak{z}$, it may be thought of as an operator that multiplies the wave function of any state in $\mathfrak{D}$ by a matrix valued Hermitian distribution $\mathbf{B}(p)$. Let $\mathfrak{z}^S$ be the subset of $\mathfrak{z}$ consisting of all elements for which $\mathbf{B}(p)$ is an infinitely-differentiable function. Note that if B is in $\mathfrak{z}^S$ it makes sense to think of B as an operator acting on momentum eigenstates, a procedure which we shall adopt.

We will now show that any element of $\mathfrak{z}$ is the sum of an infinitesimal translation and an infinitesimal internal symmetry transformation (as defined in Sec. I). The argument proceeds in three stages: In Lemmas 4, 5, and 6 we show that the traceless part of any element of $\mathfrak{z}^S$ is an infinitesimal internal symmetry transformation. In Lemma 7 we show that the trace of any such element is the sum of an infinitesimal translation and an infinitesimal internal symmetry transformation. Finally, in Lemma 8 we extend our results from $\mathfrak{z}^S$ to $\mathfrak{z}$.

If B is any element of $\mathfrak{z}$, and $f(\Lambda)$ is any test function on the homogeneous Lorentz group, then

$$B^f=\int d\Lambda f(\Lambda)U(\Lambda,0)BU(\Lambda,0)^+\,, \tag{17}$$

where $d\Lambda$ denotes the invariant integral on the homogeneous Lorentz group, is in $\mathfrak{z}^S$.

On any given mass hyperboloid, we may divide $\mathbf{B}(p)$ into a traceless part and a multiple of the identity matrix. We call the traceless part $\mathbf{B}^*(p)$. For any p, we define $K(p)$ as the set of all B's in $\mathfrak{z}^S$ for which

$$\mathbf{B}^*(p)=0\,. \tag{18}$$

All of this is for one-particle states. We may use Eq. (5) to define B on two-particle states; in the obvious notation,

$$\mathbf{B}(p,q)=[\mathbf{B}(p)\otimes\mathbf{I}]\oplus[\mathbf{I}\otimes\mathbf{B}(q)]\,. \tag{19}$$

We define $K(p,q)$ as the set of all B's in $\mathfrak{z}^S$ for which

$$\mathbf{B}^*(p,q)=0\,. \tag{20}$$

It follows from Eq. (19) that

$$K(p,q)=K(p)\cap K(q)\,. \tag{21}$$

Now, let (p,q) be any non-null pair, and let J be the generator of space rotations about any axis in the (p,q) center-of-mass frame. Assume there is a B in $K(p,q)$ such that $[B,J]$ is not in $K(p,q)$. Then

$$\mathbf{B}^*(p',q')\neq 0\,, \tag{22}$$

for any (p',q') obtained by a sufficiently small rotation from (p,q). Equation (22) implies that $\mathbf{B}(p',q')$ has at least two distinct eigenvalues. At least one of these is not equal to the single eigenvalue of $\mathbf{B}(p,q)$. Since B commutes with S, the eigenvector corresponding to this eigenvalue must have zero amplitude to scatter to any two-particle state with momentum (p,q). But this contradicts lemma 3; therefore $[B,J]$ must be in $K(p,q)$. That is to say, $K(p,q)$ is invariant under rotations in the (p,q) center-of-mass frame; this is equivalent to:

Lemma 4: $K(p,q)$ is a function only of $p+q$.

Let (p,q) and (p',q') be any two non-null pairs on the same hyperboloid such that

$$p+q=p'+q'\,. \tag{23}$$

By Lemma 4,

$$K(p,q)=K(p',q')\,. \tag{24}$$

Therefore, by Eq. (21),

$$K(p)\supset K(p,q)\,, \tag{25a}$$

and

$$K(p')\supset K(p',q')=K(p,q)\,. \tag{25b}$$

Thus,

$$K(p,p')\supset K(p,q)\,. \tag{26}$$

But $p+p'\neq p+q$. Thus we have passed from one value of the total momentum to another. It is easy to see that by iterating this procedure we can show that

$$K(k)\supset K(p,q)\,, \tag{27}$$

where k is any momentum on the hyperboloid. Thus we have established:

Lemma 5: Let (p,q) be any non-null pair on any mass hyperboloid. Let B be any element of $\mathfrak{z}^S$. Then if

$$\mathbf{B}^*(p,q)=0\,, \tag{28}$$

$\mathbf{B}^*$ vanishes on the entire hyperboloid.

The traceless parts of all B's in $\mathfrak{z}^S$, restricted to a given mass hyperboloid, form an algebra closed under

[16] If G is not assumed to be unitary, we define $\mathfrak{z}'$ to consist of the Hermitian and anti-Hermitian parts of all the elements of $\mathfrak{A}$ which commute with space-time translations. $\mathfrak{z}'$ is evidently a subset of $\mathfrak{z}$. The arguments above should then be taken as applying to $\mathfrak{z}'$ rather than $\mathfrak{z}$, until Lemma 8.

commutation, and with the Lorentz transformations as a group of automorphisms. Let us call this algebra $\mathfrak{B}$. If we pick a non-null pair (p,q), there is a natural homomorphism of $\mathfrak{B}$ into the algebra of $SU(n)\otimes SU(n)$, where n is the number of states with momentum p. Lemma 5 asserts that this homomorphism is an isomorphism. Thus $\mathfrak{B}$ is isomorphic to a subalgebra of a compact Lie algebra; therefore it is a direct sum of a compact semisimple Lie algebra and an Abelian Lie algebra.

The elements of the semisimple Lie algebra must commute with Lorentz transformations, since the connected part of the group of automorphisms of a compact semisimple Lie algebra is known to be the corresponding compact semisimple group, and the only homomorphism of the Lorentz group into such a group is the trivial one. Hence the semisimple algebra is composed of infinitesimal internal symmetry transformations.

The Abelian algebra requires a special argument. Let us go to a frame in which the space parts of p and q are aligned along the z axis. The $SO(2)$ group of rotations about this axis acts as automorphisms on the Abelian algebra. Thus we may write every element of this algebra as a linear combination of elements which transform as irreducible representations of $SO(2)$. It is easy to see that if there is an element which transforms nontrivially (i.e., changes J_z), it can not commute with its adjoint, in contradiction to the Abelian nature of the algebra. Thus, every element of $\mathfrak{B}$ commutes with this particular $O(2)$ subgroup of the homogeneous Lorentz group. But we can do the same argument for any non-null pair; thus every element of $\mathfrak{B}$ commutes with every element of the Lorentz group, i.e., is an infinitesimal internal symmetry transformation on the hyperboloid in question.

The above argument holds for every hyperboloid; thus we obtain:

Lemma 6: If B is in $\mathfrak{Z}^S$, $\mathbf{B}^*(p)$ is an infinitesimal internal symmetry transformation. [That is to say, $\mathbf{B}^*(p)$ does not depend on p and commutes with rotations.]

All that remains is to analyze $\mathrm{Tr}\mathbf{B}$. If, armed with Lemma 6, we return to the same hypothetical scattering experiment we used to establish Lemma 4, it is easy to see that the same arguments used there imply that if

$$p+q=p'+q', \tag{23}$$

then

$$\mathrm{Tr}\mathbf{B}(p)+\mathrm{Tr}\mathbf{B}(q)=\mathrm{Tr}\mathbf{B}(p')+\mathrm{Tr}\mathbf{B}(q'). \tag{29}$$

But this is only possible if $\mathrm{Tr}\mathbf{B}(p)$ is a linear function of p. Thus we obtain:

Lemma 7: For any B in $\mathfrak{Z}^S$, $\mathrm{Tr}\mathbf{B}(p)$ is a linear function of p.

Lemmas 6 and 7 have been established for elements of $\mathfrak{Z}^S$. However, for any B in $\mathfrak{Z}$, B^f, defined by Eq. (17), is in $\mathfrak{Z}^S$. Thus B^f must obey the lemmas, for any f. Hence B must obey the lemmas. Thus we obtain:

Lemma 8: Any element of $\mathfrak{Z}$ is of the form

$$\mathbf{B}(p)=a_\mu p^\mu+\mathbf{b}, \tag{30}$$

where a_μ is a constant four-vector and b is a constant Hermitian matrix that does not involve spin indices.[17]

We are now in a position to prove the theorem. We will use only Lemmas 2 and 8. Let us commute A, given by Eq. (14), with p_μ, N times:

$$[p_{\mu_1},[p_{\mu_2}\cdots A]\cdots]=\mathbf{A}^{(N)}{}_{\mu_1\cdots\mu_N}(p). \tag{31}$$

The right-hand side of this equation is in $\mathfrak{A}$, therefore in $\mathfrak{Z}$. Hence, by Lemma 8,

$$\mathbf{A}^{(N)}{}_{\mu_1\cdots\mu_N}=a_{\sigma\mu_1\cdots\mu_N}p^\sigma+b_{\mu_1\cdots\mu_N}. \tag{32}$$

Now let us commute A with $N-1$ p_μ's and one $p_\mu p^\mu$. By Eq. (15), this must give zero:

$$\begin{aligned}[p_\mu p^\mu,[p_{\mu_2}\cdots A]\cdots]&=a_{\sigma\mu_1\cdots\mu_N}p^\sigma p^{\mu_1}\\&\quad+b_{\mu_1\cdots\mu_N}p^{\mu_1}\\&=0,\end{aligned} \tag{33}$$

for every p on the mass hyperboloid. This implies

$$b_{\mu_1\cdots\mu_N}=0, \tag{34}$$

unless $N=0$, in which case we can not do the required commutation. Equation (33) also implies that

$$a_{\sigma\mu_1\cdots\mu_N}=-a_{\mu_1\sigma\cdots\mu_N}, \tag{35}$$

except again for $N=0$. However, for, $N>1$, Eq. (35) is inconsistent with the symmetry of a under the interchange of the μ's.

Therefore, N is always either zero or one. In the latter case,

$$\mathbf{A}_\mu{}^{(1)}\frac{\partial}{\partial p_\mu}=a_{\sigma\mu}p^\sigma\frac{\partial}{\partial p_\mu}\mathbf{I}, \tag{36}$$

where a is antisymmetric. But this is just the space part of an infinitesimal Lorentz transformation. Let us call this Lorentz transformation M. Then $A-M$ is in $\mathfrak{Z}$, and Lemma 8 applies again. Thus we find:

Lemma 9: Any A in $\mathfrak{A}$ is the sum of an infinitesimal Lorentz transformation, an infinitesimal translation, and an infinitesimal internal symmetry transformation.

Lemma 9 is just the infinitesimal form of our theorem. This completes the proof.

ACKNOWLEDGMENTS

One of us (S.C.) would like to thank the Theory Division of CERN and the Istanbul Summer School on High Energy Physics[18] for their hospitality. We would also like to thank Professor Freeman Dyson for orthographical advice.

[17] If G is not assumed to be unitary, this follows because every element of $\mathfrak{Z}$ is the sum of two elements of $\mathfrak{Z}'$. The matrix $\mathbf{b}$ need not be Hermitian, but a_μ must still be real; otherwise G would not consist of bounded operators.

[18] A discussion of a very rough preliminary version of the theorem proved here will appear in the *Proceedings of the Istanbul School* (W. H. Freeman and Company, San Francisco, to be published).