Large N: The Circus of Particles

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In this paper I present an introduction of the large N approximation method for certain models of quantum field theories. Naively, one expects that adding more types of particles will increase the complexity of a theory. However, under certain conditions, the opposite is true and one can even obtain nonperturbative results. I will first illustrate the formulation of the 1/N expansion by analyzing the structure of diagrammatic perturbation theory, and then discuss what new insights it may provide.

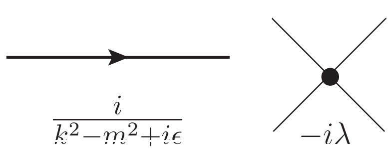
INTRODUCTION

The large N expansion is based on an observation that under certain conditions, a perturbation series is organized in such a way that it can be partially resummed to obtain nonperturbative results. Generally, in perturbation theory we start writing down Feynman diagrams and usually evaluate only diagrams which contain no more than two loops. This is partially due to the technical difficulty of evaluating higher order diagrams, but also because it is thought that the relevant physics should be apparent at the one or two loop level. Occasionally, however, it turns out that it is possible to go beyond this and actually sum an (infinite) part of the full expression. In these cases, we can obtain some qualitatively new results. Partial resummations are the basis of multiple approximation schemes, such as $1/N$ which we discuss here, or the RPA approximation in condensed matter.

First, I remind the reader of the Feynman rules for the ϕ^4 model

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m_0^2\phi^2 - \tfrac{\lambda}{4!}\phi^4 \tag{1}$$

The Feynman rules for this theory are as follows, where internal propagators are integrated over:



(a) Propagator (b) Vertex

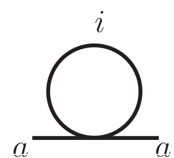
We know without doing any calculations that as we add vertices, the contributions of the diagrams become suppressed by higher powers of the small parameter λ (at least until the symmetry factor overwhelms the 'smallness' of λ^n). The situation can change when we introduce additional ϕ fields and are careful about our expansion parameter.

LARGE N FORMULATION FOR $(\phi^2)^2$

Now we consider the following model: let ϕ be a vector valued field whose components are distinct scalar fields ϕ^α with $\alpha = 1...N$. The Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi^\alpha\partial^\mu\phi^\alpha - \tfrac{1}{2}m_0^2\phi^\alpha\phi^\alpha - \tfrac{\lambda}{8}(\phi^\alpha\phi^\alpha)^2 \tag{2}$$

where the repeated index is summed over. This Lagrangian is invariant under an $O(N)$ symmetry which rotates the components of ϕ. We wish to investigate the limit of N becoming large, but before we do this, we will make two modifications. The first is that we rescale our coupling constant λ by N; let $g_0 \equiv \lambda N$, and consider the limit $N \to \infty$ as g_0 is held constant. The reason for this can be seen by considering the one loop correction to the propagator:



This diagram is proportion to λN because we need to sum over N possible field indices for the loop. Higher order corrections will go like even higher powers of N. Therefore, taking the $N \to \infty$ limit with λ fixed is not sensical. If instead we take the limit with g_0 fixed, the factor of N from the loop is cancelled. It turns out that in this case all corrections to the propagator will be $O(N^0)$ or lower, so an expansion in powers of $1/N$ is well defined and diagrams such as the one above are the only ones which contribute in the large N limit.

At second (and higher) order in g_0 there is a distinction between loops where we need to sum over the N possible particles running around, and those where the field indices are fixed by the external legs. We can make this distinction evident by decoupling the interaction using

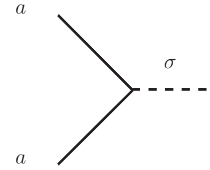
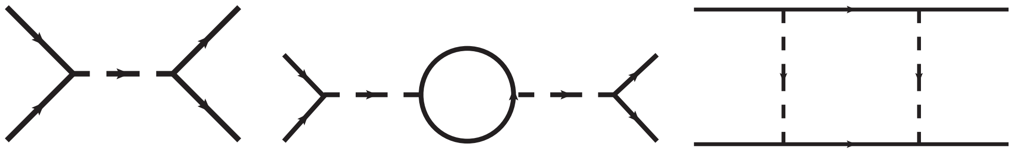


FIG. 2: The new propagator after decoupling the original interaction



(a) First order contribution. One σ propagator, no loops: $O(g_0/N)$

(b) One of the second order corrections. Two internal σ lines, one loop: $O(g_o^2/N)$

(c) The other second order correction. We have two internal σ lines, but no loops, so this diagram is $O(g_0^2/N^2)$

FIG. 3: The first order and two second order contributions to the four point function.

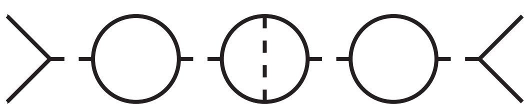
an auxiliary field. We introduce a Hubbard-Stratonovich field $\sigma(x)$ so that the Lagrangian becomes

$$\mathcal{L}' = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m_0^2\phi^2 + \frac{1}{2}\underbrace{\frac{N}{g_0}}_{\lambda^{-1}}\sigma^2 - \tfrac{1}{2}\sigma\phi^2 \tag{3}$$

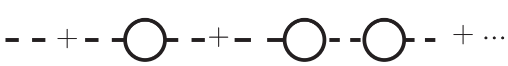
where $\phi^2 = \phi^a\phi^a$. We have traded a $(\phi^2)^2$ interaction for a $\sigma\phi^2$ interaction and an extra field to integrate over. The benefit of doing this is that all of the power counting is now contained in the σ propagator $G_\sigma = ig_0/N$, denoted by a dotted line, and the number of ϕ loops, which are each proportional to N. Now it becomes simpler to figure out what the leading contributions to certain processes are; all we do is connect the lines, and count the number of internal σ lines and ϕ loops.

Consider the four point function $G^{(4)}(\phi_1^\alpha, \phi_2^\beta, \phi_3^\gamma, \phi_4^\delta)$ where superscript indicates field index and $\phi_i \equiv \phi(x_i)$. Due to the nature of the interaction, the indices must come in pairs, i.e. $\alpha = \beta, \gamma = \delta$ or permutations. The diagram which appear at first and second order in g_0 are illustrated Figure 3. Note that the two second order corrections have different N dependence, as promised.

At higher order ($O(g_0^5)$ for the diagram shown) we will have structures like



By counting the number of σ lines and ϕ loops, we may convince ourselves of the following fact: for every loop which contains a σ, the contribution is less by a factor of N. The point is that to leading order in $1/N$, the diagrams that survive are the ones which have the most ϕ loops. They look like this (omitting external legs):



This sequence of bubbles is a geometric series which we can be evaluated explicitly. Denote by $\Pi(p^2)$ the value of the loop integral,

$$\Pi(p^2) = \int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2 - m^2 + i\epsilon}\frac{i}{(k-p)^2 - m^2 + i\epsilon} \tag{4}$$

Then the series above reads

$$\begin{aligned} G_\sigma + G_\sigma\Pi(p^2)G_\sigma + G_\sigma\Pi(p^2)G_\sigma\Pi(p^2)G_\sigma + ... \\ = G_\sigma \times \sum_{n=0}^{\infty}\left(\Pi(p^2)G_\sigma\right)^n \\ = G_\sigma \times \left[1 - G_\sigma\Pi(p^2)\right]^{-1} \end{aligned} \tag{5}$$

Restoring the external legs we find

$$G^{(4)} = \frac{1}{G_\sigma^{-1} - \Pi(p^2)} \times \prod_{n=1}^{4}\frac{i}{k_n^2 - m^2 + i\epsilon} \tag{6}$$

When we apply the LSZ reduction formula, the external legs are amputated, so we focus on the first part of the above expression. Evidently, when $G_\sigma^{-1} - \Pi(p^2) = 0$, the scattering amplitude has a pole. Thus, through the $1/N$ expansion, we have discovered a metastable bound state in the two body scattering process.

WHAT ELSE CAN HAPPEN? GROSS-NEVEU MODEL

The large N expansion can reveal other interesting physics; in this section I briefly describe its application to another theory. The Gross-Neveu model is defined in terms of a set of massless interacting Dirac fields:

$$\mathcal{L} = \bar\psi^a \partial\!\!\!/ \psi^a + \tfrac{g_0}{N}(\bar\psi^a\psi^a)^2 \tag{7}$$

This Lagrangian is $SU(N)$ invariant.

As before, we introduce an auxiliary field to decouple $(\bar{\psi}\psi)^2 \to \sigma\bar{\psi}\psi$. The derivation of the following results requires slightly more advanced technology, so I state them mostly without justification.

One can derive an effective potential for this theory by minimization of the action, which involves some functional integration tricks. The effective potential is a sum of a series of Feynman diagrams, but once again the quantity $1/N$ is the hero of the story: it allows us to restrict our attention to a subset of diagrams which may be summed exactly.

It can be shown that the minimum of the effective potential occurs at a nonzero value of $\sigma = \sigma_0$. On the other hand, this value σ_0 acts like a mass for the ψ fields due to the interaction term $\sigma\bar{\psi}\psi$! We began with massless Dirac fields, but to leading order in $1/N$, the fields acquire a mass.

CONCLUSION

I have motivated that in some field theories, when we send the number of different types of fields to infinity while keeping the coupling $g_0 = \lambda N$ fixed, there is a drastic effect on which diagrams in the perturbative expansion matter. Working to lowest order in $1/N$ allows us to resum the series to *all orders* in g_0 and obtain nonperturbative results.

This expansion has enjoyed success in QCD (even though there N is only 3) and is also frequently used in condensed matter theory. The application of this expansion to matrix valued field theories was the beginning in the historical development of the AdS/CFT correspondence. In this paper, I have sought to provide a flavor of the basics of large N methods without going into the technical details.

[1] Sidney Coleman. *Aspects of Symmetry*. Cambridge University Press, 1985. Cambridge Books Online.

[2] Gerard T Hooft. Large N Lecture Notes. (1):1–16, 2008.

[3] AV Manohar. Large N QCD. 1998.

[4] M Moshe and J Zinn-Justin. Quantum Field Theory in the Large N limit: a review. *Physics reports*, (May):1–200, 2003.