A Review of the Haag-Lopuszanski-Sohnius Theorem

Elizabeth Wicks[1]

[1] *Department of Physics, University of California at San Diego, La Jolla, CA 92093-0319*

Coleman and Mandula showed that under certain assumptions, the only allowed symmetries of the S matrix consist of the Poincare group and internal symmetries. The only way to combine these two types of symmetries is through a local direct product. Haag, Lopuszanski and Sohnius show that if the assumptions of Coleman and Mandula are relaxed to allow for anticommutation relations, there is an additional symmetry of the S matrix generated by supercharges. In this case the two symmetry types are mixed. We discuss the massive field case of the proof.

INTRODUCTION

Before Coleman and Mandula, physicists assumed that the symmetries of the S matrix could be written as local direct products of the Poincare group with a group of internal symmetries. After investigation of $SU(6)$ by the scientific community, some thought that perhaps there existed symmetry groups which had a richer structure. By proving that the symmetries of the S matrix must be locally isomorphic to the direct product of spacetime symmetries with the internal symmetry group, Coleman and Mandula put the final nail in the coffin for such theories - or so they thought.

In seeming contrast to the Coleman-Mandula Theorem, Wess and Zumino proved the existence of a supersymmetry generated by supercharges - Fermi operators that transform bosonic to fermionic fields (and vice-versa), transforming like spinors under the Lorentz group [3]. The algebra of these operators is closed under commutation and anticommutation relations, and so is a part of a pseudo Lie algebra, or superalgebra.

Haag, Lopuszanski and Sohnius sought to extend the Coleman-Mandula Theorem to include these supersymmetric operators by considering superalgebras instead of Lie algebras. Coleman and Mandula had considered Lie algebras of symmetry generators, which do not have anticommutation relations, so the supercharge operators were excluded from their analysis.

RESULTS

Haag et al. found that the algebra of symmetry operators is not a direct product of the Poincare group and another group. The addition of the Fermi supercharges mixes spacetime and internal symmetries in the following way.

Let P_μ be the energy-momentum operators, $M_{\mu\nu}$ the generators of the homogeneous Lorentz group, and B_l the Bose scalar charges, of which there are a finite number. All of these operators are Bose type.

Divide the Fermi supercharge operators by their transformation character using van der Waerden notation: Q^L_α and $\bar{Q}^L_{\dot{\alpha}}$ $(L = 1, \ldots, v; \alpha = 1, 2)$. We choose the basis of our superalgebra such that $(Q^L_\alpha)^\dagger = \bar{Q}^L_{\dot{\alpha}}$.

Then

$$\{Q^L_\alpha, Q^M_\beta\} = \epsilon_{\alpha\beta} Z^{LM}, \tag{1}$$

$$\text{where } Z^{LM} \equiv \epsilon_{\alpha\beta} \sum_l (a^l)^{LM} B_l \tag{2}$$

$$\text{and } [Z^{LM}, G] = 0 \text{ for all G in the superalgebra}, \tag{3}$$

$$\{Q^L_\alpha, (Q^M_\beta)^\dagger\} = \delta^{LM} \sigma^\mu_{\alpha\dot{\beta}} P_\mu, \tag{4}$$

$$[Q^L_\alpha, B_l] = \sum_M s_l^{LM} Q^M_\alpha, \tag{5}$$

$$[B_l, B_m] = i \sum_k c^k_{lm} B_k. \tag{6}$$

We also have the Poincare transformation properties of spinor and scalar charges as follows:

$$[Q^L_\alpha, P_\mu] = [B_l, P_\mu] = [B_l, M_{\mu\nu}] = 0, \tag{7}$$

$$[Q^L_\alpha, M_{\mu\nu}] = \frac{1}{2} (\sigma_{\mu\nu})^\beta_\alpha Q^L_\beta. \tag{8}$$

The c^k_{lm} are structure coefficients of a compact Lie group, the s_l are the Hermitian matrices of a ν-dimensional representation of the generators, and the a_l are matrices such that $s_m a^l = -a^l \bar{s}_m$.

Equations (7) and (8) follow from the properties of spinor and scalar charges. The rest of the relations are more interesting. Equation (1) says that the Fermi operators do not form their own subalgebra. In fact, the anticommutator of two such operators is in the center of the superalgebra (3).

Furthermore, the anticommutator of Q^L_α with $(Q^M_\beta)^\dagger$ is a combination of energy-momentum operators (4). This shows that the internal symmetries mix with the spacetime symmetries; that is, we cannot write the superalgebra as a direct product of these two symmetries. If we could, then $\{Q^L_\alpha, (Q^M_\beta)^\dagger\}$ would have to be contained within a subalgebra of internal symmetries, which clearly it is not.

Equation (5) gives the commutation relation between Fermi and Bose charges, and (6) shows that the Bose charges form their own Lie subalgebra.

PROOF

Equations (3)-(8) had already been verified by other sources [2]. So we must prove Equations (1) and (2).

We define a symmetry of the S matrix to be an operator G on a Hilbert space such that

$$\text{(i) } [G,S]=0, \tag{9}$$

$$\text{(ii) } G=\sum_{i,j,r,s}\int d^3p d^3p' a^{\text{in}\dagger}_{js}(p')K_{js;ir}(p',p)a^{\text{in}}_{ir}(p), \tag{10}$$

for K a c-number kernel. In plain language this means that G commutes with the S matrix (9) and that G acts additively on multi-particle states (10).

These two properties along with non-triviality of the S matrix imply a third property: G only connects particles with the same mass [1]. More specifically K is written as

$$K(p',p)=\sum_n K^{(n)}(p)\partial^n\delta(p-p'), \tag{11}$$

where ∂^n is a monomial in derivatives $\partial/\partial p_i$ and the sum is finite [1].

Because $K(p',p)$ has this form, it is easy to divide up the elements of the symmetry group into manageable chunks. Define $\mathcal{S}^{(N)}$ as the set of symmetry generators where $K(p',p)$ only contains derivatives up to and including Nth order.

Generators which commute with translations

Suppose $G\in\mathcal{S}^{(0)}$. Then G commutes with translations and we can write the kernel as

$$K_{js;ir}(p',p)=K_{js;ir}(p)\delta(p-p'). \tag{12}$$

For a Bose type generator we can borrow the results from [1], which state that a basis for the Bose operators in $\mathcal{S}^{(0)}$ can be given by the P_μ and the scalar charges B_l. The Lie algebra $\mathcal{L}$ of the scalar charges is the direct product of a semisimple and abelian subalgebra [2].

We continue to borrow results from [1] and note that Fermi type generators have the property that if $K(p)$ vanishes for p and p' on a mass hyperboloid, then it vanishes on the entire hyperboloid. $K(p)$ is the submatrix of $K_{js;ir}(p)$ for one mass multiplet. Therefore if we know two $K(p)$ matrices we can find G.

Classification of translation invariant Fermi type generators

We have a finite number of Fermi generators because $K(p)$ has finite dimensionality and G is determined by two $K(p)$ matrices. Moreover the homogeneous Lorentz transformations stabilize $\mathcal{S}^{(0)}$.

Therefore we can say that $\mathcal{S}^{(0)}$ is a finite dimensional representation of the Lorentz group, which is semisimple hence completely reducible. The irreducible representations of the Lorentz group are classified by indices (j,j'). For a irreducible representation (j,j') we will have generators of the form $Q_{\alpha_1\ldots\alpha_{2j};\beta_1\ldots\beta_{2j'}}$. Since these are Fermi operators we must have $2(j+j')$ odd. Furthermore, $\{Q,Q^\dagger\}$ is of Bose type and an element of $\mathcal{S}^{(0)}$.

Now if we consider just one component of $\{Q,Q^\dagger\}$ we notice that it is a component of a spinor with $2(j+j')$ undotted and $2(j+j')$ dotted indices, symmetric in the dotted and undotted indices. Therefore it must belong to the representation $(j+j',j+j')$. However we can only have Bose type generators in such a representation if $j+j'\leq\frac{1}{2}$. Therefore if $\{Q,Q^\dagger\}$ does not vanish then we must have Q in $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$. If $\{Q,Q^\dagger\}$ vanishes then the original Q vanishes as well.

Therefore we can say

$$\{Q_\alpha,(Q_\beta)^\dagger\}=c\sigma^\mu_{\alpha\dot{\beta}}P_\mu, \tag{13}$$

for some complex number c. The reason this true is because we have determined that $\{Q,Q^\dagger\}$ is an element of the representation $(\frac{1}{2},\frac{1}{2})$. Elements of this representation are four-vectors. The only four vectors in $\mathcal{S}^{(0)}$ (the set of generators which commute with translations) are P_μ. Therefore we can write $\{Q,Q^\dagger\}$ as some linear combination of these.

In the case of multiple charges Q^L we can generalize to

$$\{Q^L_\alpha,(Q^M_\beta)^\dagger\}=\delta^{LM}\sigma^\mu_{\alpha\dot{\beta}}P_\mu. \tag{14}$$

We have proved (4). Now consider $\{Q^L_\alpha,Q^M_\beta\}$, which is also an element of $\mathcal{S}^{(0)}$. The antisymmetric part must be a scalar, so it must be a linear combination of the Bose scalar charges B_l. The symmetric part is an element of $(1,0)$ and is a self-dual skew-symmetric tensor. However we have no such tensors in $\mathcal{S}^{(0)}$ so $\{Q^L_\alpha,Q^M_\beta\}$ is entirely antisymmetric.

Therefore we can write

$$\{Q^L_\alpha,Q^M_\beta\}=\epsilon_{\alpha\beta}Z^{LM}, \tag{15}$$

$$\text{where } Z^{LM}\equiv\epsilon_{\alpha\beta}\sum_l(a^l)^{LM}B_l \tag{16}$$

$$\text{and } (a^l)^{LM}=-(a^l)^{ML}, \tag{17}$$

since the anticommutator should be symmetric.

Generators which do not commute with translations

Here we will comment on the results on $G\in\mathcal{S}^{(N)}$ from [2] without proof. In the massive case, all Bose generators of degree one are linear combinations of $M_{\mu\nu}$. There are no Bose generators of degree higher than one.

Furthermore, there are no Fermi charges of degree 1 or higher.

This result is interesting because it states that the superalgebra of symmetry generators has fermionic part with spin $\frac{1}{2}$. This shows how we can build up everything from spin $\frac{1}{2}$, even in the supersymmetric case.

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