The Unruh Effect

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In this brief report, an intuitive and physical derivation of Davies-Unruh effect (an effect where a detector undergoing uniform acceleration a through a vacuum field, with respect to an inertial observer, perceives the vacuum as a thermal bath of temperature $T = \frac{\hbar a}{2\pi kc}$) has been given for a scalar field in 1+1 dimension. A formal proof of the aforementioned effect based on KMS condition has also been included for the sake of completeness with physical insights on how detector perceives the thermal bath.

INTRODUCTION

Davies and Unruh predicted [1, 2] that a uniformly accelerated detector (observer) moving through a conventional quantum field theory's vacuum defined with respect to usual flat (Minkowski) space-time perceives the vacuum as a thermal bath of particles with temperature (famous as *Hawking-Unruh temperature*) given by

$$T = \frac{\hbar a}{2\pi kc} = \frac{a}{2\pi k} \tag{1}$$

where a is the acceleration in the instantaneous rest frame of the detector, $c, \hbar, k$ represent speed of light in vacuum, Plank's constant and Boltzmann's constant respectively. The second equality follows upon employing choice of natural unit i.e. $\hbar = c = 1$.

Hawking [3, 4] has also shown in an earlier paper that a black hole should radiate with a temperature given by eq. (1). If we use principle of equivalence, we realise Hawking effect is intimately related to Davies-Unruh effect since, a uniform accelerated observer can be thought of an inertial observer in presence of gravity. In fact, stationary observer sitting just outside the horizon of black hole has to undergo a constant acceleration just to be at rest.

The work of Hawking, Davies and Unruh has profound implication regarding the merger of general relativity with quantum field theory. The results obtained has brought forth a host of questions which includes questions regarding the formulation of quantum theory of gravity, information loss paradox. However, we will not delve into those details in this report though. The purpose of this brief report is to present an intuitive and physical understanding of Davies-Unruh effect in 1+1 dimension along with a formal development of the topic based on KMS condition.

The write-up is organised as follows. In section II, we briefly mention the essential features of uniform acceleration along with an introduction to Rindler coordinates. In section III, Hwaking-Unruh temperature (1) is obtained via considering the Doppler effect. In Sec IV, we introduce the famous KMS condition and formally show that the two point correlation function of free scalar field theory appears to satisfy KMS condition when written in terms of co-ordinates adapted to instantaneous rest frame of accelerated observer (detector), hence define a thermal state with temperature given by eq. (1). We conclude with a discussion on the physics of Davies-Unruh effect with some important remarks.

UNIFORM ACCELERATION: RINDLER SPACE-TIME

In this section and here on, an observer moving with constant velocity in a flat space time will be referred as *Minkowski observer: M* while we will refer to a *Rindler observer: R* as one who travels with uniform acceleration with respect to the former i.e inertial observer. By uniform acceleration, we mean a constant acceleration $a > 0$ in the positive x direction, as measured in the instantaneous inertial frame of reference in which R is at rest. The orbit of R is given by:

$$t(\tau) = \frac{1}{a}\sinh(a\tau), \quad x(\tau) = \frac{1}{a}\cosh(a\tau) \tag{2}$$

where t is the time of inertial observer and τ is the proper time, as measured by R in his instantaneously rest frame.

In this connection, we will take a detour to understand Rindler space-time, which we will use in the formal development of the Davies-Unruh effect in sec IV.

Rindler Detour In this detour we will show, Rindler space-time is nothing but the Minkowski space-time described in a co-ordinate system adapted to the accelerated observer R. We will now elucidate what we mean by adapted co-ordinate.

We start with a Minkowski space-time (1+1 dimensional) with invariant distance given by

$$ds^2 = \left(dx^0\right)^2 - \left(dx^1\right)^2, \tag{3}$$

and do the following transformation:

$$x^0 = e^{\lambda}\sinh(\theta), \; x^1 = e^{\lambda}\cosh(\theta) \tag{4}$$

so that (3) can be recast into

$$ds^2 = e^{2\lambda}\left[(d\theta)^2 - (d\lambda)^2\right] \tag{5}$$

The co-ordinate transformation (4) is adapted to the accelerated observer in the sense that if we compare the transformation (4) with the orbit of Rindler observer (2), we deduce the motion of R is given by $\lambda = -ln\,(a)$ =constant upon following identification: $d\tau = e^{\lambda}d\theta = \frac{d\theta}{a}$ and $x^1 = x$. Hence, translation in θ i.e τ variable (i.e. $d\lambda = 0$) is in fact acceleration in x^1 direction.

DOPPLER EFFECT APPEARS AS THERMAL EFFECT

We will consider a plane wave with vector $\vec{K} = \pm K\hat{x}$ and frequency $\omega_K = K > 0$ in M. Now R in his instantaneous rest frame will perceive this as $\omega'_K(\tau)$ given by

$$\omega'_K(\tau) = \gamma\,(\omega_K - Kv(\tau)) = \omega_K e^{\mp a\tau} \tag{6}$$

In small τ approximation $\omega'_K \sim \omega_K\,(1 \mp a\tau)$, which is the well known Doppler shift. This very Doppler shift is what causes the observer R to observe the wave with a time dependent phase factor given by $\phi(\tau) = \int \omega'_K(\tau')d\tau' = \frac{\omega_K}{a}e^{\mp a\tau}$. As a result, the power spectrum,measured by R is given by [5]

$$S(\Omega) = |\int_{-\infty}^{\infty} d\tau e^{\imath\Omega}e^{\imath\phi(\tau)}|^2 = \frac{2\pi}{\Omega a}\frac{1}{e^{\frac{2\pi\Omega}{a}} - 1} \tag{7}$$

We identify the equation (7) as containing a factor mimicking the Bose-Einstein distribution function with the temperature T given by

$$\frac{2\pi\Omega}{a} = \frac{\Omega}{kT} \Leftrightarrow T = \frac{a}{2\pi k} \tag{8}$$

which is exactly same as (1). Similar analysis in Rindler space-time has also been put forward by T. Padmanavan and coauthors in [6] but with real cosine waves. To generalise the idea presented, we will now look at a quantum field in vacuum having components at all frequencies rather than confining our attention to a single field frequency ω_K. To be specific, we first compute a particular two point correlation function (to be defined later) of a thermal quantum field theory and compare it to the same correlation function of a quantum field theory, living in Rindler space-time i.e expectation value will be taken with respect ot the Rindler vacuum. We will show both of them bears the same Bose-Einstein distribution function.

A massless scalar field Φ can be written as:

$$\Phi = \int dK \frac{1}{\sqrt{(2\pi)(2\omega_K)}}\left[e^{-\imath Kx}a_K + e^{\imath Kx}a_K^{\dagger}\right] \tag{9}$$

Now we consider a fourier transformed operator $g(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\Phi(t,0)e^{\imath\Omega t} = \frac{1}{\sqrt{(2\pi)(2\omega_K)}}a_\Omega$ where for simplicity we have taken $\Omega > 0$ and considered the field at $x = 0$. We define the correlation function as $\langle g^\dagger g\rangle \equiv \langle g^\dagger(\Omega)g(\Omega')\rangle$, which comes out to be

$$\langle g^\dagger g\rangle = \langle a_\Omega^\dagger a_\Omega\rangle\delta\,(\Omega - \Omega') = \frac{2/\Omega}{e^{\frac{\Omega}{kT}} - 1}\delta\,(\Omega - \Omega')$$

where we have used the fact that for a Bosonic thermal state (particle content of scalar field is Bosonic) $\langle a_K^\dagger a_K\rangle = [e^{\frac{\omega_K}{kT}} - 1]^{-1}$. It deserves mention although we start with all Fourier components, yet by doing Fourier transform, we essentially reduce the calculation to the same done previously.

As discussed earlier, to a Rindler observer, the frequency gets Doppler shifted and at $x = 0$, we have

$$\Phi = \int dK \frac{1}{\sqrt{(2\pi)(2\omega_K)}}\left[e^{-\imath\phi(\tau)}a_{KR} + e^{\imath\phi(\tau)}a_{KR}^{\dagger}\right] \tag{10}$$

where the subscript R denotes these creation, annihilation operators live in Rindler space-time. The integral measure is same as in eq. (9) because of its invariance under Lorentz transformation. Now we employ results from conventional quantum field theory and use the following facts involving vacuum with respect to Rindler observer (henceforth called $|0_R\rangle$)

$$\langle a_{KR}a_{KR}^\dagger\rangle \equiv \langle 0_R|a_{KR}a_{K'R}^\dagger|0_R\rangle = \delta\,(K - K') \tag{11}$$

$$a_{KR}|0_R\rangle = 0 \tag{12}$$

and the integral over τ (we did the same to find power spectrum) to arrive at

$$\langle g^\dagger g\rangle = \frac{2/\Omega}{e^{\frac{2\pi\Omega}{a}} - 1}\delta\,(\Omega - \Omega') \tag{13}$$

Comparing the eq. (13) with the eq. (10), we arrive at the expression for Hawking-Unruh temperature i.e $T = \frac{a}{2\pi k}$.

KMS CONDITION: A PHYSICAL PERSPECTIVE

Without going into technical details of KMS condition, in this section we will try to motivate it physically and as a by product, we are going to obtain Hawking-Unruh temperature. We look at the Minkowski two point correlation function $\omega_2 \equiv \langle 0_M|\Phi(x)\Phi^\dagger(x')|0_M\rangle$ evaluated at two points on Rindler orbit, (Minkowski vacuum is referred as $|0_M\rangle$) for a massless scalar field,

$$\omega_2(x, x') = \int d\vec{K}\frac{e^{-\imath K(x-x')}}{2\omega} \tag{14}$$

and express it in terms of Rindler co-ordinates as (We recall resullts from section II. and note λ is fixed)

$$\omega_2(\theta, \theta') = \int \frac{dK}{2\omega}\exp\left[-\imath\omega e^{\lambda}(\sinh\theta - \sinh\theta')\right] \tag{15}$$

$$\times\exp\left[\imath Ke^{\lambda}(\cosh\theta - \cosh\theta')\right] \tag{16}$$

which turns out to satisfy the following condition [7]:

$$\omega_2(\theta, \theta') = \omega_2(\theta', \theta - \imath 2\pi) \tag{17}$$

Now in terms of the fourier transform $g(\Omega) \equiv \frac{1}{2\pi}\int d\tau e^{\imath\Omega(\tau-\tau')}\omega_2(\theta, \theta')$, (we note that θ is related to time of R i.e τ variable and $\omega_2(\theta,\theta') = \omega_2(\theta-\theta')$ due to Lorentz invariance since translation in θ implies boost along x direction) we can cast the eq (17) in following form:

$$g(\Omega) = e^{-\frac{2\pi\Omega}{a}} g(-\Omega) \tag{18}$$

With the identification $T = \frac{a}{2\pi k}$, this becomes

$$g(\Omega) = e^{-\frac{\Omega}{kT}} g(-\Omega) \tag{19}$$

which has a direct physical interpretation [8] as follows:

Density of states being even function of Ω (Takagi[8] showed it explicitly, but here we are not including it to keep the write up brief), we can relate $g(\Omega)$ to be proportional to the transition rate of detector to go from a state with energy E to a state with energy $E+\Omega$ where we have assumed $\Omega > 0$ without loss of generality whereas $g(-\Omega)$ represents the transition rate from level $E+\Omega$ to level E. Now the Boltzmann factor sitting in the eq (19) represents relative population of upper energy level with respect to the lower one, provided detector's internal degrees of freedom were initially in thermal equilibrium at temperature T. The eq. (19) is nothing but equation of detailed balance implying there is no net change in population of detector's energy levels after it is switched on, provided it is initially in thermal equilibrium with the thermal bath of particle it is observing.

The condition (17) is famously known as KMS condition. The two point correlation function satisfying this condition translates into a nice interpretation in terms of accelerated detector perceiving a thermal bath of particles with Hawking-Unruh temperature. Hence, we can take KMS condition (17) to be a signature of thermal state and deduce the Minkowski vacuum correlation function is indeed thermal in nature.

DISCUSSION

In this work, we have looked at the physical origin of Unruh effect. The first derivation is based on two facts that there are fluctuations in vacuum state i.e $\langle a_{KR}a^\dagger_{KR}\rangle \neq 0$ and the accelerated observer perceives a time dependent Doppler shifted frequency while the second derivation based on KMS condition gives insight to how accelerated detector responses to Minkowski vacuum.

We can in fact trace back the Unruh effect to the very definition of vacuum state, which is observer dependent. To see that, we note the orbit of accelerated observer is confined to the region $x>0, x>|t|$, bounded by asymptotes $t=\pm x$, which is called Right Rindler wedge. The left wedge can similarly be obtained for the observer accelerating in $-x$ direction. It is very easy to show that the left and the right wedge are not causally connected. Hence, the Minkowski vacuum which pervades the all space time can't be equal to the vacuum state perceived by the observer living on right Rindler wedge (or left Rindler wedge). As a result, when Rindler observer moves through the Minkowski vacuum, it does not appear to be vacuum to him, rather it is an excited state brimming with particles compared to his own Rindler vacuum. The particle distribution precisely follows Bose-Einstein statistics for scalar field theory in even space-time dimension. To be precise, the restriction of Minkowski vacuum to the right (left) Rindler wedge is a thermal state with temperature given by the eq. (1). In this connection, it deserves mention that the Bose-Einstein distribution with Hawking-Unruh temperature for scalar fields is arrived at by showing $\langle 0_M|a^\dagger_{KR}a_{KR}|0_M\rangle \sim \left[\exp\left(\frac{\Omega}{kT}\right)-1\right]^{-1}$, which is basically showing the expectation value of number operator with respect to Minkowski vacuum ($|0_M\rangle$) behaves like as if there is a thermal bath. This is what Takagi [8] refers as *thermalization theorem* and he shows that the thermalisation theorem is not equivalent to the effect where an accelerating particle detector picks up thermal character of the power spectrum of vacuum noise defined by the fourier transform of two point correlation function $\omega_2(\tau,\tau') \equiv \langle 0_M|\Phi(\tau)\Phi^\dagger(\tau')|0_M\rangle$ [5, 8]. These two effect turns out to give same result only in even space-time dimension and this fact is intimately connected to the expression for density of states [8].

It deserves mention that the Unruh effect may seem to be paradoxical since the particle content of the theory appears to depend on the observer. This apparent paradox goes away if we think the concept of particle being a mere label to certain states. In fact, the natural notion of particles with respect to an observer depends on time translation, perceived by him in his co-ordinate and it turns out that the concept of time translation is different in two frames M and R. In Rindler frame what we understand as the time translation (translation in τ or θ) is in fact acceleration in Minkowski frame, hence, not same as translation in ordinary time t. With respect to τ translation, the restriction of Minkowski vacuum to Rindler wedge appears to be thermal, brimming with particles, but it is a non thermal vacuum state with respect to t translation. This picture manifests in the second approach where we show that the correlation function satisfies the KMS condition and hence represent a thermal state. Nevertheless, we should arrive at same physical prediction, in whichever way we want to label our particle states, that there is a finite probability of particle detector carried by Rindler observer making a transition to excited state, which the accelerating observer will describe as absorption of particle by the detector (he does see the vacuum as an excited state brimming with particles) whereas an inertial observer will describe the same as emission of a particle by the detector along with radiation reaction back on the detector [9, 10].

As a last remark, we should add that for macroscopic acceleration, the Davies-Unruh effect is really small compared to any macroscopic temperature scale. For example, an observer with acceleration of $5ms^{-1}$ will perceive the Minkowski vacuum as a thermal bath of $T \sim 10^{-19}K$. But this effect could be relevant for linear particle accelerators and as we mentioned, it does have profound implication concerning the merger of quantum field theory with gravity.

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[1] P. C. W. Davies, J. Phys. A **8**, 609-616 (1975).
[2] W. G. Unruh, Phys. Rev. D. **14**, 870-892 (1976).
[3] S. W. Hawking, Nature (London) **248**, 30-31 (1974)
[4] S. W. Hawking, Commun. Math. Phys. **43**, 199-220 (1975)
[5] P. M. Alsing, P. W. Milonni, Am. J. Phys. **72** (12) 1524-1529 (2004), arXive: [quant-ph] 041170v2.
[6] K. Srinivasan, L. Sriramkumar, and T. Padmanavan, Phys. Rev. D. **56**, 6692-6694 (1997)
[7] T. Reichanbcher, The Unruh effect, Seminar on mathematical aspects of theoretical physics
[8] S. Takagi, Prog. Theor. Phy. **88**, 1-142 (1986)
[9] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago) (1984)
[10] R. M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics* (University of Chicago Press, Chicago) (1994)