The Unruh Effect

Luke Johns[1]

[1]*Department of Physics, University of California at San Diego, La Jolla, CA 92093-0319*

A derivation of the Unruh effect is sketched. The essential conceptual ideas are then brought into sharper relief by juxtaposing this derivation — which ultimately hinges on the time-dependent Doppler shift perceived by an accelerating observer — with the standard treatment of the phenomenon.

I. INTRODUCTION

In the four decades since the phenomenon was first elucidated (independently) by Unruh [1] and Davies [2], the Unruh effect has aroused vigorous discussion. What the Unruh effect alleges, in essence, is that a uniformly accelerating observer detects a thermal distribution of particles where an inertial observer detects only vacuum. This surprising idea was originally inspired by the recently postulated existence of Hawking radiation, which is thought to be emitted by black holes, but in fact the Unruh effect is a more general phenomenon that is present even in flat spacetimes. Although certain controversies surrounding the issue still linger, the existence of the Unruh effect is by now generally accepted and a number of derivations of it may be found in the literature.

In this paper we examine one particular derivation due to Alsing and Milonni [3]. In Section II we outline the relevant calculations. In Section III we then compare this treatment with the standard derivation, as exemplified by [4] and [5]. We conclude the paper in Section IV with a brief elaboration of a few additional noteworthy issues related to the Unruh effect.

II. THE ALSING–MILONNI DERIVATION

To establish the pertinent scenario, consider an inertial observer in flat spacetime and a second observer traveling with uniform acceleration along the positive z-axis with respect to the first observer. For convenience we will dub the inertial observer the "Minkowski observer" and the accelerating observer the "Rindler observer," out of deference to the coordinates in which their metrics are most conveniently expressed.

To be precise, the Rindler observer moves with constant acceleration a in a momentarily comoving reference frame, so that the Minkowski observer sees the Rindler observer accelerating at a rate

$$\frac{dv}{dt} = a\left(1 - \frac{v^2}{c^2}\right)^{3/2}, \tag{1}$$

where t is the coordinate time and v the velocity in the Minkowski frame. The three Lorentz γ factors are picked up by appropriately Lorentz-transforming the acceleration a into the Minkowski frame.

Rearranging Eq. (1) and integrating yields an expression for $v(t)$:

$$v(t) = \frac{at}{\sqrt{1 + \frac{a^2t^2}{c^2}}}, \tag{2}$$

after imposing the boundary condition $v(t = 0) = 0$. Ultimately we wish to express the Minkowski observer's coordinates z and t in terms of the Rindler observer's proper time τ, hence we avail ourselves of the relation $dt = \gamma d\tau$, where γ is a function of $v(t)$ (Eq. (2)), and integrate again to find

$$t(\tau) = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right), \tag{3}$$

having also set $t(\tau = 0) = 0$. Substitution of Eq. (3) into Eq. (2) produces $v(\tau) = c\tanh(a\tau/c)$. Using the chain rule and integrating one final time, we obtain

$$z(\tau) = \frac{c^2}{a}\cosh\left(\frac{a\tau}{c}\right), \tag{4}$$

which together with Eq. (3) supplies us with the Minkowski-frame coordinates of the accelerating observer, parametrized by the latter's proper time.

The utility of Eqs. (3) and (4) will now become apparent. Consider a plane wave with wave vector k and frequency ω_k moving parallel to the direction along which the Rindler observer accelerates. To the Minkowski observer the plane wave has phase $\varphi(t, z) = kz - \omega_k t$, but substituting the two aforementioned equations reveals that to the Rindler observer the same plane wave has phase

$$\varphi(\tau) = \left(\frac{\omega_k c}{a}\right)\exp\left(-\frac{a\tau}{c}\right). \tag{5}$$

(If the plane wave were moving anti-parallel to the Rindler observer's acceleration, the minus sign in the exponential would instead be a plus sign.) For small values of $a\tau$ we can expand this expression to get $\varphi(\tau) \approx \omega_k(1 - a\tau/c)$, which suggests that the accelerated observer witnesses a time-dependent Doppler shift of the plane-wave phase.

Let us introduce a massless scalar field ϕ quantized in a box of volume V. For simplicity we consider the field only at the origin, in which case we have

$$\phi(t) = \sum_k \left(\frac{2\pi\hbar c^2}{\omega_k V}\right)^{1/2}\left[a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right], \tag{6}$$

where a_k and $a_k^\dagger$ are the annihilation and creation operators for the mode with $k = \omega_k/c$. Suppose further that Eq. (6) is in fact the quantization of ϕ relative to the Rindler observer, so that, for instance, $a_k |0_R\rangle = 0$, where $|0_R\rangle$ is the Rindler vacuum. To rewrite Eq. (6) in terms of τ rather than t we substitute the expression in Eq. (5) for the phase $\varphi = \omega_k t$, thus obtaining $\phi(\tau) \sim a_k \exp\left[i\left(\omega_k c/a\right)\exp\left(-a\tau/c\right)\right] + a_k^\dagger \exp\left[i\left(\omega_k c/a\right)\exp\left(a\tau/c\right)\right]$.

The quantity we wish to examine now is the correlation function $\left\langle g^\dagger(\Omega) g(\Omega')\right\rangle$, where $g(\Omega)$ is the Fourier transform of ϕ, defined by

$$g(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau \phi(\tau) e^{i\Omega\tau}, \tag{7}$$

and the expectation value is taken with respect to the Rindler vacuum. The reason for considering the correlation function will become clear momentarily, but for now we merely note that it evaluates to

$$\left\langle g^\dagger(\Omega) g(\Omega')\right\rangle = \frac{2\hbar c/\Omega}{e^{2\pi\Omega c/a} - 1}\delta\left(\Omega - \Omega'\right). \tag{8}$$

(The full calculation requires several lines and can be found in [3]; we settle for citing the result rather than getting bogged down by the integrals that must be performed.) In this computation we have, crucially, used the fact that $\left\langle a_k a_{k'}^\dagger \right\rangle = \delta_{kk'}$, which follows from our earlier assertion that a_k and $a_k^\dagger$ are the annihilation and creation operators associated with the Rindler observer.

The appearance of the Planck factor $\left(e^{2\pi\Omega c/a} - 1\right)^{-1}$ is suggestive of the result we have been striving for: namely, that the accelerating observer detects a thermal spectrum of particles where the inertial observer detects vacuum. But to make this conclusion more rigorous we must verify that Eq. (8) is indeed the correlation function that would be measured by an inertial detector immersed in a bosonic system at temperature T. In this case the number operator has expectation value $\left\langle a_k^\dagger a_k \right\rangle = \left(e^{\hbar\omega_k/kT} - 1\right)^{-1}$. Computing the Fourier transform with respect to t,

$$g(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \phi(t) e^{i\Omega t}, \tag{9}$$

the correlation function evaluates to

$$\left\langle g^\dagger(\Omega) g(\Omega')\right\rangle = \frac{2\hbar c/\Omega}{e^{\hbar\Omega/kT} - 1}\delta\left(\Omega - \Omega'\right). \tag{10}$$

Comparing Eq. (8) with Eq. (10) we see that the Rindler observer does indeed detect a thermal spectrum at temperature

$$T = \frac{\hbar a}{2\pi k c}. \tag{11}$$

This result is the Unruh effect, and the temperature in Eq. (11) is known as the Unruh temperature.

III. COMPARISON WITH OTHER DERIVATIONS

The derivation sketched above (due to [3]) provides a slightly different perspective on the Unruh effect than the standard derivation (e.g., [4] and [5]) does. It is illuminating to juxtapose the two derivations; in so doing we will flesh out some of the conceptual details underlying the phenomenon.

In the standard derivation of the Unruh effect one considers flat spacetime in Rindler coordinates, given by Eqs. (3) and (4). The trajectories specified by these coordinates are confined to the spacetime region $z > c|t|$ (called the right Rindler wedge, or RRW), but it is also possible to define coordinates (t, z) given by -1 times the right-hand sides of Eqs. (3) and (4). The trajectories specified by these coordinates are in turn confined to the region $z < 0, |z| < c|t|$ (called the left Rindler wedge, or LRW). Since the LRW and RRW are causally disconnected, the positive-frequency modes in the Rindler quantization must be constructed out of two sets of modes, one with support in the RRW and one with support in the LRW. The fact that a single positive-frequency Rindler mode cannot be constructed purely out of positive-frequency Minkowski modes implies that the Rindler annihilation operators a_R must be superpositions of Minkowski annihilation and creation operators. It then follows that $a_R |0_M\rangle$, where $|0_M\rangle$ is the Minkowski vacuum, cannot be zero, and therefore the Rindler and Minkowski vacua do not coincide.

The Unruh temperature, in this treatment, can be found by explicitly constructing the Rindler modes from the Minkowski modes using the Bogolubov transformation and then calculating $\langle 0_M | a_R^\dagger a_R |0_M\rangle$. The result reproduces the thermal spectrum we derived in the previous section, but the calculation is more laborious. Conceptually, the difference between the standard derivation and the one presented in Section II can be phrased as follows: The standard derivation computes the spectrum of particles detected by the Rindler observer in the Minkowski vacuum, whereas the derivation outlined in Section II demonstrates that the correlation function for a Rindler observer in his own vacuum is identical to that of a Minkowski observer in a thermal state. The correlation function, in a sense, is just a calculational convenience for showing this equivalence; the key ingredient in the derivation, ultimately, is the time-dependent Doppler shift (Eq. (5)) perceived by the Rindler observer due to his acceleration.

IV. FURTHER DISCUSSION

As noted in the Introduction, a connection exists between the Unruh effect and Hawking radiation, since a stationary observer outside the horizon of a black hole is accelerating relative to an observer freely falling into a black hole. Indeed, the temperature of radiation outside of a black hole is expected to be precisely the Unruh temperature, with a replaced by the gravitational acceleration at the surface of the black hole. However, as pointed out in [6], since the Schwarzschild metric becomes inertial at large distances from the source, the particles radiated by a black hole are in fact detectable by inertial observers, unlike in the flat-spacetime Unruh effect where the particles are artifacts of the acceleration that are undetectable by inertial observers.

Some authors (e.g., [4]) have argued that the Unruh effect, which is necessitated by the internal consistency of free-field quantum field theory, requires no experimental verification beyond that of quantum field theory itself. Nonetheless, there has been much discussion of the possibility of detecting the Unruh effect in the laboratory. The remoteness of the Unruh effect from everyday experience can easily be seen by computing the magnitude of the effect due to the Earth's gravitational field: Setting $a = g$ in Eq. (11) yields a minuscule Unruh temperature of 4×10^{-20} K. As one might therefore expect, experiments aimed at producing a detectable Unruh effect — such as using ultra-intense lasers to accelerate electrons extremely rapidly [7] — have proved to be very challenging to pull off.

We end this paper by addressing one final puzzle [5] relating to the Unruh effect. If the Minkowski observer sees an energy–momentum tensor with expectation value $\langle T_{\mu\nu} \rangle = 0$, how is it possible that the Rindler observer can detect particles? The solution is that work is being done on the Rindler detector in order to sustain its uniform acceleration. The energy that the Rindler observer associates with the thermal distribution of particles is accounted for by the Minkowski observer as the energy supplied by whatever agent is accelerating the detector. Following [6], we can take this line of thinking a bit further by supposing that the detector carried by the Rindler observer is charged, in order to detect charged particles that are "produced" by the Unruh effect. What we have seen in this paper is that the Rindler observer believes his detector registers particles because he is immersed in a thermal spectrum of particles rather than vacuum. But the Minkowski observer, on the other hand, believes that the detections are registered due to the radiation reaction from accelerating the charged detector.

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