The T.D. Lee Model

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This paper presents a simple introduction to the T.D. Lee model for solvable field theories. First, the framework for the model is laid out and the key assumptions made are discussed. It is then shown how the model develops scattering states and renormalizes the coupling constant to find a scattering amplitude free of divergence. Finally, applications and comparissons of the model to traditional methods are discussed.

INTRODUCTION

A common issue among quantum field theories is that of renormalization. Often divergent terms appear and prevent the theory from having a usable form, while still being a correct theory. The hope, then, is to find some way to renormalize the theory in such a way that the divergent terms don't appear in any physical measured quantity. The most thorough and general way to do so is the perturbation power series method of computing Feynman diagrams in powers of the coupling constant. While accurate, this method is cumbersome and difficult so the preferred method is to alter the model in such a way that straightforward calculations of physical quantities end up having all divergent terms suppressed. The T.D. Lee model accomplishes just this for a class of field theories. The original implementation of the model solves a scattering process involving a single boson but has since been applied to further field theories.

THE T.D. LEE MODEL

The example to which the T.D. Lee model was introduced is a scattering process between two fermions, labeled V and N, and a single scalar boson, θ. The masses for each particle are m_V, m_N, and μ respectively. The free field Hamiltonian density for these particles is

$$\mathcal{H}_0 = m_V \psi_V^\dagger \psi_V + m_N \psi_N^\dagger \psi_N + \frac{1}{2}\left[\pi^2 + (\nabla\phi)^2 + \mu^2\phi^2\right] \tag{1}$$

where ψ_V and ψ_N are the annihilation operators associated to the particles V and N respectively, and ϕ is the usual scalar boson field operator [1]. Something important to note is that the V and N particles have no kinetic terms so they are treated essentially as fixed objects in this model. Some treatments of the T.D. Lee model even hold the θ particle fixed as well [2] but in following Lee's original the work kinetic terms for θ were left in. The ϕ field can be expressed as a sum of the $\phi^{(+)}$ and $\phi^{(-)}$ operators summing over the allowed wavevectors $\vec{k}$ in a finite volume Ω.

$$\phi(x) = \sum_k (2E_k\Omega)^{-\frac{1}{2}}\left[\alpha_k e^{ik\cdot r} + \alpha_k^\dagger e^{-ik\cdot r}\right] \tag{2}$$

From here we examine the scattering process of $V \rightleftarrows N + \theta$, which has the following interaction Hamiltonian density,

$$\mathcal{H}_1 = g\left[\psi_V^\dagger \psi_N \phi^{(+)} + \psi_V \psi_N^\dagger \phi^{(-)}\right] + \delta m_V \psi_V^\dagger \psi_V \tag{3}$$

Here g is the coupling constant of the interaction and δm_V is used to correct for any change in mass of the V particle due to the interaction and will be solved for later. The form of the Hamiltonian is what we'd intuitively expect from the $V \rightleftarrows N + \theta$ interaction as the first term describes annihilating a N and θ particle in exchange for the creation of a V particle, while the second term describes the reverse process. This leads us to observe the following two conserved quantities:

$$\mathcal{N}_V \; + \; \mathcal{N}_N \tag{4}$$

$$\mathcal{N}_V \; + \; \mathcal{N}_\theta \tag{5}$$

where $\mathcal{N}_i$ is the number operator of the i particle. With the help of these conserved quantities we can now diagonalize the total Hamiltonian, $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$, in terms of the eigenstates of the free field Hamiltonian. The eigenstates of $\mathcal{H}_0$ are just the V, N, and θ particle states denoted as $|V\rangle$, $|N\rangle$, and $(\alpha_k |0\rangle)$ respectively. By examination we can see that $|N\rangle$ and $(\alpha_k |0\rangle)$ are still eigenstates of the total Hamiltonian, but the eigenstate we will associate with the V particle changes. A guess for the new V particle eigenstate, denoted $|\boldsymbol{V}\rangle$, would be one that is a superposition of $|V\rangle$ and $(\alpha_k |N\rangle)$ as these are the terms that swap places under the interaction Hamiltonian. We'll start with an arbitrary superposition of these states of the form,

$$|\boldsymbol{V}\rangle = Z^{\frac{1}{2}}\left(|V\rangle + g\sum_k f(k)\alpha_k^\dagger |N\rangle\right) \tag{6}$$

where $f(k)$ is some probability distribution and Z is a normalization constant. We know this state solves the

Schrodinger equation so we can operate on this state by the total Hamiltonian and enforce that it has the eigenvalue m_V, which will constrain $f(k)$ and δm_V. Doing so yields,

$$\delta m_V = -g^2 \sum_k (2E_k\Omega)^{-1}(m_V - m_N - E_k)^{-1} \quad (7)$$

$$f(k) = (2E_k\Omega)^{-\frac{1}{2}}(m_V - m_N - E_k)^{-1} \quad (8)$$

Now we can normalize $|\boldsymbol{V}\rangle$ to find,

$$Z^{-1} = 1 + g^2 \sum_k (2E_k\Omega)^{-1}(m_V - m_N - E_k)^{-2} \quad (9)$$

Note that Z is a divergent quantity so we'll need to address it at some point in the future.

We now have what we need to apply this model to various scattering processes and see that we can express the scattering amplitudes without any divergent terms. Consider the scattering process $N + \theta \to N + \theta$. We can use a similar process as before where we guess an eigenfunction for $|N + \theta\rangle$ of the form,

$$|N + \theta\rangle = c\,|\boldsymbol{V}\rangle + \sum_k \chi(k)\alpha_k^\dagger\,|N\rangle \quad (10)$$

and enforce that it is an eigenstate of the Hamiltonian with eigenvalue, $m_N + E_{k'}$, to constrain c and $\chi(k)$. The result is

$$c = -\langle\boldsymbol{V}|\sum_k \chi(k)\alpha_k^\dagger\,|N\rangle \quad (11)$$

$$\chi(k) = g^2 Z\,(E_k - E_k')^{-1}\int W(k,k'')\chi(k'')d^3k'' \quad (12)$$

where

$$W(k,k'') = \frac{(m_V - m_N - E_k)(4E_kE_{k''})^{-\frac{1}{2}}}{8\pi^3(m_V - m_N - E_k)(m_V - m_N - E_k'')} \quad (13)$$

From here we can use this form of the eigenstate to calculate scattering amplitudes and phase shifts as Lee does in his original paper [1]. But the most important thing to note at this time is whether or not this form of $|N + \theta\rangle$ is divergent. We now have to address the divergence of Z as it appears in the expression for $\chi(k)$. From (9) we see that evaluating the sum in Z gives us $Z \to -\infty$. But if we define a renormalized coupling constant, $g_c^2 = g^2 Z$, it is possible to keep this finite by letting the unrenormalized coupling constant, g, approach 0 in such a way that keeps g_c finite. Something to note, however, is that to make g_c positive g needs to be imaginary, which breaks the Hermiticity of the original Hamiltonian. We resolve this by noting that the scattering matrix would only depend on the real quantity g_c so it is still unitary and we can just apply a similarity transformation to the Hamiltonian to restore it to being hermitian [2].

DISCUSSION

We've now seen how the Lee model produces renormalized elements of the scattering matrix but it's interesting to compare Lee's treatment to the traditional model studied in [3]. The standard interaction Hamiltonian for the scattering process examined in the previous section is,

$$\mathcal{H}_1 = g\left[\psi_V^\dagger\psi_N + \psi_V\psi_N^\dagger\right]\phi \quad (14)$$

The only difference between this and (3) is that the exchange of a V particle with an N particle is coupled to both the creation and anihilation of a θ particle. This model is solvable but with considerably more work. In terms of Feynman diagrams, the Lee model offers a simplification to (14) by breaking the symmetry that allows for an infinite number of self energy diagrams in the traditional model [3].

An important bennefit of the Lee model is that it allows us to compute the commutation relation,

$$\left\{\psi_V, \psi_V^\dagger\right\} = Z^{-1} \quad (15)$$

With this we can find various propagators is the LSZ formalism [4]. For instance we can find

$$\langle 0|\,T\left[\psi_V(t')\psi_V^\dagger(t)\right]|0\rangle \quad (16)$$

entirely in terms of renormalized quantities.

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