Large N Limit in 1/N Expansion

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This paper summarizes works in chapter $1/N$ of Coleman's lecture notes. $1/N$ expansion for ϕ^4 theory, the Gross-Neveu model, and the $\mathbb{C}P^{N-1}$ model are summarized.

INTRODUCTION

Interactive field theories with quartic interactions can be expanded using Feynman diagrams in the orders of $1/N$. When N is large, the leading term, terms proportional to $1/N$, in this expansion dominates. This theory can be used to show the symmetry breaking in asymptotically free field theories (Gross-Neveu model), and is consistent with the $\mathbb{C}P^{N-1}$ model.

1/N EXPANSION FOR ϕ^4 THEORY

To see the emergence of orders in $1/N$, consider a system with a set of N scalar fields, ϕ^a, $a = 1...N$, and its Lagrangian density is

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}\mu_0^2\phi^a\phi^a - \frac{1}{8}\lambda_0(\phi^a\phi^a)^2. \quad (1)$$

The scattering by perturbation theory is shown by diagrams in Fig. 1. The first diagram is of order $O(\lambda_0)$, the second one is $O(\lambda_0^2 N)$ since there are N possible choices for the internal index c, and the third one is $O(\lambda_0^2)$ due to fixed internal indices. However, if we define

$$g_0 \equiv \lambda_0 N, \quad (2)$$

and set it fixed as taking the limit of large-N, then the orders of the diagrams in Fig. 1 become $O(g_0/N)$, $O(g_0^2/N)$, and $O(g_0^2/N^2)$, respectively.

Then, following Ref. [1] and Ref. [2], by introducing an auxiliary field, σ, we can add a term to the Lagrangian density without affecting the dynamics of the theory:

$$\mathscr{L} \to \mathscr{L} + \frac{1}{2}\frac{N}{g_0}\left(\sigma - \frac{1}{2}\frac{g_0}{N}\phi^a\phi^a\right)^2. \quad (3)$$

The equation of motion for the auxiliary field, σ, is

$$\sigma = \frac{1}{2}\frac{g_0}{N}\phi^a\phi^a, \quad (4)$$

which is an equation of constraint. As a result, the Feynman rules of the new Lagrangian are different. Note that the new Lagrangian is

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}\mu_0^2\phi^a\phi^a + \frac{1}{2}\frac{N}{g_0}\sigma^2 - \frac{1}{2}\sigma\phi^a\phi^a, \quad (5)$$

where the only non-trivial interaction is the $\sigma\phi^a\phi^a$ coupling term. The corresponded graphs to this Lagrangian is shown in Fig. 2. Further, keeping only those ϕ lines in closed loops, plus doing all the momentum integrals over the closed ϕ loops, we get graphs with only σ lines, as shown by Fig. 3. For such graphs, there is an effective action, $S_{eff}(\sigma)$, which is obtained by functional integral over ϕs:

$$e^{iS_{eff}(\sigma)} = \int \prod_a [d\phi^a] e^{iS(\phi^a,\sigma)}. \quad (6)$$

Each term in the above effective action is proportional to N such that

$$S_{eff}(\sigma, N) = NS_{eff}(\sigma, 1). \quad (7)$$

In addition, each external and internal line carries a factor of $1/N$ since the propagator is obtained by inverting the quadratic part of the Lagrangian; each vertex carries a factor of N. Thus the order of a given graph is N^{V-I-E}. Using the relation $L = I - V + 1$, the order is finally N^{-E-L+1}. Consequently, the leading power of $1/N$ is given by graphs with no loops (i.e. tree graphs) and with the minimum number of external lines. As a result, the leading order of the above meson-meson scattering case is $1/N$.

In order to eliminate the infinite number of tree graphs with two external lines (i.e. the linear vertex for S_{eff}), we define a shifted field

$$\sigma' \equiv \sigma - \sigma_0, \quad (8)$$

where σ_0 is a constant chosen such that $\left.\frac{\delta S_{eff}}{\delta\sigma}\right|_{\sigma=\sigma_0} = 0$. Thus, the newly defined Lagrangian can be expressed, in terms of σ', as

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}\mu_1^2\phi^a\phi^a + \frac{1}{2}\frac{N}{g_0}\sigma'^2 - \frac{1}{2}\sigma'\phi^a\phi^a + \frac{N}{g_0}\sigma_0\sigma', \quad (9)$$

where the constant term is omitted and $\mu_1^2 \equiv \mu_0^2 + \sigma_0$. In the above Lagrangian, there is an additional linear vertex emerging from the last term which can be used to eliminate the linear (in σ) vertex of the previous statements.

THE GROSS-NEVEU MODEL

The Gross-Neveu model is a 2-dimension theory defined by

$$\mathscr{L} = \bar{\Psi}^a i\partial_\mu\gamma^\mu\Psi^a + \frac{g_0}{N}(\bar{\Psi}^a\Psi^a)^2, \quad (10)$$

where Ψ and $\bar{\Psi}$ are Dirac fields.

Duplicating the construction of $1/N$ approximation for ϕ^4 theory, introducing an auxiliary field, σ, we get the new Lagrangian from (11):

$$\mathscr{L} \to \mathscr{L} - \frac{N}{2g_0}\left(\sigma - \frac{g_0}{N}\bar{\Psi}^a\Psi^a\right)^2 = \bar{\Psi}^a i\partial_\mu\gamma^\mu\Psi^a - \frac{N}{2g_0}\sigma^2 + \sigma\bar{\Psi}^a\Psi^a. \tag{11}$$

Sequently, $S_{eff}(\sigma)$ can be obtained by integrating over Ψ loops. Since the trace of odd powers of Dirac matrices is zero, only even powers of the auxiliary field, σ, survive. The diagrams for consequent $S_{eff}(\sigma)$ are shown in Fig. 4. This is also followed by a conclusion that $\sigma = 0$ is a stationary point of $S_{eff}(\sigma)$.

Next, we'll show that other stationary points exist. To do this, we compute S_{eff} for constant σ. Define

$$-V(\sigma) = \lim_{L,T\to\infty} S_{eff}(\sigma)/LT. \tag{12}$$

Each stationary point of V leads to a $1/N$ expansion, and thus defines a possible vacuum state (in terms of $1/N$). In this philosophy, summing the Feynman diagrams in Fig. 4, we can calculate V to be

$$V = N\left[\frac{\sigma^2}{2g_0} - \int \frac{d^2p_E}{(2\pi)^2}\ln\left(1 + \frac{\sigma^2}{p_E^2}\right)\right]. \tag{13}$$

Note that the momentum integral is ultraviolet divergent, so we cut it off by restricting $p_E^2 \leqslant \Lambda^2$, where Λ is some large number. As a result, we have

$$V = N\left[\frac{\sigma^2}{2g_0} + \frac{1}{4\pi}\sigma^2\left(\ln\frac{\sigma^2}{\Lambda^2} - 1\right)\right]. \tag{14}$$

Further, rewrite this in terms of a renormalized coupling constant, g, which is conveniently defined by

$$\frac{1}{g} \equiv N^{-1}\left.\frac{d^2V}{d\sigma^2}\right|_M = \frac{1}{g_0} + \frac{1}{2\pi}\ln\frac{M^2}{\Lambda^2} + \frac{1}{\pi}, \tag{15}$$

such that V is given by

$$V = N\left[\frac{\sigma^2}{2g} + \frac{1}{4\pi}\sigma^2\left(\ln\frac{\sigma^2}{M^2} - 3\right)\right]. \tag{16}$$

By taking derivative of V over σ and setting it to be zero, we can find the stationary point

$$\sigma^2 = \sigma_0^2 \equiv M^2 exp\left(2 - \frac{2\pi}{g}\right), \tag{17}$$

and the effective potential at this point

$$V = -N\sigma_0^2/4\pi. \tag{18}$$

With the effective potential at this stationary point less than $V(0)$, this gives rise to the breakdown of symmetry, and to a nonzero mass, σ_0, of fermions.

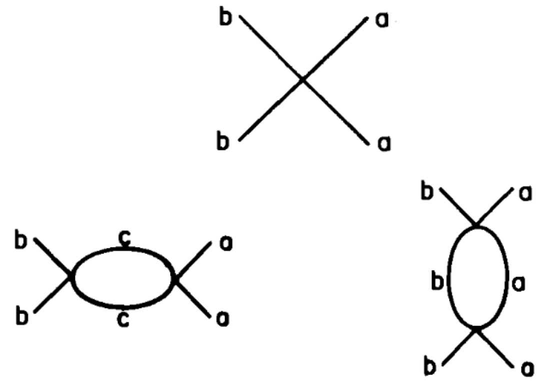

FIG. 1: First few diagrams for scattering of two mesons of type a into two mesons of type b with the Lagrangian given by (1).

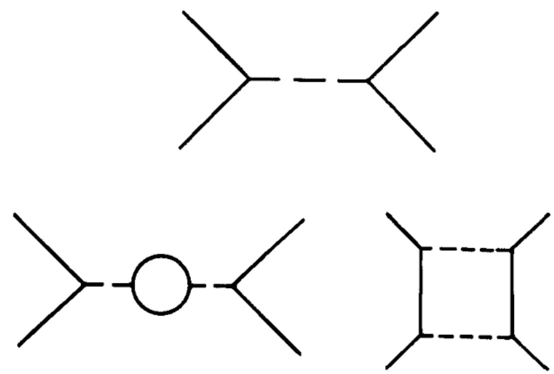

FIG. 2: Corresponded graphs with the Lagrangian given by (5). Dashed lines are σ propagators.

THE $\mathbb{C}P^{N-1}$ MODEL

The $\mathbb{C}P^{N-1}$ model is a formal limit of a linear theory. The linear theory is a theory of N^2-1 scalar fields, assembled into a $N \times N$ traceless Hermitian matrix, ϕ, with Lagrangian

$$\mathscr{L} = \frac{1}{2}\operatorname{Tr}\partial_\mu\phi\partial^\mu\phi - \lambda\operatorname{Tr}P(\phi), \tag{19}$$

where P is a polynomial in ϕ. This theory has SU(N) symmetry. Choose P such that the minima of $\operatorname{Tr}P$ are matrices with $N-1$ equal eigenvalues and one unequal eigenvalue. The ground states are then constant fields

$$\phi = g_0^{-1}[N^{\frac{1}{2}}zz^\dagger - N^{-\frac{1}{2}}I], \tag{20}$$

where z is constrained to be of unit length, i.e. $z^\dagger z = 1$.

As λ goes to infinity, the formal limit of the above theory turns into the $\mathbb{C}P^{N-1}$ model. Rescaling $z \to g_0 N^{-\frac{1}{2}}z$, the Lagrangian is simplified to

$$\mathscr{L} = \frac{1}{2}\operatorname{Tr}\partial_\mu\phi\partial^\mu\phi = (N/g_0^2)(\partial_\mu z^\dagger\partial^\mu z - j_\mu j^\mu), \tag{21}$$

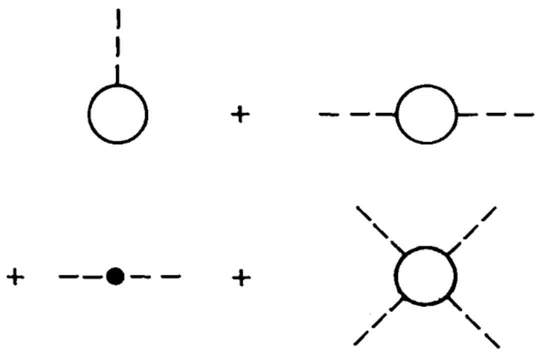

FIG. 3: Graphs for the effective action, $S_{eff}(\sigma)$.

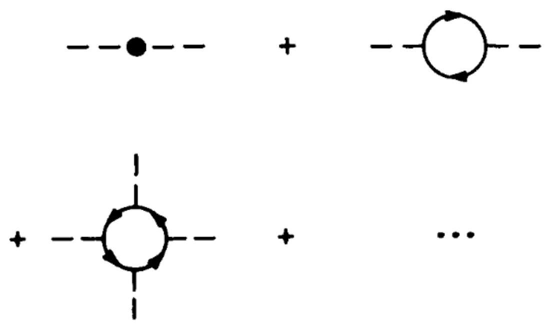

FIG. 4: Graphs for the effective action, $S_{eff}(\sigma)$, of Gross-Neveu model. Graphs with odd number of σ vertices vanish.

where $j_\mu \equiv (2i)^{-1}[z^\dagger \partial_\mu z - (\partial_\mu z^\dagger)z]$, and the constraint equation is

$$z^\dagger z = N/g_0^2. \tag{22}$$

With this Lagrangian, we can follow the same steps as in previous two sections, introducing an auxiliary field to eliminate quartic interactions, and then shifting to a stationary point of effective potential. The difference is that the auxiliary field is a vector field since the interaction term in the Lagrangian is in terms of vectors. The Lagrangian with the auxiliary vector field, A_μ, is

$$\mathscr{L} = (\partial_\mu - iA_\mu)z^\dagger(\partial_\mu + iA_\mu)z. \tag{23}$$

To get the Lagrangian satisfying the constraint, add another auxiliary field, σ, which is a scalar field and acts as a Lagrange multiplier. This leads to

$$\mathscr{L} = (\partial_\mu - iA_\mu)z^\dagger(\partial_\mu + iA_\mu)z - \sigma[z^\dagger z - g_0^{-2}N]. \tag{24}$$

The process of computing V is almost the same except for that now there are two auxiliary fields, A_μ and σ. Thus, with the coupling constant, g, renormalized, the effective potential is

$$V = -N\left[\frac{\sigma}{g^2} + \frac{\sigma}{4\pi}\left(\ln\frac{\sigma}{M^2} - 1\right)\right]. \tag{25}$$

As as result, the stationary point is at $\sigma = \sigma_0 \equiv M^2 exp(-4\pi/g^2)$ where $V = -\frac{N\sigma}{4\pi}\left(\ln\frac{\sigma}{\sigma_0} - 1\right)$.

[1] S. Coleman, "Notes from Sidney Coleman's Physics 253a: Quantum Field Theory," arXiv:1110.5013 [physics.ed-ph].

[2] S. R. Coleman, R. Jackiw and H. D. Politzer, Phys. Rev. D **10**, 2491 (1974).

[3] D. J. Gross and A. Neveu, Phys. Rev. D **10**, 3235 (1974).