Fiber Bundles and Gauge Theory

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Gauge theory studies the redundant degrees of freedom in a physical system by determining which transformations keep the action of the theory invariant. These redundant degrees of freedom are mathematically described by the theory of fiber bundles. The connections between fiber bundles and gauge theory are discussed, and these ideas are then applied to an example from scalar $O(n)$ gauge theory.

INTRODUCTION

Gauge theory studies the redundant degrees of freedom in a physical system by determining which local transformations keep the action of the theory invariant. Gauge theories are physically important for a number of reasons, potentially the most important being they serve as the basis for our understanding of the Standard Model. The $SU(3) \times SU(2) \times U(1)$ gauge group, for instance, predicts and is consistent with the existence of the fundamental particles we know to exist.

Gauge theories are mathematically described by objects called fiber bundles, which are generally studied as topological or geometric spaces. Viewing gauge theories as a geometric structure of fiber bundles gives us a deeper understanding of the physics.

The next section includes a few mathematical definitions that will be relevant for the discussion. If the reader is not familiar with these definitions, or other mathematical concepts that are introduced, refer to [2], [4] for a further discussion on these topics.

The following section develops scalar $O(n)$ gauge theory with the theory of fiber bundles. The remaining section is included as a summary of the connections between the mathematics and physics.

MATHEMATICAL PRELIMINARIES

A few mathematical definitions and results are necessary before the connection between fiber bundles and gauge theory can be made.

Definition: A *fiber bundle* is a structure $(\mathbb{E}, \mathbb{M}, \mathbb{F}, G, \pi)$, where:

- $\mathbb{E}$ is the "total space"
- $\mathbb{M}$ is the "base space" (in this paper, this corresponds to an $\mathbb{R}^4$ space-time, though it need not be this in general)
- $\mathbb{F}$ is the "fiber space". The fiber above a single point is denoted $\mathbb{F}_x = \pi^{-1}(x)$
- G is the "structure group" with left action on $\mathbb{F}$
- $\pi : \mathbb{E} \to \mathbb{M}$ is a projection map

These objects are best understood graphically (see Figure 1). The fiber space can be thought of as strands sticking off of the base space. Any point along a single fiber will get mapped to the same base point under the projection map. Although the fibers are typically drawn in as being one-dimensional, they can be of arbitrary dimension. Mathematically, $\pi(g) = x \quad \forall \quad g \in \mathbb{F}_x$. The structure group is used to move along points in $\mathbb{F}_x$.

To give an example take $\mathbb{M} = S^1$, the circle in $\mathbb{R}^2$. Take the fiber at each point to be a finite line segment, which can be visualized as sticking off of S^1 into $\mathbb{R}^3$. The trivial fiber space is a cylinder; the non-trivial fiber space is the Mobius strip. In this case, the difference between trivial and non-trivial is that in the latter case, the fibers twist around in $\mathbb{R}^3$, whereas in the former case they do not.

Definition: A fiber bundle that has $\mathbb{F}_x \cong G$ is called a *principle bundle*, where the equality is taken as a homeomorphism. Fiber bundles will be assumed to be principal from now on. Note that if this is the case, G is a Lie Group.

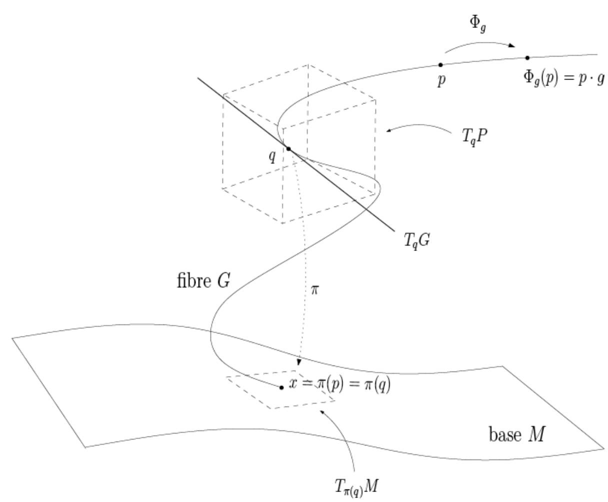

FIG. 1: A principle bundle which shows: $\mathbb{M}$, $\mathbb{F}_x$, G, π. Image from [1].

Definition: A *connection* on $\mathbb{E}$ is a unique separation of the tangent space into a horizontal and vertical subspace, that satisfies various "nice" properties. A connection is used to move a vector around $\mathbb{E}$ in a well defined way. Well defined here just means that the connection allows us to compare vectors in different parts of $\mathbb{E}$.

Definition: A *local section* is a continuous map $s : U \to \mathbb{E},\ U \in \mathbb{M}$, such that $(\pi \circ s)|_U = id_U$. See Figure 2.

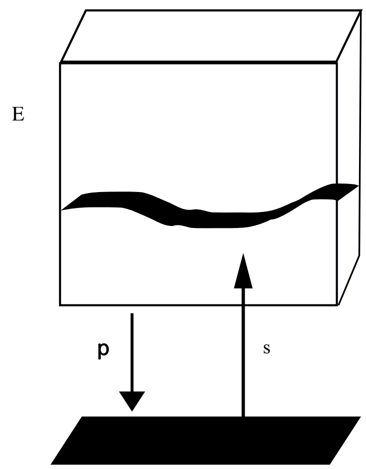

FIG. 2: A section of $\mathbb{E}$ and corresponding π, $\mathbb{M}$. Image from [3].

EXAMPLE: SCALAR $O(n)$ GAUGE THEORY

To elaborate on the connection between gauge theory and fiber bundles, scalar $O(n)$ gauge theory will be discussed in this section. Assuming there are n real scalar fields of equal mass, the Lagrangian density takes the form

$$\mathcal{L}_{glob} = \frac{1}{2}(\partial^\mu \Phi)^T(\partial_\mu \Phi) - \frac{1}{2}m^2\Phi^T\Phi \tag{1}$$

where $\Phi^T \equiv (\phi_1, \phi_2, ..., \phi_n)$. For a gauge transformation $\Phi \to \Phi' = G\Phi$, $G \in O(n)$, the Lagrangian density will automatically remain invariant if G does not vary over $\mathbb{E}$. If G is allowed to vary over $\mathbb{E}$, a little more work is required to maintain Lagrangian invariance. In the former case, the theory is said to have a global symmetry; the latter theory is said to have a gauge symmetry. Explicitly:

Case 1: Global symmetry.
$\partial_\mu G = 0\ \forall\ x \in \mathbb{E} \Rightarrow (\partial_\mu \Phi)' = \partial_\mu{}'\Phi' = G\partial_\mu\Phi$.
Lagrangian invariance is automatically maintained. To show this, simply use $(AB)^T = B^T A^T$. Geometrically, this means the connection is trivial.

Case 2: Gauge symmetry.
$\partial_\mu G \neq 0\ \forall\ x \in \mathbb{E} \Rightarrow (\partial_\mu \Phi)' = \partial_\mu{}'\Phi' \neq G(\partial_\mu\Phi)$.
Lagrangian invariance is not automatically maintained. Geometrically, this means a nontrivial connection needs to be established.

Since gauge symmetries are global symmetries as well, only Case 2 will be discussed further. It will turn out that enforcing a gauge symmetry will produce gauge bosons, which mathematically are seen as generators of the connection. To start out, let's try to reproduce the global symmetry statement by introducing a new differential operator so that

$$(D_\mu\Phi)' = D_\mu{}'\Phi' \equiv GD_\mu\Phi$$

If such a D_μ exists, then $\mathcal{L}$ will trivially remain invariant. In this case, $\mathcal{L}$ is seen to be an extension of Equation 1:

$$\mathcal{L}_{gauge} = \frac{1}{2}(D^\mu \Phi)^T(D_\mu \Phi) - \frac{1}{2}m^2\Phi^T\Phi \tag{2}$$

D_μ is called the gauge covariant derivative. Since D_μ should involve ∂_μ in some way, the naive approach would be to add a vector, let's call it η_μ, to ∂_μ. This turns out to be the correct approach in scalar $O(n)$ gauge theory. With no proof, η_μ turns out to be proportional to the gauge field:

$$D_\mu = \partial_\mu + \eta_\mu = \partial_\mu + igA_\mu(x)$$

where g is a physical coupling constant, and i is included as a convention. Plugging D_μ into the above definition gives:

$$A'_\mu = GA_\mu G^{-1} + \frac{i}{g}(\partial_\mu G)G^{-1}$$

This transformation is equivalent to how connection one-forms are defined in $\mathbb{E}$, so A_μ is seen to be a connection. Specifically, it is a Lie-Algebra valued differential one-form. We see that a gauge transformation $\Phi \to \Phi' = G\Phi$ forces A_μ to transform as given above. So requiring a gauge symmetry produces interactions in the scalar fields. This can be seen by taking the difference of the gauge symmetric and globally symmetric Lagrangians, which is called the "interaction Lagrangian":

$$\begin{aligned}&\mathcal{L}_{gauge} - \mathcal{L}_{glob} \equiv \mathcal{L}_{int}\\ &= i\frac{g}{2}\Phi^T A_\mu^T(\partial^\mu\Phi) + i\frac{g}{2}(\partial_\mu\Phi)^T A^\mu\Phi - \frac{g^2}{2}(A_\mu\Phi)^T A^\mu\Phi\end{aligned} \tag{3}$$

Since there are no derivatives of A_μ in this theory, this field is static. If we want to impose dynamics on it, a term needs to be added that is gauge and Lorentz invariant, and involves derivatives of A_μ. One way to do this is to introduce the Yang-Mills field Lagrangian:

$$\mathcal{L}_{YM} = -\frac{1}{2}Tr(F^{\mu\nu}F_{\mu\nu}), \quad F^{\mu\nu} \equiv \frac{1}{ig}[D_\mu, D_\nu]$$

So, we are left with a gauge symmetric theory with a dynamical gauge field:

$$\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{YM} = \mathcal{L}_{glob} + \mathcal{L}_{int} + \mathcal{L}_{YM} \qquad (4)$$

CONNECTION TO FIBER BUNDLES

Some connections were made in the previous section between scalar $O(n)$ gauge theory and fiber bundles. A further discussion of these connections follows. Table I summarizes these connections.

$\mathbb{E}$ is seen to be the "arena" in which most of the concepts of gauge theory live. Gauge transformations push points along each $\mathbb{F}_x$ to other points in the same fiber. A connection (also seen to be the gauge field) in $\mathbb{E}$ establishes sections in $\mathbb{E}$. The physics of the scalar fields can either be studied in $\mathbb{M}$ where they are defined to "live", or in a local section of $\mathbb{E}$. Geometrically, this can be visualized as the gauge transformation pushing the section up or down in $\mathbb{E}$.

$A_\mu(x)$ can be decomposed as a sum of generators of G, so there are as many gauge fields as there are generators: $A_\mu = \sum_a A^a_\mu T^a$. For instance, taking an $SO(2)$ gauge theory, we see that there is one gauge field produced by the single generator. Since $SO(2)$ is isomorphic to $U(1)$, which is the gauge in electromagnetism, this gauge field corresponds to the photon.

The connection $A_\mu(x)$ can be interpreted as a differential one-form. A generalization to curvature in differential geometry is the covariant derivative of the connection one-form, which in the case of Riemannian geometry, turns out to be the Riemann curvature tensor. So, $F_{\mu\nu}$ can be considered a local curvature at points in $\mathbb{E}$.

Fiber theory objects	Gauge theory objects
$\mathbb{E}$	Where sections are defined
$\mathbb{F}_x$	Where gauge and global symmetries take place
$\mathbb{M}$	The original manifold where real scalar fields are defined, i.e. space-time. This can be $\mathbb{R}^4$ or some non-trivial manifold.
Local section	Where the real scalar fields can also be studied
G	Generates gauge bosons
Connection	Gauge potential
Curvature	$F_{\mu\nu}$

TABLE I: Relationships between mathematical and physical concepts

Acknowledgements I would like to thank Professor Benjamin Grinstein for his valuable insights into this topic.

[1] R. Way, "Introduction to Connections on Principal Fibre Bundles," March 2010
[2] M. Nakahara, "Geometry, Topology and Physics," Second Edition
[3] http://en.wikipedia.org/wiki/Section_(fiber_bundle)
[4] C. Nash, S. Sen, "Topology and Geometry for Physicists," First Edition